$\boxed{8\text{-}1}$ SU(2) is not enough

a)
$$
\begin{aligned}
[T^+, T^-] &= [T^1 + iT^2,\ T^1 - iT^2] \\
&= -i[T^1, T^2] + i[T^2, T^1] \\
&= -2i \cdot iT^3 \qquad\qquad \because\ [T^i, T^j] = i\epsilon^{ijk}T^k \quad (i = 1,2,3)\\
&= 2T^3
\end{aligned}
$$

$$
\begin{aligned}
[T^3, T^\pm] &= [T^3,\ T^1 \pm iT^2] \\
&= iT^2 \pm i(-iT^1) \\
&= \pm T^\pm
\end{aligned}
$$

b) The Noether charges :

$$T^+(t) = \frac{1}{2}\int d^3x\ J_0(x) = \frac{1}{2}\int d^3x\ \bar{\nu}_e(x)\gamma_0(1-\gamma_5)e(x) = \frac{1}{2}\int d^3x\ \nu_e^\dagger(x)(1-\gamma_5)e(x)$$

$$T^-(t) = (T^+(t))^\dagger = \frac{1}{2}\int d^3x\ e^\dagger(x)(1-\gamma_5)\nu_e(x) \qquad (\because\ \bar{\psi} = \psi^\dagger\gamma^0,\ \gamma^{0\,2} = 1)$$

$$Q(t) = \int d^3x\ J_0^{em.}(x) = -\int d^3x\ e^\dagger(x)e(x)$$

$$[T^+, T^-] = \frac{1}{4}\int d^3x\,d^3y\ \left[\nu_e^\dagger(x)(1-\gamma_5)e(x),\ e^\dagger(y)(1-\gamma_5)\nu_e(y)\right]$$

Using the canonical (equal-time) anti-commutation relations for Dirac fields ψ and $\psi^\dagger$:

$$\{\psi_a(t,\vec{x}),\ \psi_b^\dagger(t,\vec{y})\} = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y}),$$
$$\{\psi_a(t,\vec{x}),\ \psi_b(t,\vec{y})\} = \{\psi_a^\dagger(t,\vec{x}),\ \psi_b^\dagger(t,\vec{y})\} = 0,$$

$$
\begin{aligned}
[T^+, T^-] &= \frac{1}{4}\int d^3x\,d^3y\ \Big\{\nu_{e_a}^*(x)(1-\gamma_5)_{ab}\left\{\delta_{bc}\,\delta^{(3)}(\vec{x}-\vec{y}) - e_c^*(y)\,e_b(x)\right\}(1-\gamma_5)_{cd}\,\nu_{e_d}(y) \\
&\qquad - e_c^*(y)(1-\gamma_5)_{cd}\left\{\delta_{da}\,\delta^{(3)}(\vec{y}-\vec{x}) - \nu_{e_a}^*(x)\,\nu_{e_d}(y)\right\}(1-\gamma_5)_{ab}\,e_b(x)\Big\} \\
&= \frac{1}{4}\int d^3x\ 2\left\{\nu_e^\dagger(x)(1-\gamma_5)\nu_e(x) - e^\dagger(x)(1-\gamma_5)e(x)\right\} \qquad \left(\because\ (1-\gamma_5)^2 = 2(1-\gamma_5)\right)\\
&\quad + \frac{1}{4}\int d^3x\,d^3y\ \Big\{-e_c^*(y)\left(\nu_e^\dagger(x)\cancel{(1-\gamma_5)e(x)}\right)(1-\gamma_5)_{cd}\,\nu_{e_d}(y) \\
&\qquad + \nu_{e_a}^*(x)\left(e^\dagger(y)\cancel{(1-\gamma_5)\nu_e(y)}\right)(1-\gamma_5)_{ab}\,e_b(x)\Big\} \\
&= 2T^3 \neq 2Q
\end{aligned}
$$

Therefore, $T^\pm$, Q do not form a closed algebra.

c) • In order for Q to be a generator of $SU(2)$, the charges of a complete multiplet must add up to zero, corresponding to the requirement that the generators for $SU(2)$ must be traceless. A doublet out of ν_e and e clearly do not satisfy this condition.

