d) The Higgs mechanism – Abelian

* Consider a gauge theory & SSB :

  gauged $G \longrightarrow$ gauged H

The Higgs effect is the phenomenon that the $(\dim G - \dim H)$ Goldstone bosons are "eaten" = absorbed by $(\dim G - \dim H)$ vector bosons, which thereby become massive.

i.e. the Goldstone bosons constitute the 3rd d.o.f. of the massive gauge bosons $A^\mu_{\hat{a}}$ where $T_{\hat{a}}$ generate G/H.

* It relies on the local character of the gauged G & H.

  Goldstone bosons $\Longleftrightarrow$ gauge trafo $T_{\hat{a}}$ that do <u>not</u> leave vacuum invariant

  $\downarrow$

  can be gauged away

* different gauges are used to show unitarity & renormalisability

  <u>unitary gauge</u>: particle content manifest

  <u>renormalisable gauge</u>: renorm. manifest, particle content obscure

Consider $U(1)$ gauge theory w/ complex boson Φ :

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \phi)^* (D_\mu \phi) - V(\phi^* \phi)$$

$$D_\mu \phi = \partial_\mu \phi - i e A_\mu \phi \qquad A_\mu \to A_\mu + \frac{1}{e} \partial_\mu \chi$$

$$\Phi \to e^{i\chi(x)} \Phi$$

$$V(\phi^*\phi) = \frac{g}{2}\left(\phi^*\phi - \frac{1}{2}v^2\right)^2 \Longrightarrow \text{SSB}$$

Vacuum: $\phi_0^* \phi_0 = \frac{1}{2} v^2$ breaks $U(1)$ symmetry

Idea: Isolate the Goldstone mode corresponding to broken $U(1)$ trafo

$$(*) \quad \boxed{\phi(x) = \frac{1}{\sqrt{2}} \left(v + f(x) \right) e^{-i \frac{\Theta(x)}{v}}}$$

$\Theta(x) \longleftrightarrow U(1)$ trafo
$\longleftrightarrow$ Goldstone

$(v, f : \text{real})$

$$\boxed{A_\mu = B_\mu - \frac{1}{e} \frac{\partial_\mu \Theta(x)}{v}}$$

Note: $\phi(x) = \frac{1}{\sqrt{2}} \left(v + f(x) - i\, \Theta(x) + \dots \right) \Rightarrow f, \Theta$: 2 real d.o.f.

If $v = 0$ the ansatz $(*)$ only gives 1 mode $f(x)$ & interaction $f(x)\, \Theta(x) \rightarrow \Theta(x)$ no d.o.f.

Gauge trafo : $\frac{\Theta}{v} \longrightarrow \frac{\Theta}{v} - \chi$ $\quad B_\mu, f$: gauge invariant

Use gauge trafo to gauge away would-be Goldstone $\Theta(x)$

$$\rightarrow \quad \left.\begin{aligned} \phi &\rightarrow \frac{1}{\sqrt{2}} \left(v + f(x) \right) \\ A_\mu &\rightarrow B_\mu \end{aligned}\right\} (**)$$

For original $(*)$: $\quad D_\mu \phi = \frac{1}{\sqrt{2}} e^{-i\frac{\Theta}{v}} \left(\partial_\mu f - i e B_\mu (v + f) \right)$

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F^2 + \frac{1}{2} e^2 (v + f)^2 B^\mu B_\mu + \frac{1}{2} \partial_\mu f \partial^\mu f$$
$$- \frac{1}{8} g \left(2 v f + f^2 \right)^2$$

Note: * Cross-terms $\partial^\mu f B_\mu$ dropped out

* B^μ is massive vector field

$m_B^2 = e^2 v^2 \Longrightarrow$ 3 d.o.f.

