e) Quarks

In the quark sector we need to distinguish between

~ electroweak eigenstates = states for which generalised/gauged kinetic terms are diagonal

~ mass eigenstates

* El.-weak eigenstates: 3 families of quarks (u', d') (c', s') (t', b') in representations:

$$L_1 = \begin{pmatrix} u' \\ d' \end{pmatrix}_L \qquad L_2 = \begin{pmatrix} c' \\ s' \end{pmatrix}_L \qquad L_3 = \begin{pmatrix} t' \\ b' \end{pmatrix}_L \qquad \Big\} \; T = \tfrac{1}{2} \; SU(2)$$

$$\left.\begin{matrix} R_{1+} = u'_R & R_{2+} = c'_R & R_{3+} = t'_R \\ R_{1-} = d'_R & R_{2-} = s'_R & R_{3-} = b'_R \end{matrix}\right\} \; SU(2) \text{ singlets}$$

Compare: For leptonic sector without ν_R only 1 right-handed field $R_1 = e_R$ etc.

* Hypercharge follows from el.-magn. charge

L_N: $Y = \frac{1}{6}$ $\longrightarrow$ $Q_{u'} = 2/3$, $Q_{d'} = -1/3$

R_{N+}: $Y = \frac{2}{3}$

R_{N-}: $Y = -\frac{1}{3}$

$\longrightarrow$ Gauge invariant kinetic Lagrangian:

$$\mathcal{L}_{quark} = \overline{L}_N i\gamma^\mu D_\mu L_N + \overline{R}_{N+} i\gamma^\mu D_\mu R_{N+} + \overline{R}_{N-} i\gamma^\mu D_\mu R_{N-}$$

Note: With this Y assignment $U(1)_Y$ is free of anomalies (see tutorial!)

Quark masses

Possible Yukawas are constrained by hypercharge

* Down-type quarks acquire masses in same manner as leptons:

$$\mathcal{L}_{\text{quark},\Phi} = -\sqrt{2}\left(\underset{\downarrow}{\overline{L}_N}\, f^{(-)}_{NM}\, \underset{\downarrow}{\Phi}\, R_{M-} + \overline{R}_{M-}\, f^{*(-)}_{NM}\, \Phi^{+} L_N\right)$$

$$Y=-\tfrac{1}{6} \qquad Y=\tfrac{1}{2} \qquad Y=-\tfrac{1}{3} \quad \checkmark$$

* Up-type quarks:

$$\left.\begin{array}{ll} \overline{L}: & Y=-1/6 \\ R_{M+}: & Y=2/3 \end{array}\right\} \text{ need scalar with } Y=-1/2 \text{ in } T=1/2 \text{ rep.}$$

Solution: $$\phi^c = i\sigma_2\,\phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix} = \begin{pmatrix} \phi_2^+ \\ -\phi_1^+ \end{pmatrix}$$

$$\Rightarrow Y(\phi^c) = -\tfrac{1}{2}, \quad T = \tfrac{1}{2} \quad \checkmark$$

Proof: Use $i\sigma_2\, \underline{\sigma}^*\, i\sigma_2 = \underline{\sigma}$ to show
$i\sigma_2\left(\exp \tfrac{i}{2}\,\underline{\alpha}\cdot\underline{\sigma}\right)^* = \exp\left(i\,\underline{\alpha}\,\underline{\sigma}\right) i\sigma_2$ ✓

$$\mathcal{L}_{\text{quark}^+,\Phi} = -\sqrt{2}\left(\overline{L}_N\, f^{(+)}_{NM}\, \phi^c\, R_{M+} + \text{h.c.}\right)$$

