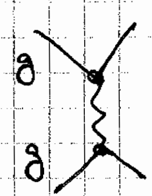
## 2) The problem of massive gauge fields

- Motivation: Effective 4-fermi interaction describable by massive bosons:

$$\mathcal{L}_W = -\frac{1}{4}\left(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu\right)\left(\partial^\mu W^\nu - \partial^\nu W^\mu\right) + M_W^2 \, W^+_\mu W^\mu$$

$$\mathcal{L}_{int} = g \cdot \left(J^\mu W_\mu + c.c.\right)$$

heuristic propagator: $$\tilde{\Delta}_{\mu\nu}(k^2) = (-i)\,\frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{M_W^2}}{k^2 - M_W^2 + i\varepsilon}$$




as $k^2 \ll M_W^2$ $\sim \frac{g^2}{M_W^2} = + \frac{G_F}{\sqrt{2}} \simeq 10^{-5}\left(\frac{1}{GeV}\right)^2$

- How does one formulate massive gauge theories?

Problem: mass terms breaks gauge symmetry manifestly, but gauge symmetry needed for quantisation

- In fact, for $U(1)$ gauge theory a bare mass term is perfectly fine:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu A^\mu$$

E.o.m.: $$\partial_\mu F^{\mu\nu} + M^2 A^\nu = 0 \quad \Big/ \cdot \partial_\nu$$

$$\rightarrow M^2 \partial_\nu A^\nu = 0$$

$\rightarrow$ replaces gauge fixing condition & allows us to eliminate the timelike d.o.f.

But: For massless gauge theory $\exists$ residual gauge symmetry

$A_\mu \rightarrow A_\mu + \partial_\mu \chi \quad \partial^2 \chi = 0 \quad \longrightarrow$ removes another d.o.f.

massless $U(1)$ boson: 2 d.o.f.

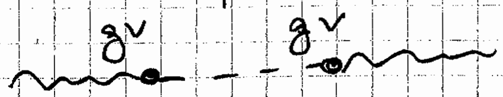
massive 3 d.o.f.

- Renormalisability $\longrightarrow$ tutorial
- This logic does not go through for non-abelian theories:

E.o.m.: $D_\mu F^{\mu\nu} + M_W^2 W^\nu = 0 \to \partial_\mu W^\mu$ does <u>not</u> follow

$\rightsquigarrow$ timelike modes do not decouple

- Way out: Coupling of gauge field to exactly massless scalar can produce consistent mass terms




$$\tilde{\Delta}_{\mu\nu} = (-i)\,\frac{g_{\mu\nu} - k_\mu k_\nu / k^2}{k^2 - g^2 v^2}$$

But: A priori mass of scalar is arbitrary and $m=0$ is not protected by symmetry

$\Rightarrow$ Need:

i) Rationale why scalars with mass $m=0$ appear
ii) Explain coupling such as to produce effective mass for gauge field

i) by spontaneous symmetry breaking & Goldstone's theorem

ii) by Higgs-mechanism

# 3) Spontaneous Symmetry Breaking (SSB)

* occurs when a physical theory has some symmetry but the ground state is degenerate & a particular choice of vacuum does not respect this symmetry

* is a universal phenomenon in physics
  - ⇝ ferromagnetism by alignment of atomic spins
  - ⇝ superconductivity
  - ⇝ QFT

## a) Discrete Symmetries

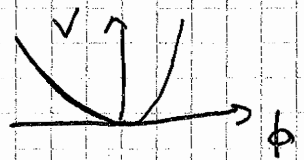
$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - V(\phi) \qquad \phi \text{ real scalar}$$

Assume $\mathbb{Z}_2$ symmetry $\phi \to -\phi$

$\rightarrow V(\phi) = V(-\phi)$ : $\mathbb{Z}_2$ symmetry

Ex.: $V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} g \phi^4 \qquad g > 0$

i) $m^2 > 0$




- classical minimum at $\phi_0 = 0$ respects $\mathbb{Z}_2$ symmetry
- quantum mechanically: $\mathbb{Z}_2 \rightsquigarrow U$ unitary operator

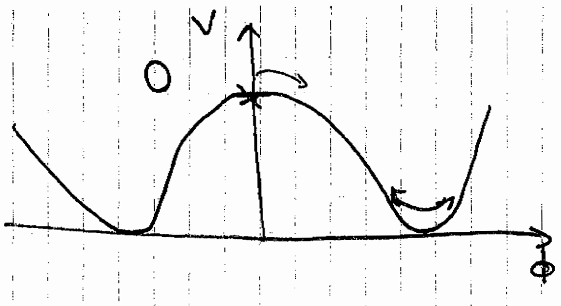
  $U^\dagger H U = H \quad \& \quad U|0\rangle = |0\rangle$

  Can find energy eigenstates $U|\psi_\pm\rangle = \pm|\psi_\pm\rangle$

ii) $m^2 < 0$

Up to additive constant can rewrite $V(\phi)$ as:

$$V(\phi) = \frac{1}{4!} g \left(\phi^2 - v^2\right)^2$$




double-well potential

$\phi = 0 \longrightarrow$ tachyonic excitation!

