4) Standard Model of electro-weak interactions

a) Electro-weak gauge group

* Observationally: enlargement of $U(1)_Q$ – electromagn. theory w/ massless photon A_μ – by at least 2 massive, charged gauge bosons W_μ, W_μ^+

* Simplest choice of gauge group would be $U(1)_Q \subset SU(2)$

<u>But:</u> Not consistent w/ observations because $J^\mu_{e.m.}, J^\mu_{weak}, (J^\mu_{weak})^+$ do not close to $SU(2)$

* The Standard Model (SM) is based on second simplest choice:

$$G = SU(2) \times U(1)_Y \qquad U(1)_Y: \text{hypercharge}$$

$SU(2)$ generated by (T_1, T_2, T_3): $[T_a, T_b] = i\varepsilon_{abc} T_c$

$$T_a = \frac{1}{2}\sigma_a \qquad \sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$U(1)_Y$ generated by Y

Electro-magnetic $U(1)_Q$ embedded into $SU(2) \times U(1)_Y$:

$$\boxed{Q = T_3 + Y} \qquad \text{(mind different conventions)}$$

* $SU(2)$ gauge fields:

$$\underline{A}_\mu \cdot \underline{T} = A_{\mu a} T_a$$

$$\underline{A}_\mu \cdot \underline{T} \longrightarrow u\, \underline{A}\, \underline{T}\, u^{-1} + \frac{i}{g}(\partial_\mu u) u^{-1}$$

$$u = \exp\left(i\, \underline{\alpha}(x) \cdot \underline{T}\right) \qquad g: SU(2) \text{ coupling}$$

* $U(1)_Y$: $B_\mu(x) \longrightarrow B_\mu(x) - \frac{1}{g'} \partial_\mu \beta(x)$

g' : $U(1)_Y$ - coupling

⇝ $SU(2) \times U(1)_Y$ ~~trafo~~ : $g(\underline{\alpha}, \beta) = \exp(i\, \underline{\alpha} \cdot \underline{T}) \exp(i \beta Y)$

<u>Remark</u>: There is no <u>actual unification</u> of el-magn. & weak interaction due to 2 independent couplings g, g'

* Field strengths:

$SU(2)$: $\underline{F}_{\mu\nu} = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + \underbrace{g\, \underline{A}_\mu \times \underline{A}_\nu}_{= g\, \varepsilon_{abc} A_{\mu b} A_{\nu c}}$

$U(1)_Y$ = $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$

$\mathcal{L}_{gauge} = -\frac{1}{4} \underline{F}_{\mu\nu} \cdot \underline{F}^{\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu}$

b) Electroweak SSB

* Since $U(1)_Q$ is observed to be massless, the other 3 combinations of vector bosons acquire mass

SSB : $SU(2) \times U(1)_Y \longrightarrow U(1)_Q$

* One assumes a Higgs field $\underline{\Phi}$ in non-trivial repr. of $SU(2)$

simplest irrep.: fundamental = doublet

$$\underline{\Phi}(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \qquad \phi_1, \phi_2 : \text{ complex fields}$$

<u>Note</u>: Next simplest would be 3-component symmetric 2-tensor. But this turns out not to describe data.

$\underline{\Phi}$ must also carry $U(1)_Y$ charge so that a component of $\underline{\Phi}$ can remain $U(1)_Q$ - neutral.

Assignment : $Y = \frac{1}{2}$ $\Rightarrow$ $\phi_2(x)$ is $U(1)_Q$ neutral

$$\boxed{D_\mu \underline{\Phi}(x) = \left(\partial_\mu + i \cdot g \, \underline{A}_\mu \cdot \frac{\underline{\sigma}}{2} + i g' \frac{1}{2} B_\mu \right) \underline{\Phi}}$$

* SSB requires potential $V(\underline{\Phi})$ in

$$\mathcal{L}_{Higgs} = (D^\mu \underline{\Phi}(x))^\dagger (D_\mu \phi(x)) - V(\phi(x))$$

$$V(\underline{\Phi}(x)) = f(\underline{\Phi}^\dagger \underline{\Phi}) \longleftarrow SU(2) \times U(1)_Y \text{ invariant}$$

To be renormalisable, V can at most be quartic in $\underline{\Phi}$.

Take : $\boxed{V(\underline{\Phi}) = \frac{1}{2} \cdot \lambda \cdot \left(\underline{\Phi}^\dagger \underline{\Phi} - \frac{1}{2} v^2 \right)^2}$ (mind conventions)

* Ground state: $\phi_0^\dagger \phi_0 = \frac{1}{2} v^2$

Take : $\underline{\Phi}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$ v real $v > 0$

$\underline{\Phi}_0$ is chosen to have charge 0 under $U(1)_Q$:

$$Q \underline{\Phi}_0 = (T_3 + Y) \phi_0 = \left(\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \underline{\Phi}_0 =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \underline{\Phi}_0 = 0$$

$\rightsquigarrow$ $U(1)_Q$ remains unbroken / massless

* Unitary gauge:

$$\underline{\Phi}(x) = \frac{1}{\sqrt{2}} (v + H(x)) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$H(x)$: real scalar field
= Higgs particle

Gauge corresponds to

$$\phi^\dagger \sigma \phi_0 - \phi_0^\dagger \sigma \phi = 0 \qquad \phi^\dagger \phi_0 - \phi_0^\dagger \phi = 0$$

