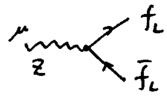
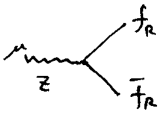
9-1 Total Z-width Γ_Z

From Ex. 8-2 d), the Feynman rule of the f-$\bar{f}$-Z vertex is given by

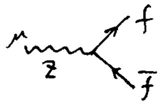


$$\sim \frac{-ig}{C_W}\gamma^\mu (T^3 - S_W^2 Q)$$



$$\sim \frac{-ig}{C_W}\gamma^\mu (-S_W^2 Q)$$

$$\xrightarrow{} \quad \begin{pmatrix} f_L = \frac{1}{2}(1-\gamma_5)f \\ f_R = \frac{1}{2}(1+\gamma_5)f \end{pmatrix}$$



$$\sim \frac{-ig}{C_W}\gamma^\mu \frac{1}{2}\left[(T^3 - S_W^2 Q)(1-\gamma_5) - S_W^2 Q(1+\gamma_5)\right]$$
$$= \frac{-ig}{C_W}\gamma^\mu \frac{1}{2}\left[\underbrace{(T^3 - 2S_W^2 Q)}_{= g_V} - \underbrace{T^3}_{= g_A}\gamma_5\right]$$

The amplitude for $Z(q) \to f(p_1) + \bar{f}(p_2)$ is

$$i\mathcal{M} = \frac{-ig}{2C_W}\,\bar{u}(p_1)\gamma^\mu (g_V - g_A\gamma_5) v(p_2)\, \varepsilon_\mu(q)$$

The spin-summed squared matrix element is

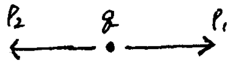
$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g^2}{4C_W^2}\sum_{\text{spins}} \bar{u}(p_1)\gamma^\mu (g_V - g_A\gamma_5) v(p_2)\bar{v}(p_2)\gamma^\nu(g_V - g_A\gamma_5)u(p_1)\,\varepsilon_\mu(q)\varepsilon_\nu^*(q)$$
$$= \frac{g^2}{4C_W^2}\, \underbrace{\mathrm{tr}[\not{p}_1\gamma^\mu(g_V - g_A\gamma_5)\not{p}_2\gamma^\nu(g_V-g_A\gamma_5)]}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M^2}\right)$$
$$= (g_V^2 + g_A^2)\,\mathrm{tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] - 2g_Vg_A\,\mathrm{tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu\gamma_5]$$
$$= 4(g_V^2+g_A^2)\{p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1\cdot p_2)g^{\mu\nu}\} + 8i\,g_Vg_A\,\epsilon^{\rho\mu\sigma\nu}p_{1\rho}p_{2\sigma}$$

does not contribute due to anti-symmetry.

$$= \frac{g^2}{C_W^2}(g_V^2+g_A^2)\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1\cdot p_2)g^{\mu\nu}\right)\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M^2}\right)$$
$$= \frac{g^2}{C_W^2}(g_V^2+g_A^2)\left\{2(p_1\cdot p_2) + \frac{1}{M^2}\left[2(p_1\cdot q)(p_2\cdot q) - (p_1\cdot p_2)q^2\right]\right\}$$

Using kinematics at the Z-boson rest frame:

$$\begin{cases} q = (M,0,0,0) \\ p_1 = \frac{M}{2}(1,0,0,1) \\ p_2 = \frac{M}{2}(1,0,0,-1) \end{cases}$$



$$\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{g^2}{C_W^2}(g_V^2+g_A^2)\left\{M^2 + \frac{1}{M^2}\left[\cancel{2\cdot\frac{M^2}{2}\frac{M^2}{2}} - \cancel{\frac{M^2}{2}\cdot M^2}\right]\right\}$$