SSB: $$\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \iff \phi_0^c = \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix}$$

$$\mathcal{L}_{\text{quark},m} = -\left(\overline{u}_L', \overline{c}_L', \overline{t}_L'\right) \mathcal{M}^+ \begin{pmatrix} u_R' \\ c_R' \\ t_R' \end{pmatrix} + \text{h.c.}$$

$$\qquad - \left(\overline{d}_L', \overline{s}_L', \overline{b}_L'\right) \mathcal{M}^- \begin{pmatrix} d_R' \\ s_R' \\ b_R' \end{pmatrix} + \text{h.c}$$

where: $$\left.\begin{array}{l} \mathcal{M}^+ \equiv \mathcal{M}^+_{NM} = v\, f^{(+)} \\ \mathcal{M}^- = v\, f^{(-)} \end{array}\right\} \text{ 3x3 complex matrices}$$

Mass eigenstates:

Diagonalise

$$\mathcal{M}^{(+)} \longrightarrow \mathcal{M}^{+}_{(D)} = V^{\dagger}_{(+)} \mathcal{M}^{(+)} U_{(+)} = \begin{pmatrix} m_u & & 0 \\ & m_c & \\ 0 & & m_t \end{pmatrix}$$

$$\mathcal{M}^{(-)} \longrightarrow \mathcal{M}^{-}_{(D)} = V^{\dagger}_{(-)} \mathcal{M}^{(-)} U_{(-)} = \begin{pmatrix} m_d & & 0 \\ & m_s & \\ 0 & & m_b \end{pmatrix}$$

Eigenstates:

$$\begin{pmatrix} u \\ c \\ t \end{pmatrix}_R = U^{\dagger}_{(+)} \begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_R , \quad \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L = V^{\dagger}_{(+)} \begin{pmatrix} u' \\ c' \\ s' \end{pmatrix}_L$$

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix}_R = U^{\dagger}_{(-)} \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_R , \quad \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L = V^{\dagger}_{(-)} \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L$$

<u>Note</u>: Due to appearance of $V_{(+)}$ & $V_{(-)}$ diagonalisation <u>not</u> compatible w/ $SU(2)$ doublet form.

EW Quark interactions $\longleftrightarrow$ from gauged kinetic terms

$$\mathcal{L}^{(I)}_{quark} = -\frac{g}{2\sqrt{2}} \left(J^{\mu} W_{\mu} + J^{\mu\dagger} W^{\dagger}_{\mu} \right) - e\, j^{\mu}_{e.m.} A_{\mu} - \frac{g}{2\cos\theta_W} J^{\mu}_{n} Z_{\mu}$$

* charged weak current:

$$J^{\mu} = 2 \bar{L}_N \gamma^{\mu} \sigma_+ L_N = 2 \left(\bar{u}' \; \bar{c}' \; \bar{t}' \right)_L \gamma^{\mu} \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L$$

$$= \left(\bar{u} \; \bar{c} \; \bar{t} \right) \gamma^{\mu} (1-\gamma_5)\, V \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

$V = V_{(+)}^{\dagger} V_{(-)}$ Cabbibo – Kobayashi – Maskawa (CKM) matrix

The CKM matrix leads to flavour-changing weak charged currents (see below)

Note: In lepton sector w/out ν_R: $V = \mathbb{1}$

$\rightsquigarrow$ did get $V_{(-)}$ analogue, but no $V_{(+)}$ because need not need to introduce mass eigenstates for neutrinos (more precisely: all neutrinos are mass degenerate)

$\rightsquigarrow$ V_- absorbed into definition of neutrinos

* el.-magn. current:

Find:
$$j^{\mu}_{e.m.} = -\frac{1}{3}\left(\bar{d}' \; \bar{s}' \; \bar{b}'\right)\gamma^{\mu}\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} + \frac{2}{3}\left(\bar{u}' \; \bar{s}' \; \bar{b}'\right)\gamma^{\mu}\begin{pmatrix} u' \\ c' \\ b' \end{pmatrix} =$$
$$= (q' \to q)$$

