d) Leptons

* We need to identify the correct $SU(2) \times U(1)_Y$ representations of the leptons

$$\psi(x) \longmapsto e^{i \underline{\alpha}(x) \cdot \underline{I} + i \beta(x) Y} \psi(x)$$

$\underline{I} \leftrightarrow SU(2)$ representation $\qquad Y: U(1)_Y$ charge

* Consider just one family of leptons:

$$e_R \quad , \quad \nu_L, e_L \qquad (\nu_R \text{ will be dealt with later})$$

where: $e_R = \frac{1}{2}(1+\gamma_5)e$, $e_L = \frac{1}{2}(1-\gamma_5)e$ etc.

* Experimentally: Weak interaction couples only to left-handed states

$\Rightarrow$ Right-handed leptons are $SU(2)$ singlets

$$e_R(x) = R(x) \qquad \text{invariant under } SU(2) \text{ trafo}$$

If don't want to introduce extra states, the ν_L & e_L must form $SU(2)$ doublet:

$$L(x) = \begin{pmatrix} \nu_e(x) \\ e_L(x) \end{pmatrix}$$

* $Q_{e.m.} = T_3 + Y \qquad T_3 = \frac{1}{2}\sigma_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$

Experimentally:

$$\left.\begin{aligned} Q_{e.m.}(\nu_e) &= 0 \\ Q_{e.m.}(e_L) &= -1 \end{aligned}\right\} L: Y = -1/2$$

$$\rightarrow Q = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} - \frac{1}{2}\mathbb{1} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Q_{e.m.}(e_R) = -1 \qquad \Rightarrow R: Y = \underline{-1}$$

Covariant derivatives:

$$D_\mu L(x) = \left(\partial_\mu + ig\,\tfrac{1}{2}\,\underline{A}_\mu(x)\circ\underline{\sigma} - ig'\cdot\tfrac{1}{2}\,B_\mu(x)\right)L(x)$$
$$D_\mu R(x) = \left(\partial_\mu - ig' B_\mu(x)\right)R(x)$$

$\rightarrow$ purely leptonic terms:

$$\mathcal{L}_{lept.} = \bar{L}(x)\, i\gamma^\mu D_\mu L(x) + \bar{R}(x)\, i\gamma^\mu D_\mu R(x)$$

Expansion leads to the leptonic electro-weak interactions:

$$\mathcal{L}_{lept} = \bar{L}(x)\, i\gamma\cdot\partial\, L(x) + \bar{R}(x)\, i\gamma\cdot\partial\, R(x)$$
$$- g\left(\bar{L}\gamma^\mu\cdot\tfrac{1}{2}\underline{\sigma}\, L\right)\underline{A}_\mu + g'\left(\tfrac{1}{2}\bar{L}\gamma^\mu L + \bar{R}\gamma^\mu R\right)B_\mu$$

Rewrite these in terms of $W_\mu, W_\mu^+, Z_\mu, A_\mu$ using:

$$A_1 = \frac{1}{\sqrt{2}}(W + W^+) \qquad A_3 = \cos\Theta_W Z + \sin\Theta_W A$$
$$A_2 = \frac{i}{\sqrt{2}}(W - W^+) \qquad B = -\sin\Theta_W Z + \cos\Theta_W A$$

and $g' = \tan\Theta_W \cdot g$

Result:

$$\mathcal{L}_{lept} = \mathcal{L}_{kin} - \frac{g}{2\sqrt{2}}\left(J^\mu W_\mu + J^{\mu +} W_\mu^+\right) - e\, j^\mu_{e.m.} A_\mu$$
$$- \frac{g}{2\cos\Theta_W} J^\mu_n \cdot Z_\mu$$

where:

* $e = g\cdot\sin\Theta_W$
* J^μ: charged lepton-weak current

$$J^\mu = 2\,\bar{L}\,\gamma^\mu\,\sigma_+\,L \qquad \sigma_+ = \tfrac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= 2\cdot(\bar{\nu}_L, \bar{e}_L)\,\gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

$$= 2\,\bar{\nu}_L\,\gamma^\mu\,e_L = 2\,\bar{\nu}\,\tfrac{1}{2}(1+\gamma_5)\,\gamma^\mu\,\tfrac{1}{2}(1-\gamma_5)\,e$$

$$\boxed{J^\mu = \bar{\nu}\,\gamma^\mu(1-\gamma_5)\,e}$$

$\Longrightarrow$ $(V-A)$ like, couples to charged W^μ

* $j^\mu_{e.m.}$: electro-magnetic current

$$j^\mu_{e.m.} = \bar{L}\,\gamma^\mu\left(\tfrac{1}{2}\sigma_3 - \tfrac{1}{2}\mathbb{1}\right)L - \bar{R}\,\gamma^\mu R$$

$$= -\bar{e}_L\,\gamma^\mu\,e_L - \bar{e}_R\,\gamma^\mu\,e_R$$

$$\boxed{j^\mu_{e.m.} = -\bar{e}(x)\,\gamma^\mu\,e(x)}$$

$\rightsquigarrow$ purely V-like

$\rightsquigarrow$ The appearance of $g\cdot\sin\Theta_W = e$ justifies interpretation of $g\cdot\sin\Theta_W$ as electric charge

$\rightsquigarrow$ ν_L is uncharged under Q & thus does not couple to A_μ

$\Longrightarrow$ J^μ, $j^\mu_{e.m.}$ reproduce observed couplings.