• $T^\pm$: V−A $\Longleftrightarrow$ Q: V

d) The interactions of leptons with gauge bosons are dictated by the covariant derivative:

$$D_\mu = \partial_\mu + i g T^i A^i_\mu(x) + i g' Y B_\mu(x)$$

where

$$\begin{cases} g / g' : & SU(2)_L / U(1)_Y \text{ gauge coupling} \\ T^i / Y : & \text{" generators} \end{cases} \qquad [T^i, T^j] = i\epsilon^{ijk} T^k$$

The covariant derivative in terms of the physical electroweak gauge bosons, $W^\pm, Z, A$:

$$D_\mu = \partial_\mu + ig(T^1 A^1_\mu + T^2 A^2_\mu) + ig T^3 A^3_\mu + i g' Y B_\mu \qquad (※\ Q = T^3 + Y)$$

$$\left(※\ W^\pm_\mu = \frac{A^1_\mu \mp i A^2_\mu}{\sqrt{2}}\right) \quad = \partial_\mu + i\frac{g}{\sqrt{2}}(T^+ W^+_\mu + T^- W^-_\mu) + igT^3(C_W Z_\mu + S_W A_\mu) + ig'(Q - T^3)(-S_W Z_\mu + C_W A_\mu)$$

$$= \partial_\mu + i\frac{g}{\sqrt{2}}(T^+ W^+_\mu + T^- W^-_\mu) + i\frac{g}{C_W}(T^3 - S_W^2 Q) Z_\mu + ieQA_\mu$$

where

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} = g \sin\theta_W = g' \cos\theta_W$$

Weinberg angle (weak-mixing)

$$\left(※\ \begin{array}{l} \psi_L = \frac{1-\gamma_5}{2}\psi \\ \psi_R = \frac{1+\gamma_5}{2}\psi \end{array}\right)$$

The Lagrangian for leptons (one generation):

$$\mathcal{L}_{\text{lepton}} = \sum_\ell \bar{\psi}_\ell i \gamma^\mu D_\mu \psi_\ell \qquad \ell = \left\{ L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \ R = e_R \right\}$$

- The charged current int.:

$$\mathcal{L}_{c.c} = -\frac{g}{\sqrt{2}}\left[(\bar{\nu}_e\ \bar{e})_L \gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L W^+_\mu + (\bar{\nu}_e\ \bar{e})_L \gamma^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L W^-_\mu \right]$$

$$= -\frac{g}{2\sqrt{2}}\left[\underbrace{\bar{\nu}_e \gamma^\mu (1-\gamma_5) e}_{= J^{+\mu}} W^+_\mu + \underbrace{\bar{e}\gamma^\mu(1-\gamma_5)\nu_e}_{= J^{-\mu}} W^-_\mu \right]$$

- The neutral current int. (only Z):

$$\mathcal{L}^Z_{n.c.} = -\frac{g}{C_W}(\bar{\nu}_e\ \bar{e})_L \gamma^\mu \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} + S_W^2 \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L Z_\mu - \frac{g}{C_W} \bar{e}_R \gamma^\mu S_W^2 e_R Z_\mu$$

$$= -\frac{g}{C_W} \underbrace{\frac{1}{4}\left[\bar{\nu}_e \gamma^\mu (1-\gamma_5)\nu_e - \bar{e}\gamma^\mu(1 - \gamma_5 - 4S_W^2) e \right]}_{= J^{n\,\mu} = J^{3\mu} - \sin^2\theta_W J^{em,\mu}} Z_\mu$$

$$\xrightarrow[g'=0]{} -g \underbrace{\frac{1}{4}\left[\bar{\nu}_e \gamma^\mu (1-\gamma_5)\nu_e - \bar{e}\gamma^\mu(1-\gamma_5) e \right]}_{= J^{3\mu}} \underset{A^3_\mu}{\overset{\parallel}{Z_\mu}}$$