Since $Q\phi_0 = 0$ ⇝ only 3 constraints

$$* \; D_\mu \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{i}{2} g \cdot (v+H)(A_{\mu 1} - i A_{\mu 2}) \\ \partial_\mu H - \frac{i}{2}(v+H)(g A_{\mu 3} - g' B_\mu) \end{pmatrix}$$

The natural combinations of gauge fields are:

$$\boxed{\begin{aligned} W_\mu &= \frac{1}{\sqrt{2}}(A_{\mu 1} - i A_{\mu 2}) \\ Z_\mu &= \cos\Theta_W A_{3\mu} - \sin\Theta_W B_\mu \end{aligned}} \; (*) \quad \text{complex vector}$$

where: $\boxed{\tan\Theta_W = \frac{g'}{g}}$ $\quad \cos\Theta_W = \frac{g}{(g^2 + (g')^2)^{1/2}}$

Θ_W: Weinberg angle

$$\mathcal{L}_{Higgs} = \frac{1}{2}\partial_\mu H \partial^\mu H + \frac{1}{4} g^2 (v+H)^2 \left(\frac{1}{\cos^2\Theta_W} \cdot \frac{1}{2} Z^\mu Z_\mu + W^{\mu\dagger} W_\mu \right) - \frac{1}{8}\lambda (H^2 + 2vH)^2$$

→ (*) diagonalises vector mass terms

<u>i) Physical particle content in Higgs-gauge sector</u>

* Higgs field $H(x)$ — 1 real scalar
  mass: $m_H^2 = \lambda \cdot v^2$

* massive bosons: W_μ : 1 charged/complex vector
  Z_μ : 1 real vector

$$m_W^2 W_\mu^\dagger W^\mu + \frac{1}{2} m_Z^2 Z_\mu Z^\mu$$

$$m_W^2 = \frac{1}{4} g^2 v^2 \qquad m_Z^2 = \frac{1}{4}(g^2 + g'^2) v^2$$

$$m_W^2 = \cos^2\Theta_W \cdot m_Z^2$$

<u>Important</u>: Θ_W defined as $\tan\Theta_W = \frac{g'}{g}$

couplings g', g accessible experimentally from couplings of leptons (see later)

$\rightsquigarrow$ <u>prediction</u> for ratio $\frac{m_Z^2}{m_W^2} = \frac{1}{\cos^2\Theta_W}$

Note: If Φ were to be in different irrep of $SU(2)$ this would be different!

The relation $m_W^2 = \cos^2\Theta_W \cdot m_Z^2$ is true classically

Including loop-corrections:

$$\boxed{m_W^2 = \cos^2\Theta_W \cdot m_Z^2 \cdot \rho \qquad \rho \approx 1.01}$$

Exp.: $m_W \cong 80$ GeV $\qquad m_Z \approx 91$ GeV

$\sin^2\Theta_W = 0.23 \Longrightarrow$ triumph of el.-weak theory

$\Longrightarrow v \cong 250$ GeV (in absence of exotic matter)

Note: m_H is set by unknown coupling λ

In SM: upper bound from el.-weak precision due to Higgs loops: $m_H \lesssim 250$ GeV

experimental lower bound (LEP): $m_H \gtrsim 115.2$ GeV

* Gauge field orthogonal to Z_μ:

$$A_\mu = \sin\Theta_W A_{3\mu} + \cos\Theta_W B_\mu$$

$$\begin{pmatrix} A_3 \\ B \end{pmatrix} = \begin{pmatrix} \cos\Theta_W & \sin\Theta_W \\ -\sin\Theta_W & \cos\Theta_W \end{pmatrix} \begin{pmatrix} Z \\ A_3 \end{pmatrix}$$

A_μ is massless $\Longrightarrow$ A_μ is photon of $U(1)_Q$

* Full gauge part $\mathcal{L}_{gauge}$ in terms of physical combinations:

$$F^A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F^Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$F^W_{\mu\nu} = d_\mu W_\nu - d_\nu W_\mu$$

$$d_\mu = \partial_\mu + ie A_\mu + ig\cos\Theta_W Z_\mu$$

where $\boxed{e = g \cdot \sin\Theta_W}$

$$\mathcal{L}_{gauge} = -\frac{1}{2}\left(F^W_{\mu\nu}\right)^+ F^{\mu\nu W} - \frac{1}{4}\left(F^A_{\mu\nu}\right)^2 - \frac{1}{4}\left(F^Z_{\mu\nu}\right)^2$$
$$+ \; i \cdot W^\mu W^{\nu +}\left(e F^A_{\mu\nu} + g\cos\Theta_W F^Z_{\mu\nu}\right)$$
$$+ \; \frac{1}{2} g^2 \left(W^2 (W^+)^2 - (W \cdot W^+)^2 \right)$$

* W_μ, W_μ^+ couples to A_μ w/ coupling $+e$, $-e$ $= g \cdot \sin\Theta_W$
  $\longrightarrow$ charged under $U(1)_{el.-magn.}$

* Z_μ carries no $U(1)_{em}$ charge

Summary : 4 parameters (g, g', v, λ)

4 physical quantities:

$$m_Z = \frac{m_W}{\cos\Theta_W}$$

$$m_W = \frac{1}{2} g \cdot v \qquad \tan\Theta_W = \frac{g'}{g}$$

$$m_H = \sqrt{\lambda} \cdot v$$

$$e = g \cdot \sin\Theta_W$$