f) Discrete Symmetries

parity P, charge conjugation C, time reversal T

i) Parity:
$$x = (x_0, \vec{x}) \longmapsto x_P = (x_0, -\vec{x})$$

$\hat{P}$: unitary operator

* scalar field: $\hat{P}\, \phi(x)\, \hat{P}^{-1} = \eta_P\, \phi(x_P)$

  η_P: intrinsic parity, $|\eta_P| = 1$

  ϕ hermitian: $\eta_P = +1$ scalar
  $\eta_P = -1$ pseudo-scalar

  (ϕ complex: η_P can be absorbed in phase of ϕ)

  in modes: $\hat{P}\, a(p)\, \hat{P}^{-1} = \eta_P\, a(p_P)$
  $\hat{P}\, b^\dagger(p)\, \hat{P}^{-1} = \eta_P\, b^\dagger(p_P)$

  $\hat{P}|0\rangle = |0\rangle \implies \hat{P}|p\rangle = \eta_P^* |p_P\rangle$

* vector fields: $\hat{P}\, V^\mu(x)\, \hat{P}^{-1} = \eta_P\, V_\mu(x_P)$

  V hermitian: $\eta_P = 1$ vector
  $\eta_P = -1$ pseudo/axial vector

* Dirac field: $\hat{P}\, \psi(x)\, \hat{P}^{-1} = \eta_P\, \gamma^0\, \psi(x_P)$

  $\gamma^0 \iff$ compatibility with Dirac eqn.

Bilinears:
$$\bar{\psi}(x)\psi(x) \to \bar{\psi}(x_P)\psi(x_P) \quad \text{scalar}$$
$$\bar{\psi}(x)\gamma_5\psi(x) \longrightarrow -\bar{\psi}(x_P)\gamma_5\psi(x_P) \quad \text{pseudo-scalar}$$
$$\bar{\psi}(x)\gamma^\mu\psi(x) \longrightarrow \bar{\psi}(x_P)\gamma_\mu\psi(x_P) \quad \text{vector}$$
$$\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) \longrightarrow -\bar{\psi}(x_P)\gamma_\mu\gamma^5\psi(x_P) \quad \text{axial vector}$$

iii) Charge conjugation

$$\hat{C}\,|p, \text{particle}\rangle = \eta_c^* \,|p, \text{anti-particle}\rangle \qquad \hat{C}: \text{unitary}$$
$$|\eta_c| = 1$$

* scalar: $\hat{C}\,\phi(x)\,\hat{C}^{-1} = \eta_c\,\phi(x)^\dagger$
  $\hat{C}\,\phi^\dagger(x)\,\hat{C}^{-1} = \eta_c^*\,\phi(x)$

  ϕ hermitian: $\eta_c = \pm 1$ (otherwise η_c absorbed into phase of Φ)

* Dirac fermion

Ansatz: $\psi(x) \longrightarrow \eta_c \psi^c(x) = \eta_c\, C\, \bar{\psi}(x)^t$

with C s.t. ψ^c satisfies Dirac eqn.

Result: need $C(\gamma^\mu)^t C^{-1} = -\gamma^\mu$

$\Rightarrow C^\dagger = C^{-1}, \quad C^T = -C$ (see tutorial)

$\hookrightarrow$ Dirac repr.: $C = i\gamma^0\gamma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}$

$\rightarrow \quad \hat{C}\,\psi(x)\,\hat{C}^{-1} = \eta_c\, C\, \bar{\psi}^t$

$\hat{C}\,\bar{\psi}(x)\,\hat{C}^{-1} = -\eta_c^*\, \psi^t C^{-1}$

Find: $\hat{C}\, a(p,\lambda)\, \hat{C}^{-1} = \eta_c\, b(p,\lambda)$
$\hat{C}\, b^\dagger(p,\lambda)\, \hat{C}^{-1} = \eta_c\, a^\dagger(p,\lambda)$