The partial decay width is

$$\Gamma(Z\to f_i\bar{f}_i) = \frac{1}{2M}\int d\Phi_2\, \overset{\text{color sum \& spin average}}{\frac{C}{3}}\sum_{\text{spins}}|\mathcal{M}|^2 \qquad (※\ C=1 \text{ for leptons},\ C=3 \text{ for quarks})$$
$$= \frac{1}{2M}\frac{1}{8\pi}\frac{C}{3}\frac{g^2}{C_W^2}(g_V^{i2}+g_A^{i2})M^2 \qquad \left(※\ d\Phi_2 = \frac{1}{8\pi}\frac{d\cos\theta}{2}\frac{d\phi}{2\pi}\right)$$
$$= \frac{CM}{48\pi}\frac{g^2}{C_W^2}(g_V^{i2}+g_A^{i2})$$
$$= \frac{C\alpha M}{12S_W^2C_W^2}(g_V^{i2}+g_A^{i2}) \qquad \begin{pmatrix} e = g\sin\theta_W \\ \alpha = \frac{e^2}{4\pi}\end{pmatrix}$$
$$\left(= \frac{CG_FM^3}{6\sqrt{2}\pi}(g_V^{i2}+g_A^{i2})\right) \qquad \left(\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{g^2}{8m_Z^2C_W^2}\right)$$

※ The V−A couplings :

	ν	ℓ^-	u-quark	d-quark	
g_V	$\frac{1}{2}$	$-\frac{1}{2}+2S_w^2$	$\frac{1}{2}-\frac{4}{3}S_w^2$	$-\frac{1}{2}+\frac{2}{3}S_w^2$	: $g_V = T^3 - 2S_w^2 Q$
g_A	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	: $g_A = T^3$

The relative branching ratio for one generation is

$$\Gamma_\nu : \Gamma_{e^-} : \Gamma_u : \Gamma_d$$
$$= 2 : 1+(1-4S_w^2)^2 : \overset{\text{color}}{3}\left\{1+\left(1-\tfrac{8}{3}S_w^2\right)^2\right\} : \overset{\text{color}}{3}\left\{1+\left(1-\tfrac{4}{3}S_w^2\right)^2\right\}$$
$$= 2 : 1.006 : 3.44 : 4.44 \qquad (@ \sin^2\theta_W = 0.231)$$

The total width is

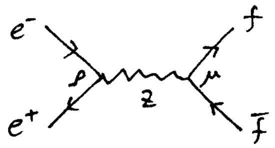
$$\Gamma_{tot} = 3\Gamma_\nu + \overset{e,\mu,\tau}{3}\Gamma_{e^-} + \overset{u,c}{2}\Gamma_u + \overset{d,s,b}{3}\Gamma_d$$
$$= \Gamma_\nu \frac{1}{2}\left[6 + 3\{1+(1-4S_w^2)^2\} + 6\left\{1+\left(1-\tfrac{8}{3}S_w^2\right)^2\right\} + 9\left\{1+\left(1-\tfrac{4}{3}S_w^2\right)^2\right\}\right]$$
$$= \Gamma_\nu\left[21 - 40S_w^2 + \frac{160}{3}S_w^4\right]$$
$$= 0.167\ \text{GeV} \times 14.61$$
$$= 2.44\ \text{GeV}$$
$$\sim 2.4952 \pm 0.0023\ \text{GeV} \quad @\ \text{LEP}$$

(※ $M_Z = 91.188$ GeV
$\sin^2\theta_W = 0.231$
$\alpha = 1/128$)

9-2 Forward-backward asymmetry for $b\bar{b}$ and $c\bar{c}$ events

At the Z-pole ($\sqrt{s} = M_Z$), the Z contribution is dominant. The amplitude for

$$e^-(k_1) + e^+(k_2) \to Z \to f(P_1) + \bar{f}(P_2)$$



is given as

$$iM = \left(\frac{-ig}{2C_w}\right)^2 \bar{u}(P_1)\gamma^\mu(g_V^f - g_A^f\gamma_5)v(P_2)\frac{-i\left(g_{\mu\rho} - \frac{g_\mu g_\rho}{M^2}\right)}{g^2 - M^2 + iM\Gamma}\bar{v}(k_2)\gamma^\rho(g_V^e - g_A^e\gamma_5)u(k_1)$$