- The electro-magnetic int:

$$\mathcal{L}_{e.m.} = e\bar{e}\gamma^\mu e A_\mu = -e \underbrace{(-\bar{e}\gamma^\mu e)}_{= J^{em\,\mu}} A_\mu$$

One can see that the current algebra of J^+, J^- and J^3 does close.

$$\mathcal{L}_{\text{lepton}} = \mathcal{L}_{\text{kin}} - \frac{g}{2\sqrt{2}}(J^{+\mu} W^+_\mu + J^{-\mu} W^-_\mu) - \frac{g}{\cos\theta_W} J^{n\mu} Z_\mu - e J^{em,\mu} A_\mu$$

8-2 Gauge field interactions

a) The mass matrix for the EW gauge bosons $X_\mu^a = (A_\mu^1, A_\mu^2, A_\mu^3, B_\mu)$:

$$\mathcal{L}_{mass} = |D_\mu \Phi_0|^2 = \frac{1}{2} M_{ab}^2 X_\mu^a X^{b\mu} \qquad \left(※ \; D_\mu = \partial_\mu + ig\frac{\sigma^i}{2}A_\mu^i + ig'\frac{1}{2}B_\mu, \quad \Phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} \right)$$

where

$$M^2 = \frac{v^2}{4}\begin{pmatrix} g^2 & & & \\ & g^2 & & \\ & & g^2 & -gg' \\ & & -gg' & g'^2 \end{pmatrix} \xrightarrow[\text{diagonalize}]{} U^+ M^2 U = diag(m_w^2, m_w^2, m_z^2, 0)$$

The physical EW gauge bosons $Y_\mu^a = (W_\mu^+, W_\mu^-, Z_\mu, A_\mu)$:

$$X_\mu^a = U_{ab} Y_\mu^b = \begin{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix} & \\ & \begin{pmatrix}\cos\theta_W & \sin\theta_W\\ -\sin\theta_W & \cos\theta_W\end{pmatrix}\end{pmatrix}\begin{pmatrix}W_\mu^+\\ W_\mu^-\\ Z_\mu\\ A_\mu\end{pmatrix} \qquad \left(※ \; \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \right)$$

b) The pure $SU(2)_L \times U(1)_Y$ Yang-Mills Lagrangian :

$$\mathcal{L}_{gauge} = -\frac{1}{4}F_{\mu\nu}^i F^{i\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$

where

$$\begin{cases} F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i - g\,\epsilon^{ijk}A_\mu^j A_\nu^k & (i=1,2,3) \\ G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \end{cases}$$

The field strength tensors in terms of the physical gauge bosons :