Majorana spinor: $\psi(x) = \psi^c(x)$ — particle its anti-particle

Find: current $\bar{\psi}(x)\gamma_\mu\psi(x) = j_\mu$

$$\hat{C}\, j_\mu(x)\, \hat{C}^{-1} = -j_\mu(x)$$

* Vector: $\hat{C} A_\mu(x) \hat{C}^{-1} = - A_\mu$ so that QED C invariant

$C \longleftrightarrow$ helicity:

$$(\psi_L)^c = C \, \overline{\psi_L}^t = C \cdot \frac{1}{2}(1+\gamma_5^t)\,\overline{\psi}^t = \frac{1}{2}(1+\gamma_5)\psi^c$$

$$(C \gamma_5^t C^{-1} = \gamma_5)$$

$$\boxed{(\psi_L)^c = (\psi^c)_R}$$

charge conjugation reverses parity!

Useful identities:

$$C^\dagger = C^{-1} \quad , \quad C^t = -C$$

$$(\gamma^\mu)^t = - C^{-1} \gamma^\mu C$$

$$\psi^c = C \gamma^0 \psi^*$$

Time reversal

* exchanges initial & final states with identical position & opposite velocities/momenta
* Consistent action on S-matrix requires $\hat{T}$ to be anti-linear: $\hat{T}(a|\psi\rangle) = a^* \hat{T}|\psi\rangle$
* **Results**: * $\hat{T}\phi(x)\hat{T}^{-1} = \eta_T \phi(x_T)$ $\quad x_T = (-x^0, x^i)$

$$\Leftrightarrow \hat{T} a(p) \hat{T}^{-1} = \eta_T\, a(p_P)$$

* $\hat{T}\psi(x)\hat{T}^{-1} = \eta_T B^{-1}\psi(x_T)$ $\quad B = \gamma^5 C$

$$\Rightarrow \hat{T}\bar{\psi}(x)\psi(x)\hat{T}^{-1} = \bar{\psi}(x_T)\psi(x_T) \quad \text{scalar}$$

$$\hat{T}\, j^\mu(x)\hat{T}^{-1} = \bar{\psi}(x_T) B (\gamma^\mu)^* B^{-1}\psi(x_T)$$

$$\Rightarrow \hat{T} j^0(x)\hat{T}^{-1} = j^0(x_T)$$

$$\hat{T} j^i \hat{T}^{-1} = -j^i(x_T)$$

* CPT theorem

Any Lorentz invariant Lagrangian $\mathcal{L}(x)$ formed from products of quantum fields at point x (locality) is invariant under CPT.

* The axial interaction of the SM violates C & P separately:

e.g. $\mathcal{L}_{int} = j^\mu_V A_\mu(x) - j^\mu_A(x) A_\mu(x)$

$$\xrightarrow{C} j^\mu_V A_\mu + j^\mu_A A_\mu$$

* The complex phase of the CKM matrix is responsible for violation even of CP:

$$\mathcal{L}_q = -\frac{g}{2\sqrt{2}} \left(\bar{u}_i \gamma^\mu (1-\gamma^5) V_{ij} d_j W_\mu + \bar{d}_j \gamma^\mu (1-\gamma^5) V_{ij}^* u_i W_\mu^+ \right)$$

CP: $\bar{u}_i \gamma^\mu (1-\gamma^5) d_j W_\mu \longrightarrow \bar{d}_j \gamma^\mu (1-\gamma^5) u_i W_\mu^+$

$\bar{d}_j \gamma^\mu (1-\gamma^5) u_i W_\mu^+ \longrightarrow \bar{u}_i \gamma^\mu (1-\gamma^5) d_j W_\mu$

$$\Rightarrow \mathcal{L}_q \longrightarrow -\frac{g}{2\sqrt{2}} \left(\bar{u}_i \gamma^\mu (1-\gamma^5) V_{ij}^* d_j W_\mu + \bar{d}_i \gamma^\mu (1-\gamma^5) V_{ij} u_j W_\mu^+ \right)$$

Since $V_{ij} \neq V_{ij}^*$ $\longleftrightarrow$ CP violation