* neutral:
$$j^{\mu}_{n} = \frac{1}{2}\left(\bar{u} \; \bar{c} \; \bar{t}\right)\gamma^{\mu}\left(1 - \gamma_5 - \frac{8}{3}\sin^2\Theta\right)\begin{pmatrix} u \\ c \\ t \end{pmatrix} - \frac{1}{2}\left(\bar{d} \; \bar{s} \; \bar{b}\right)\gamma^{\mu}\left(1 - \gamma_5 - \frac{4}{3}\sin^2\Theta\right)\begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

$\Rightarrow$ no flavour changing el.-magn. current

$\Rightarrow$ no flavour changing neutral current

$\Leftarrow$ Glashow – Iliopoulos – Maiani (GIM) mechanism

A closer look at the CKM matrix

$V = V_+^{(\dagger)} V_-$ $\qquad V = V^\dagger$ unitary

Consider theory w/ N families $\Rightarrow$ 2N quark fields

A unitary $N\times N$ matrix has N^2 real parameters.

Some of them are redundant:

$\rightsquigarrow$ each quark field absorbs 1 real phase because kinetic terms $\bar{\psi}_i \, i\gamma \cdot \partial \, \psi_i$ invariant under $\psi_i \to e^{i\alpha_i}\psi_i$

$\rightsquigarrow$ out of these 2N phases, one overall rotation

$$q_{iL} \to e^{i\alpha} q_{iL} \,, \quad q_{iR} \to e^{i\alpha} q_{iR} \quad \text{leaves } \mathcal{J}^\mu \text{ inv.}$$

$$\Rightarrow \# \text{ d.o.f of } V = N^2 - (2N-1) = N^2 - 2N + 1$$

Ex.: $N = 2 \Rightarrow 1$ parameter $\longleftrightarrow$ rotation by 1 angle

$N = 3 \Rightarrow 4$ real parameters

$\Rightarrow V$ can be written as real orthogonal 3×3 matrix + 1 complex phase

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{td} & V_{tb} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \cdot \begin{pmatrix} c_3 & 0 & s_3 e^{-i\delta} \\ 0 & 1 & 0 \\ -s_3 e^{i\delta} & 0 & c_3 \end{pmatrix} \cdot \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

w/: $c_i = \cos\Theta_i \,, \quad s_i = \sin\Theta_i$

$\Leftrightarrow$ pairwise family mixing + 1 phase δ

Note: We will see that the complex phase is responsible for CP violation in EW sector.
$\Rightarrow$ CP violation requires at least 3 families!

* Experimentally: Mixing decreases with distance in family space:

$|V_{ud}| = 0.975 \quad |V_{cs}| = 0.974 \quad |V_{tb}| = 0.999$

$|V_{us}| = 0.22 \quad \longleftrightarrow$ 1-2 mixing

$|V_{cb}| = 0.04 \quad \longleftrightarrow$ 2-3 mixing

$|V_{ub}| = 0.003 \quad \longleftrightarrow$ 1-3 mixing

$\longleftrightarrow \quad \Theta_1 \gg \Theta_2 \gg \Theta_3$

* Unitarity constraint: $V^\dagger V = \mathbb{1}$ implies:

$$V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0$$

$\Rightarrow \quad a + b + c = 0 \qquad a, b, c \in \mathbb{C}$

$\Longleftrightarrow$ a, b, c form closed triangle (unitarity triangle)
$\longleftrightarrow$ deviations from closure parametrise BSM physics

* At energies $\ll m_t, m_b$ can effectively work in 2-family approximation:

$$V = \begin{pmatrix} \cos\Theta_c & \sin\Theta_c \\ -\sin\Theta_c & \cos\Theta_c \end{pmatrix}$$

Θ_c: Cabbibo angle
$\Theta_c \simeq 13°$

* Wolfenstein parametrisation:

$$V \cong \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

$A \simeq 1$
$\lambda \simeq \sin\Theta_c$
ρ, η: order 1
poorly determined