The spin-summed squared matrix element is (at $\sqrt{s} = g^2 = M^2$)

$$\Sigma|M|^2 = \frac{g^4}{16C_w^4}\frac{1}{M^2\Gamma^2}\,\mathrm{tr}[\not{P}_1\gamma^\mu(g_V^f - g_A^f\gamma_5)\not{P}_2\gamma^\nu(g_V^f - g_A^f\gamma_5)]\left(-g_{\mu\rho} + \frac{g_\mu g_\rho}{M^2}\right)$$
$$\times\,\mathrm{tr}[\not{k}_2\gamma^\rho(g_V^e - g_A^e\gamma_5)\not{k}_1\gamma^\sigma(g_V^e - g_A^e\gamma_5)]\left(-g_{\nu\sigma} + \frac{g_\nu g_\sigma}{M^2}\right)$$

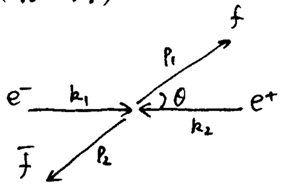
Due to the $g_\mu\,\mathrm{tr}[\not{P}_1\gamma^\mu\not{P}_2\gamma^\nu(1-\gamma_5)] = g_\rho\,\mathrm{tr}[\not{k}_2\gamma^\rho\not{k}_1\gamma^\sigma(1-\gamma_5)] = 0$, only $g_{\mu\rho}g_{\nu\sigma}$ remains.

$$\Sigma|M|^2 = \frac{g^4}{16C_w^4}\frac{16}{M^2\Gamma^2}\left[(g_V^{f2} + g_A^{f2})(P_1^\mu P_2^\nu + P_1^\nu P_2^\mu - (P_1\cdot P_2)g^{\mu\nu}) + 2i\,g_V^f g_A^f\,\epsilon^{\alpha\mu\beta\nu}P_{1\alpha}P_{2\beta}\right]$$
$$\times\left[(g_V^{e2} + g_A^{e2})(k_{2\mu}k_{1\nu} + k_{2\nu}k_{1\mu} - (k_1\cdot k_2)g_{\mu\nu}) + 2i\,g_V^e g_A^e\,\epsilon_{\gamma\mu\delta\nu}k_2^\gamma k_1^\delta\right]$$
$$= \frac{g^4}{C_w^4M^2\Gamma^2}\Big\{(g_V^{f2} + g_A^{f2})(g_V^{e2} + g_A^{e2})\left[2(P_1\cdot k_2)(P_2\cdot k_1) + 2(P_1\cdot k_1)(P_2\cdot k_2)\right]$$
$$-4g_V^f g_A^f g_V^e g_A^e\,\underbrace{\epsilon^{\alpha\mu\beta\nu}\epsilon_{\gamma\mu\delta\nu}}_{= -2(\delta^\alpha_\gamma\delta^\beta_\delta - \delta^\alpha_\delta\delta^\beta_\gamma)} P_{1\alpha}P_{2\beta}k_2^\gamma k_1^\delta\Big\}$$

$$= \frac{2g^4}{C_w^4M^2\Gamma^2}\Big\{(g_V^{f2} + g_A^{f2})(g_V^{e2} + g_A^{e2})\left[(P_1\cdot k_2)(P_2\cdot k_1) + (P_1\cdot k_1)(P_2\cdot k_2)\right]$$
$$+ 4g_V^f g_A^f g_V^e g_A^e\left[(P_1\cdot k_2)(P_2\cdot k_1) - (P_1\cdot k_1)(P_2\cdot k_2)\right]\Big\}$$

Using kinematics at the e^+e^- CM frame: ($\sqrt{s} = M$)

$$\begin{cases} k_1 = \frac{\sqrt{s}}{2}(1, 0, 0, 1) \\ k_2 = \frac{\sqrt{s}}{2}(1, 0, 0, -1) \\ P_1 = \frac{\sqrt{s}}{2}(1, \sin\theta, 0, \cos\theta) \\ P_2 = \frac{\sqrt{s}}{2}(1, -\sin\theta, 0, -\cos\theta) \end{cases}$$