$$\begin{aligned} F_{\mu\nu}^1 &= \partial_\mu A_\nu^1 - \partial_\nu A_\mu^1 - g(A_\mu^2 A_\nu^3 - A_\mu^3 A_\nu^2) \\ &= \frac{1}{\sqrt{2}}\left\{ \partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ + \partial_\mu W_\nu^- - \partial_\nu W_\mu^- - ig\left[W_\mu^+ A_\nu^3 - A_\mu^3 W_\nu^+ - W_\mu^- A_\nu^3 + A_\mu^3 W_\nu^-\right]\right\} \\ &= \frac{1}{\sqrt{2}}\left\{(d_\mu W_\nu^+ - d_\nu W_\mu^+) + (d_\mu^* W_\nu^- - d_\nu^* W_\mu^-)\right\} \\ F_{\mu\nu}^2 &= \partial_\mu A_\nu^2 - \partial_\nu A_\mu^2 - g(A_\mu^3 A_\nu^1 - A_\mu^1 A_\nu^3) \\ &= \frac{i}{\sqrt{2}}\left\{ \partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ - \partial_\mu W_\nu^- + \partial_\nu W_\mu^- + ig\left[A_\mu^3 W_\nu^+ - W_\mu^+ A_\nu^3 + A_\mu^3 W_\nu^- - W_\mu^- A_\nu^3\right]\right\} \\ &= \frac{i}{\sqrt{2}}\left\{(d_\mu W_\nu^+ - d_\nu W_\mu^+) - (d_\mu^* W_\nu^- - d_\nu^* W_\mu^-)\right\} \equiv \frac{i}{\sqrt{2}}\left(F_{\mu\nu}^W - (F_{\mu\nu}^W)^+\right) \\ F_{\mu\nu}^3 &= \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 - g(A_\mu^1 A_\nu^2 - A_\mu^2 A_\nu^1) \\ &= \cos\theta_W(\partial_\mu Z_\nu - \partial_\nu Z_\mu) + \sin\theta_W(\partial_\mu A_\nu - \partial_\nu A_\mu) + ig(W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+) \\ G_{\mu\nu} &= -\sin\theta_W(\partial_\mu Z_\nu - \partial_\nu Z_\mu) + \cos\theta_W(\partial_\mu A_\nu - \partial_\nu A_\mu) \equiv -\sin\theta_W F_{\mu\nu}^Z + \cos\theta_W F_{\mu\nu}^A \end{aligned}$$

where

$$d_\mu = \partial_\mu + igA_\mu^3 = \partial_\mu + ig\cos\theta_W Z_\mu + \underbrace{ig\sin\theta_W}_{=e} A_\mu$$

Therefore, we find that

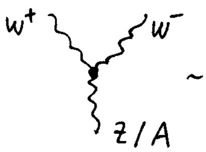
$$
\begin{aligned}
&F^1_{\mu\nu} F^{1\mu\nu} + F^2_{\mu\nu} F^{2\mu\nu} \\
&\quad = \frac{1}{2}\left(F^W_{\mu\nu} + (F^W_{\mu\nu})^\dagger \right)\left(F^{W\mu\nu} + (F^{W\mu\nu})^\dagger \right) - \frac{1}{2}\left(F^W_{\mu\nu} - (F^W_{\mu\nu})^\dagger \right)\left(F^{W\mu\nu} - (F^{W\mu\nu})^\dagger \right) \\
&\quad = 2 \, (F^W_{\mu\nu})^\dagger F^{W\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
&F^3_{\mu\nu} F^{3\mu\nu} + G_{\mu\nu} G^{\mu\nu} \\
&\quad = F^Z_{\mu\nu} F^{Z\mu\nu} + F^A_{\mu\nu} F^{A\mu\nu} \\
&\qquad + 2 F^{A^3}_{\mu\nu} \, i g \left(W^{+\mu} W^{-\nu} - W^{-\mu} W^{+\nu} \right) \\
&\qquad + (ig)^2 \left(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu \right)\left(W^{+\mu} W^{-\nu} - W^{-\mu} W^{+\nu} \right) \\
&\quad = F^Z_{\mu\nu} F^{Z\mu\nu} + F^A_{\mu\nu} F^{A\mu\nu} \\
&\qquad + 4i \left(g \cos\theta_W F^Z_{\mu\nu} + e F^A_{\mu\nu} \right) W^{+\mu} W^{-\nu} \\
&\qquad - 2 g^2 \left\{ (W^+ \cdot W^+)(W^- \cdot W^-) - (W^+ \cdot W^-)^2 \right\}
\end{aligned}
$$

Finally,

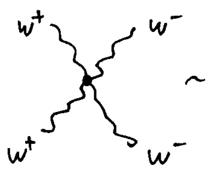
$$
\begin{aligned}
\mathcal{L}_{gauge} = &- \frac{1}{2} (F^W_{\mu\nu})^\dagger F^{W\mu\nu} - \frac{1}{4} F^Z_{\mu\nu} F^{Z\mu\nu} - \frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} \\
&- i \left(g \cos\theta_W F^Z_{\mu\nu} + e F^A_{\mu\nu} \right) W^{+\mu} W^{-\nu} \\
&+ \frac{1}{2} g^2 \left(W^{+2} W^{-2} - (W^+ \cdot W^-)^2 \right)
\end{aligned}
$$