* f is massive scalar: $m_f^2 = g v^2$

* <u>No massless scalar left!</u>

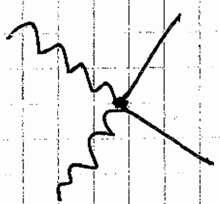
<u>Reason</u>: $U(1)$ boson has absorbed world-be Goldstone mode $\Theta(x)$

The Goldstone mode can be gauged away by going to gauge $(**)$: $\phi^* = \phi$

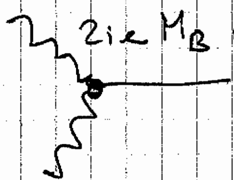
$\Longrightarrow$ <u>unitary</u> gauge : Gauge in which Lagrangian contains only physical degrees of freedom

* The interaction part of the resulting Lagrangian involving only physical d.o.f. is:

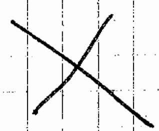
$$\mathcal{L}_{int} = \frac{e^2}{2} B^\mu B_\mu f^2 + e M_B B^\mu B_\mu f - \frac{g}{8} f^4 - \frac{1}{2} m_f \sqrt{g} f^3$$



$2ie^2 g_{\mu\nu}$



$2ie M_B g_{\mu\nu}$



$-3ig$

$-3im_f \sqrt{g}$

* The propagator in unitary gauge is:

$$D_{\mu\nu}(k^2) = (-i)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_B^2}\right) \frac{1}{k^2 - M_B^2 + i\varepsilon}$$

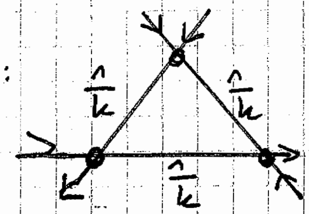
$\Longrightarrow D_{\mu\nu}(k^2) \longrightarrow \frac{1}{M_B^2}$ as $k^2 \to \infty$ & renormalisability is not manifest in the gauge

However, one can go to different gauges where unitarity is not manifest, but renormalisability is.

Insert: Renormalisability

* Consider 4-fermi interaction $G_F \cdot \psi^4$

6 particle @ 1-loop:



$\sim G_F^3 \cdot \int \frac{d^4k}{k^3} \sim G_F^3 \cdot \int dk$

$\longrightarrow$ divergent in UV!

Divergences regularised by cutoff $\Lambda \Longrightarrow$

$$G_F^3 \int_0^\Lambda d^4k \cdot \frac{1}{k^3} \sim G_F^3 \cdot \Lambda$$

& introduce counter-term : $\psi^6 \, c(\Lambda)$

s.t. divergence from 1-loop is cancelled

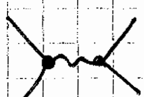
Renormalisable : All divergences can be cancelled by finite number of counter-terms

* Here: New ψ^6-interaction leads to divergent 8-point amplitude @ 1-loop $\rightsquigarrow$ need new counter term $\Longrightarrow$ not renormalisable

* Find: If Theory not renormalisable, the couplings have negative mass dimension

e.g. $[G_F] = \left(\frac{1}{\text{mass}}\right)^2$

$\Longrightarrow$ if $D_{\mu\nu}(k^2) \longrightarrow \frac{1}{M_B^2}$ as $k^2 \longrightarrow \infty$ then

effective 4-fermi coupling  not renormalisable!

* However one can go to different gauge where unitarity is not manifest, but renormalisability is!

E.g.: $\phi = \frac{1}{\sqrt{2}}\left(v + f + i v\right)$

$\rightsquigarrow$ find: $\mathcal{L} \ni \frac{1}{2} e^2 v^2 A^\mu A_\mu + ev \underbrace{A^\mu \partial_\mu \varphi}_{\text{absent in unitary gauge!}}$

Find: $D_{\mu\nu}(k^2) = (-i)\,\dfrac{\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}{k^2 - M_A^2}$ (Dyson resummation)

$\Rightarrow$ renormalisability ✓

$\Rightarrow$ unphysical massless d.o.f. φ present

e) The Higgs mechanism — Non-abelian

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_\phi$$

$$\mathcal{L}_\phi = \frac{1}{2} (D_\mu \phi)^\dagger (D_\mu \phi) - V(\phi^\dagger \phi)$$

$$D_\mu \phi = \partial_\mu \phi + ig\, A_{\mu a} T_a \phi \qquad T_a \hookrightarrow G$$

Vacuum ϕ_0 inv. under $H \subset G$

Decompose: $T_a = (t_i, T_{\hat{a}})$ $\qquad A_{\mu a} = (A_{\mu i}, A_{\mu \hat{a}})$

$$t_i \phi_0 = 0 \qquad (t_i = t_i^\dagger)$$

Analogue of unitary gauge $\phi = \phi^*$ in $U(1)$ case:

$$\boxed{\phi^\dagger T_a \phi_0 - \phi_0^\dagger T_a \phi = 0}$$

Since $t_i \phi_0 = 0 \Rightarrow (\dim G - \dim H)$ conditions for $T_{\hat{a}}$