In addition the theory predicts hitherto unseen couplings transmitted by Z_μ :

* neutral current J^μ_n

$$J^\mu_n = \bar{L}\gamma^\mu\left(\cos^2\Theta_W\,\sigma_3 + \sin^2\Theta_W\cdot\mathbb{1}\right)L$$
$$+ 2\sin^2\Theta_W\,\bar{R}\,\gamma^\mu R$$

$$J^\mu_n = \frac{1}{2}\left[\bar{\nu}_e\gamma^\mu(1-\gamma_5)\nu_e - \bar{e}\gamma^\mu(1-\gamma_5 - 4\sin^2\Theta_W)e\right]$$

$\Rightarrow$ Z_μ can decay to $\bar{\nu}_e\nu_e$ & $\bar{e}e$

discovered after SM was proposed $\Rightarrow$ important confirmation

Note: $e-e$ a) via $j^\mu_{e.m.} \leftrightarrow A^\mu$: V

b) via $J^\mu_N \leftrightarrow Z^\mu$: V & A part

Lepton mass

× gauge invariance forbids direct coupling of $\bar{L}(x)$ and $R(x)$

× mass terms are generated by coupling to Higgs doublet: Φ with $Y = +\frac{1}{2}$ after SSB:

$$\mathcal{L}_{lept,\Phi} = -\sqrt{2}\,\lambda_e\left(\underset{Y=+\frac{1}{2}}{\bar{L}(x)}\,\underset{Y=+\frac{1}{2}}{\Phi(x)}\,\underset{Y=-1}{R(x)} + \bar{R}(x)\Phi^\dagger(x)L(x)\right)$$

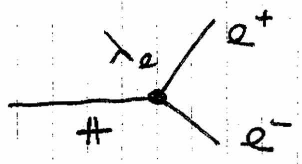
Such trilinear terms of type $\bar{\psi}\psi\phi$ are called Yukawa couplings.

In unitary gauge: $\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H\end{pmatrix}$

$$\Rightarrow \mathcal{L}_{lept,\Phi} = -\lambda_e(v+H)(\bar{e}_L e_R + \bar{e}_R e_L)$$
$$= -\lambda_e(v+H)\,\bar{e}e$$

$$\Rightarrow \boxed{m_e = \lambda_e \cdot v}$$

In addition: couplings



Note: $\mathcal{L}_{gauge} + \mathcal{L}_{lept} + \mathcal{L}_{lep,\Phi} + \mathcal{L}_{Higgs}$ contains all terms allowed by

* renormalisability
* $SU(2) \times U(1)_Y$ gauge invariance

Lepton family replication

* There have been observed <u>3 families of leptons</u>

$$L_1 = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \qquad L_2 = \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \qquad L_3 = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$$

$$R_1 = e_R \qquad R_2 = \mu_R \qquad R_3 = \tau_R$$

$$m_e = 511\ \text{keV} \qquad m_\mu = 100\ \text{MeV} \qquad m_\tau = 1.8\ \text{GeV}$$

* The gauge-covariant kinetic terms can always be brought into <u>family-diagonal form</u> = flavour-diagonal form

$$\mathcal{L}_{lept} = \sum_{N=1,2,3} \left(\bar{L}_N i\gamma^\mu D_\mu L_N + \bar{R}_N i\gamma^\mu D_\mu R_N \right)$$

* Then the most general Higgs interaction is of the form

$$\mathcal{L}_{lep,\Phi} = -\sqrt{2} \left(\bar{L}_N f_{NM} \Phi R_M + \bar{R}_M f^*_{NM} \Phi^+ L_N \right)$$

f_{NM}: general 3×3 family matrix

The <u>mass eigenstates</u> follow by diagonalising f_{NM}

Note: A general 3×3 matrix can always be diagonalised via a bi-unitary trafo

$$f \longrightarrow U^+ f V \qquad U, V \in U(3)$$

i.e.
$$R \longrightarrow V^+ R$$
$$L \longrightarrow U^+ L \qquad (*)$$

and $U^+ f V$ has positive eigenvalues

A priori this might render the kinetic terms non-diagonal.

However, $\mathcal{L}_{lept}$ is invariant under $(*)$

Reason: $(*)$ acts on full $SU(2)$ multiplets

$\rightarrow$ $\mathcal{L}_{lept,\Phi}$ & $\mathcal{L}_{lept}$ both diagonal simultaneously

$\Rightarrow$ $\boxed{\text{electro-weak eigenstates} = \text{mass eigenstates}}$

$\rightarrow$ no <u>flavour</u>-changing interactions in lepton sector.

<u>Note</u>: This will no longer be true once we introduce right-handed neutrinos & non-degenerate masses for neutrinos. (see later)