$$\Sigma|M|^2 = \frac{2g^4}{C_w^4M^2\Gamma^2}\Big\{(g_V^{f2} + g_A^{f2})(g_V^{e2} + g_A^{e2})\left[\left(\frac{M^2}{4}(1+\cos\theta)\right)^2 + \left(\frac{M^2}{4}(1-\cos\theta)\right)^2\right]$$
$$+ 4g_V^f g_A^f g_V^e g_A^e\left[\left(\frac{M^2}{4}(1+\cos\theta)\right)^2 - \left(\frac{M^2}{4}(1-\cos\theta)^2\right)^2\right]$$
$$= \frac{2g^4}{C_w^4M^2\Gamma^2}\left\{\frac{M^4}{8}(g_V^{f2} + g_A^{f2})(g_V^{e2} + g_A^{e2})(1 + \cos^2\theta) + M^4 g_V^f g_A^f g_V^e g_A^e\cos\theta\right\}$$
$$= \frac{g^4M^2}{4C_w^4\Gamma^2}\left\{(g_V^{f2} + g_A^{f2})(g_V^{e2} + g_A^{e2})(1 + \cos^2\theta) + 8g_V^f g_A^f g_V^e g_A^e\cos\theta\right\}$$

The differential cross section is

$$d\sigma = \frac{1}{2s}\underbrace{\frac{1}{4}}_{\text{spin average}}\Sigma|M|^2\,d\Phi_2, \quad d\Phi_2 = \frac{1}{8\pi}\frac{d\cos\theta}{2}\frac{d\phi}{2\pi}$$

($\because e = \sqrt{4\pi\alpha} = g\sin\theta_w$)

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s}\frac{1}{4}\frac{1}{16\pi}\frac{M^2}{4C_w^4\Gamma^2}\frac{(4\pi\alpha)^2}{S_w^4}\{\;''\;\}$$
$$= \frac{\pi\alpha^2}{2s}\frac{M^2}{\Gamma^2}\frac{1}{16C_w^2S_w^2}\left\{(g_V^{e2} + g_A^{e2})(g_V^{f2} + g_A^{f2})(1 + \cos^2\theta) + 8g_V^e g_A^e g_V^f g_A^f\cos\theta\right\}$$

The forward-backward asymmetry is defined as

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{\int_0^1 d\cos\theta \frac{d\sigma}{d\cos\theta} - \int_{-1}^0 d\cos\theta \frac{d\sigma}{d\cos\theta}}{\int_{-1}^1 d\cos\theta \frac{d\sigma}{d\cos\theta}}$$

$$= \frac{8 g_V^e g_A^e g_V^f g_A^f \cdot \frac{1}{2} \cdot 2}{(g_V^{e^2} + g_A^{e^2})(g_V^{f^2} + g_A^{f^2}) \cdot \frac{8}{3}}$$

$$= \frac{3}{4} \cdot \frac{2 g_V^e g_A^e}{(g_V^{e2} + g_A^{e2})} \cdot \frac{2 g_V^f g_A^f}{(g_V^{f2} + g_A^{f2})} \equiv \frac{3}{4} A_e A_f$$

- $A_e = 0.1511$ @ $\sin^2\theta_W = 0.231$
- $A_b = 0.9359$
- $A_c = 0.6693$

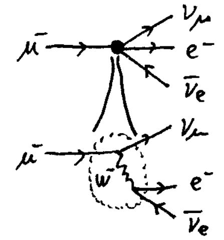
$\left(\text{※ } g_V = T^3 - 2 S_W^2 Q, \quad g_A = T^3 \quad \text{See Ex.9-1} \right)$

$\Longrightarrow A_{FB}^{b\bar{b}} = 0.106$ cf. 0.0992 ± 0.0016 (Exp.)

$A_{FB}^{c\bar{c}} = 0.076$ $\quad 0.0707 \pm 0.0035$ (Exp.)