Degenerate ground state: $\phi_0 = \pm v$ classically

QM: 2 vacua: $|0_\pm\rangle$ : $\langle 0_\pm | \phi | 0_\pm \rangle = v_\pm^{(R)}$

$\longrightarrow$ 2 orthogonal/independent Hilbert spaces $\mathcal{H}_\pm$

Perturbation theory starts from a choice of vacuum:

Define the excitation around that vacuum by shifting

$$\phi = v + f$$

$$\rightarrow \mathcal{L} = \frac{1}{2} \partial_\mu f \, \partial^\mu f - \frac{1}{6} g \left( v^2 f^2 + v f^3 + \frac{1}{4} f^4 \right)$$

- no term linear in $f$!
- $f$ is massive excitation $\quad m_f^2 = \frac{g v^3}{3}$

- classically: $f = 0$ at vacuum $\phi = v$
- QM: $\langle 0_v | f | 0_v \rangle = 0$

  $\longrightarrow$ $f$ has creation/annihilation modes
- Note: Theory of $f$ has no $\mathbb{Z}_2$ symmetry left!

b) Continuous symmetries

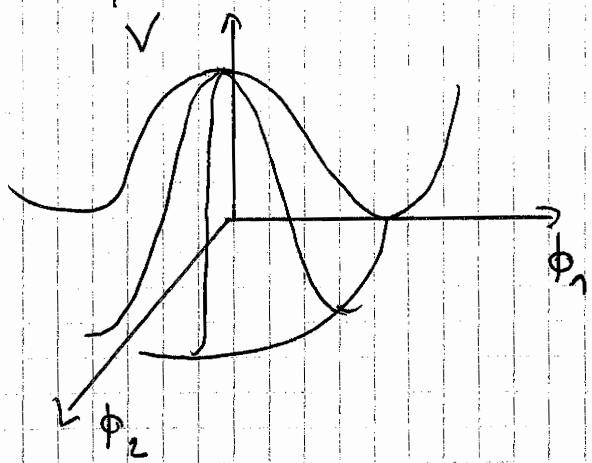
For SSB of a continuous symmetry a new feature occurs: the existence of massless modes = Goldstone modes.

Ex.: Consider real scalar fields $\phi = (\phi_1, \dots, \phi_m)$

with $\phi^2 = \phi \cdot \phi = \sum_r \phi_r \phi_r$

$$\mathcal{L} = \frac{1}{2}\,\partial^\mu \phi \cdot \partial_\mu \phi - V(\phi) \qquad V(\phi) = \frac{1}{8} g \left(\phi^2 - v^2\right)^2,$$

$$g > 0$$



"Mexican hat potential"

- full symmetry group of $V$ : $\underline{G = O(n)}$
- classical ground state: $\phi = \phi_0$ s.t. $\phi_0^2 = v^2$

$\rightarrow$ <u>vacuum manifold</u> $M_0 = \{\phi_0 : V(\phi_0) = V_{min}\}$

$M_0 = S^{n-1}$

- Even after SSB, i.e. in the vacuum $\phi_0$, there is a <u>residual symmetry</u> corresponding to <u>remaining flat directions</u>:

Take $\phi_0 = (\overbrace{0, \dots, 0}^{n-1}, v) \Rightarrow \phi_0$ inv. under $\underline{H = O(n-1)}$

- Expand $\phi$ about a choice of vacuum $\phi_0$ as above:

$$\phi = (\phi_\perp, v + f) \qquad \phi_\perp = (\phi_1, \dots, \phi_{n-1})$$

$$V(\phi) = \frac{1}{2} g v^2 f^2 + \frac{1}{2} g v \left(\phi_\perp^2 + f^2\right) f + \frac{1}{8} g \left(\phi_\perp^2 + f^2\right)^2$$

$\Rightarrow$ $f$ is massive $m_f^2 = g v^2$ $\Big\}$ $(n-1)$ massless field

$\phi_\perp$ has no mass-term $\quad$ $1$ massive field

The massless fields are called <u>Goldstone modes</u> or <u>Goldstone bosons</u>

Observe: $(n-1) = \dim O(n) - \dim O(n-1)$

$\qquad\qquad = \dim G - \dim H$

More generally:

* starting with $V$ and symmetry $G$:

$$V\left(\mathcal{U}(g)\phi\right) = V(\phi) \qquad \forall\ g \in G$$

* SSB to vacuum manifold

$$M_0 = \{\phi_0 \text{ s.t. } V(\phi_0) = V_{min}\}$$

stability group $H \subset G$: $\quad \mathcal{U}(h)\,\phi_0 = \phi_0 \quad \forall\ h \in H$

* Assume that any 2 $\phi_0, \phi_0' \in M_0$ related via $G$:

$$\phi_0' = \mathcal{U}(g)\,\phi_0 \qquad \Longrightarrow \quad H_{\phi_0} \cong H_{\phi_0'}$$

& then: $\quad M_0 = G/H \quad$ coset

defined as set of equivalence classes $g_1 \cong g_2 h$

Ex.: $\quad S^{n-1} = O(n)/O(n-1)$

## c) Goldstone's theorem

Theorem: Given a quantum field theory with SSB from $G$ to $H$ as above, then there exist $(\dim G - \dim H)$ zero mass scalars = Goldstone bosons.

Classical proof:

Consider a theory of real scalars with component fields $\phi_r$.
Symmetry under $G$ means:

$$V(\phi + \delta\phi) = V(\phi) \quad \text{for } \delta\phi = \sum_a \chi_a T_a \Phi$$

$T_a$: hermitian generators of $G$ $\quad a = 1, \dots, \dim G$

$$\Rightarrow \sum_r \frac{\delta}{\delta \phi_r} V(\phi) (T_a \phi)_r = 0 \quad \forall a \quad (*)$$

We are interested in the mass matrix in a vacuum $\phi_0 \in M_0$:

$$M_{sr} = \frac{\delta^2}{\delta \phi_r \delta \phi_s} V(\phi) \Big|_{\phi = \phi_0}$$

Now take $\frac{\delta}{\delta \phi_s}$ of $(*)$

$$\Rightarrow \sum_r \frac{\delta^2}{\delta \phi_s \delta \phi_r} V(\phi) (T_a \phi)_r + \sum_r \frac{\delta V(\phi)}{\delta \phi_r} \cdot (T_a)_{rs} = 0 \quad (**)$$

At $\phi = \phi_0$ we have $\frac{\delta V(\phi)}{\delta \phi_r}\Big|_{\phi=\phi_0} = 0$ because $\phi_0$ is a minimum.

$\Rightarrow (**)$ evaluated at $\phi = \phi_0$ yields:

$$\sum_r M_{sr} (T_a \phi_0)_r = 0 \iff (T_a \phi_0)_r \text{ is zero mass eigenstate}$$

How many linearly independent modes are there?

By assumption $\phi_0$ is invariant under group $H$:

$$t_i \phi_0 = 0 \qquad i = 1, \dots, \dim H$$

For compact semi-simple group $G$ $\exists$ positive definite group invariant scalar product: $\operatorname{tr}(T_a T_b)$

$\rightarrow$ split $T_a = (t_i, T_{\hat{a}})$ $\qquad t_i \in H$

$T_{\hat{a}} \in G/H$

$\operatorname{tr}(t_i T_{\hat{a}}) = 0$

$\Rightarrow \exists$ (dim $G$ - dim $H$) lin. indep. zero modes given by

$(T_{\hat{a}}\, \phi_0)$ $\qquad T_{\hat{a}} \in G/H$

Reason: If $\sum_{\hat{a}} f_{\hat{a}} T_{\hat{a}} \phi_0 = 0$ for some $f_{\hat{a}}$, then by assumption $\sum_{\hat{a}} f_{\hat{a}} T_{\hat{a}} \in H$, in contradiction to decomposition $(t_i, T_{\hat{a}})$.

Example: $G = O(n)$ $\quad \dim G = \binom{n}{2} = n(n-1)\cdot \frac{1}{2}$

$H = O(n-1)$ $\quad \dim H = \binom{n-1}{2} = (n-1)(n-2)\cdot\frac{1}{2}$

$\dim G - \dim H = n-1$ ✓

Proof quantum:

- A similar proof exists for quantum field theory with SSB leading to $|0_\xi\rangle$ : $u(h)|0_\xi\rangle = 0 \quad \forall h \in H$ & Hilbert space $\mathcal{H}_\xi$

* It rests on 2 assumptions:

a) manifest Lorentz invariance

b) manifest unitarity: positive definite Hilbert space

* If SSB occurs for a local, i.e. gauge theory with gauge group $\mathcal{G}$, then these 2 assumptions don't hold simultaneously:

I.e. impose covariant gauge — e.g. $\partial_\mu A^\mu = 0$ for $U(1)$
$\leadsto$ unitarity not manifest due to longitudinal modes

or non-covariant gauge $\longrightarrow$ manifest unitarity