Ansatz: $\quad \mathcal{U} = e^{i \eta_{\hat{a}}(x) T_{\hat{a}}} \qquad G/H$ gauge trafo

* $\Phi = \mathcal{U}^{-1} (\Phi_0 + f)$ s.t. $f^\dagger T_{\hat{a}} \phi_0 - \phi_0^\dagger T_{\hat{a}} f = 0$
* $A^\mu_{\hat{a}} T_{\hat{a}} = \mathcal{U}^{-1} B^\mu_{\hat{a}} T_{\hat{a}} \mathcal{U} + \frac{i}{g} (\partial_\mu \mathcal{U}^{-1}) \mathcal{U}$

$\Rightarrow \eta_{\hat{a}}(x)$: would-be Goldstones

As before these are gauged away by applying gauge trafo $\mathcal{U}$:

$$\phi \longrightarrow \phi_0 + f$$
$$A^\mu{}_{\hat a} \longrightarrow B^\mu{}_{\hat a}$$
$$A^\mu{}_i \longrightarrow A^\mu{}_i \qquad \text{(unchanged)}$$

In this unitary gauge & using $t_i \phi_0 = 0$:

* $D_\mu \phi = \Delta_\mu f + i g B_{\mu \hat a} T_{\hat a} (\phi_0 + f)$

where $\Delta_\mu f = \partial_\mu f + i g A_{\mu i} t_i f$ H-covariant derivative

* in $(D^\mu \phi)^+ (D_\mu \phi)$: cross-terms $\partial^\mu f \, B_{\mu \hat a}$ drop out

* full $\mathcal{L}_\phi \rightsquigarrow$ tutorial

* Quadratic terms in $\mathcal{L}_{gauge} + \mathcal{L}_\phi$:

$$\mathcal{L}^{(2)} = -\frac{1}{4}\left(\partial_\mu A_{\nu i} t_i - \partial_\nu A_{\mu i} t_i\right)^2$$
$$-\frac{1}{4}\left(\partial_\mu B_{\nu \hat a} T_{\hat a} - \partial_\nu B_{\mu \hat a} T_{\hat a}\right)^2$$
$$+\frac{1}{2}(\partial_\mu f)^+(\partial^\mu f) - \frac{1}{2} f^+ \mathcal{M}_f f$$
$$+\frac{1}{2} M_{\hat a \hat b} B^\mu{}_{\hat a} B_{\mu \hat b}$$

scalar mass matrix : $\mathcal{M}_f = \left.\frac{\delta^2 V}{\delta\phi_r \, \delta\phi_s}\right|_{\phi_0}$ as before

vector mass matrix : $M_{\hat a \hat b} = g^2 \left(\phi_0^+ T_{\hat a}\right)\left(T_{\hat b} \phi_0\right)$ $M = M^+$

Note:

$M = M^+ \Rightarrow$ real eigenvalues λ & orthogonal eigenvectors $\omega_{\hat{a}}$

$$M_{\hat{a}\hat{b}}\, \omega_{\hat{b}} = \lambda\, \omega_{\hat{a}}$$

$$\Rightarrow 0 \leq g^2 \left(\sum_{\hat{a}} \omega^+_{\hat{a}}\, \phi_0^+ T_{\hat{a}}\right)\left(\sum_{\hat{b}} T_{\hat{b}}\, \phi_0\, \omega_{\hat{b}}\right)$$

$$= \omega^+_{\hat{a}} M_{\hat{a}\hat{b}}\, \omega_{\hat{b}} = \sum_{\hat{a}} \lambda\, \omega^+_{\hat{a}}\, \omega_{\hat{a}} \Rightarrow \lambda \geq 0$$

If $\lambda = 0$ then $\omega_{\hat{a}} T_{\hat{a}}\, \phi_0 = 0 \Rightarrow$ not happening for $T_{\hat{b}} \in G/H$!

$\Rightarrow$ All eigenvectors are real & positive

Summary:

* $(\dim G - \dim H)$ massive vector bosons $B_{\mu\hat{a}}$
* $\dim H$ massless $A_{\mu i}$
  $\longleftrightarrow$ unbroken gauge theory H
* # of massive Higgs bosons f depends on representation of f!