9-3 Weak and electromagnetic coupling constants

Relation between the Fermi int. and the weak int. : (e.g. μ-decay)



$$\sim i\frac{G_F}{\sqrt{2}}\,\bar\nu_\mu\gamma^\mu(1-\gamma_5)\mu\cdot\bar e\,\gamma_\mu(1-\gamma_5)\,\nu_e$$

$$\sim\left(\frac{-ig}{\sqrt{2}}\right)^2\bar\nu_\mu\gamma^\mu\frac{1-\gamma_5}{2}\mu\;\underbrace{\frac{-i\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{M^2}\right)}{q^2-M^2}}_{\xrightarrow{q^2\ll M^2}\ \frac{ig_{\mu\nu}}{M^2}}\;\bar e\,\gamma^\nu\frac{1-\gamma_5}{2}\nu_e$$

$$\Rightarrow\ \frac{G_F}{\sqrt{2}}=\frac{g^2}{8M_W^2}$$

$$G_F=\frac{\sqrt{2}}{8M_W^2}\frac{e^2}{\sin^2\theta_W}=\frac{\pi\alpha}{\sqrt{2}M_W^2\sin^2\theta_W}\qquad\left(※\ e=g\sin\theta_W,\ \ \alpha=\frac{e^2}{4\pi}\right)$$

9-4 Effective couplings and electro-weak mixing angle

The effective couplings :

$$\begin{cases}\bar g_A=\sqrt{\bar\rho}\,T_3\\ \bar g_V=\sqrt{\bar\rho}\,(T_3-2Q\sin^2\theta_{eff})\end{cases}\ \xrightarrow[T_3=-\frac12,\ Q=-1]{\text{charged leptons}}\ \begin{cases}\bar g_A=-\frac12\sqrt{\bar\rho}\\ \bar g_V=\sqrt{\bar\rho}\,(-\frac12+2\sin^2\theta_{eff})\end{cases}$$

Therefore,
$$\sin^2\theta_{eff}=(\bar g_V-\bar g_A)/2\sqrt{\bar\rho}=0.2311$$

$$\begin{cases}\bar g_A=-0.50123\pm0.00026\\ \bar g_V=-0.03783\pm0.00041\\ \bar\rho=1.0050\pm0.0010\end{cases}$$

The tree-level (on-shell) definition of $\sin^2\theta_W$:

$$\sin^2\theta_W\equiv1-\frac{M_W^2}{M_Z^2}=0.2226$$

$$\begin{cases}M_W=80.399\pm0.023\ \text{GeV}\\ M_Z=91.1875\pm0.0021\ \text{GeV}\end{cases}$$

※ The differences between different definitions of $\sin^2\theta_W$ appear at the level of one-loop computations, i.e., those are predictable.

9-6 Parameter of the electroweak Standard Model

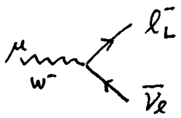
The EW standard model has three parameters (not counting the Higgs boson mass, the fermion masses and mixings) :

- the two gauge couplings ($SU(2)_L\times U(1)_Y$), g and g'
- the vacuum expectation value of the Higgs field, v

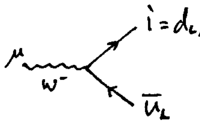
A useful set is (α_{EM}, G_F, M_Z), which can be determined precisely by experiments.

$$\begin{cases}e=\sqrt{4\pi\alpha_{EM}}=g\sin\theta_W=\dfrac{gg'}{\sqrt{g^2+g'^2}}\\ G_F=\dfrac{\sqrt{2}g^2}{8M_W^2}=\dfrac{1}{\sqrt{2}v^2}\\ M_Z=M_W\cos\theta_W=\dfrac12 gv\dfrac{g}{\sqrt{g^2+g'^2}}\end{cases}$$

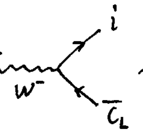
$\boxed{9\text{-}5}$ W decays



$\sim \frac{-ig}{\sqrt{2}}\gamma^\mu$



$\sim \frac{-ig}{\sqrt{2}} V_{ui}\gamma^\mu$



$\sim \frac{-ig}{\sqrt{2}} V_{ci}\gamma^\mu$

The relative branching ratio is

$$\begin{aligned}\Gamma_\ell : \Gamma_h &= \underset{\text{generation}}{3}\,\Gamma(W^-\to e^-\bar\nu_e) : \underset{\text{color}}{3}\sum_i\left(\Gamma(W^-\to i\bar u) + \Gamma(W^-\to i\bar c)\right)\\ &= 3\Gamma_e : 3\sum_i\left(|V_{ui}|^2 + |V_{ci}|^2\right)\Gamma_e\\ &= 1 : 2\end{aligned}$$