- Three point gauge boson interactions :

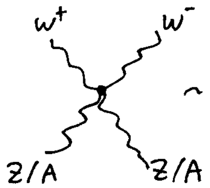


$\sim g \cos\theta_W / e$

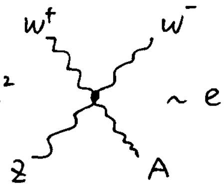
- Four point gauge boson interactions :



$\sim g^2$



$\sim g^2 \cos^2\theta_W / e^2$



$\sim e g \cos\theta_W$

$\boxed{8\text{-}3}$ Symmetries of the Higgs potential

a) $Q = T^3 + Y \;\rightarrow\; Q(\Phi_1) = +\frac{1}{2} + \frac{1}{2} = +1$

$$Q(\Phi_2) = -\frac{1}{2} + \frac{1}{2} = 0$$

b) $\Phi^\dagger(x)\Phi(x)$ is clearly invariant under the $SU(2)_L \times U(1)_Y$ gauge transformation,

$$\Phi(x) \rightarrow U(x)\Phi(x) = \exp\left[i\theta^i(x)T^i + i\theta_Y(x)Y\right]\Phi(x), \qquad \left(\because\ T^i = \frac{1}{2}\sigma^i,\ Y = \frac{1}{2}\right)$$

c)
$$\begin{aligned}\Phi^\dagger(x)\Phi(x) &= (\varphi_1 - i\varphi_2,\ \varphi_3 - i\varphi_4)\begin{pmatrix}\varphi_1 + i\varphi_2\\ \varphi_3 + i\varphi_4\end{pmatrix}\\ &= \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2\\ &\equiv |\vec{\varphi}|^2 \quad \Longrightarrow \text{The Higgs potential } V(\Phi)\ (= V(\vec{\varphi})) \text{ has } O(4) \text{ global symmetry.}\end{aligned}$$

The potential $V(\vec{\varphi})$ is minimized for any $\vec{\varphi}_0$ that satisfies $|\vec{\varphi}_0|^2 = \frac{v^2}{2}$. This condition determines only the length of the vector $\vec{\varphi}_0$; its direction is arbitrary. We choose $\vec{\varphi}_0$ as

$$\vec{\varphi}_0 = \left(0, 0, 0, \frac{v}{\sqrt{2}}\right),$$

and define a set of shifted fields as

$$\vec{\varphi}(x) = \left(\pi_1(x), \pi_2(x), \pi_3(x), \frac{v}{\sqrt{2}} + \sigma(x)\right).$$

The Higgs potential in terms of $\pi_i(x)$ and $\sigma(x)$ is

$$V(\vec{\varphi}) = \frac{\lambda}{2}\Big[\underbrace{\pi_1^2 + \pi_2^2 + \pi_3^2}_{O(3)\ \text{global symmetry}} + \sigma^2 + \sqrt{2}v\sigma\Big]^2.$$

$$\xrightarrow{\text{mass term}}\ \frac{1}{2}(2\lambda v^2)\sigma^2(x) = \frac{1}{2}m_\sigma^2\sigma^2(x)$$

We obtain a massive σ field and 3 massless π fields.

The number of would-be Goldstone modes is

$$\dim \tilde{G}/\tilde{H} = \dim O(4) - \dim O(3) = \frac{4\cdot 3}{2} - \frac{3\cdot 2}{2} = 3$$

where

$$\begin{aligned}&\tilde{G} = O(4) \sim SU(2) \times \underbrace{SU(2)}_{\text{gauged}} \supset G\\ &\downarrow\\ &\tilde{H} = O(3) \sim SU(2)\end{aligned}$$

↖ called 'custodial symmetry'