$\left(※\ \sum_i |V_{ui}|^2 = \sum_i |V_{ci}|^2 = 1\right)$ the unitarity property

The amplitude for $W^-(q) \to \ell^-(p_1) + \bar\nu_e(p_2)$ is

$$i\mathcal{M} = -\frac{ig}{2\sqrt{2}}\,\bar u(p_1)\gamma^\mu(1-\gamma_5)v(p_2)\,\varepsilon_\mu(q)$$

The spin-summed squared matrix element is

$$\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{g^2}{8}\,\mathrm{tr}\left[\not p_1\gamma^\mu(1-\gamma_5)\not p_2\gamma^\nu(1-\gamma_5)\right]\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M^2}\right) = g^2M^2$$

(See Ex. 9-1)

The partial decay width $\Gamma(W^-\to e^-\bar\nu_e)$ is

$$\begin{aligned}\Gamma_e &= \frac{1}{2M}\int d\Phi_2\,\frac{1}{3}\sum_{\text{spins}}|\mathcal{M}|^2\\ &= \frac{1}{2M}\frac{1}{8\pi}\frac{1}{3}g^2M^2 = \frac{M}{48\pi}g^2 = \frac{\alpha M}{12\,s_W^2}\left(= \frac{G_F M^3}{6\sqrt{2}\pi}\right) = 0.2266\ \text{GeV}\end{aligned}$$

$\left(※\ M_W = 80.399,\ \sin^2\theta_W = 0.231,\ \alpha = 1/128\right)$

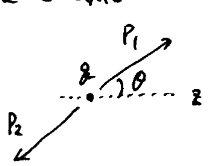
The total width is

$$\Gamma_{tot} = 9\times\Gamma_e = 2.039\ \text{GeV} \quad \sim 2.141 \pm 0.041\ \text{GeV (Exp.)}$$

※ Polarized W decays

We take the polarization axis along the z-axis :

$$\begin{cases} q = (M, 0, 0, 0)\\ p_1 = \frac{M}{2}(1, \sin\theta, 0, \cos\theta)\\ p_2 = \frac{M}{2}(1, -\sin\theta, 0, -\cos\theta)\end{cases}$$



In this frame the polarization vectors are written as

$$\begin{cases}\varepsilon^\pm(q) = \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0)\\ \varepsilon^0(q) = (0, 0, 0, 1)\end{cases}$$

$$※\ \sum_\lambda \varepsilon^\lambda_\mu(q)\varepsilon^{\lambda *}_\nu(q) = \begin{pmatrix}0&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} = \begin{pmatrix}-1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} + \begin{pmatrix}1&&&\\&0&&\\&&0&\\&&&0\end{pmatrix} = -g_{\mu\nu} + \frac{q_\mu q_\nu}{M^2}$$

The spin-summed (except W) squared matrix element is

$$\sum_{\text{spins}}|\mathcal{M}^\lambda|^2 = g^2\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1\cdot p_2)g^{\mu\nu}\right)\varepsilon^\lambda_\mu(q)\varepsilon^{\lambda *}_\nu(q)$$

$$\cdot\ \sum_{\text{spins}}|\mathcal{M}^\pm|^2 = g^2\left[-2\left(\frac{M}{2\sqrt{2}}\sin\theta\right)^2 + \frac{M^2}{2}\right] = \frac{M^2}{4}g^2(1+\cos^2\theta)$$

$$\cdot\ \sum_{\text{spins}}|\mathcal{M}^0|^2 = g^2\left[-2\left(\frac{M}{2}\cos\theta\right)^2 + \frac{M^2}{2}\right] = \frac{M^2}{2}g^2(1-\cos^2\theta)$$

$$\sum_{\text{spins}}|\mathcal{M}|^2 = \sum_\lambda\sum_{\text{spins}}|\mathcal{M}^\lambda|^2 = \sum_{\text{spins}}\left[|\mathcal{M}^+|^2 + |\mathcal{M}^-|^2 + |\mathcal{M}^0|^2\right] = g^2M^2$$