7-1 Massive abelian gauge theories

From Ex. 6-3 d), the propagator of a massive vector boson is

$$\Delta_{\mu\nu}(P, M_A) = \frac{-i}{P^2 - M_A^2 + i\varepsilon}\left[g_{\mu\nu} - \frac{P_\mu P_\nu}{M_A^2}\right].$$

The tensor structure represents a gauge boson polarization sum. Let us consider the vector boson is on-shell and boost to its rest frame, i.e.

$$p^\mu = (M_A, 0, 0, 0).$$

The polarization sum is

$$\sum_\lambda \varepsilon_\mu(P,\lambda)\,\varepsilon_\nu^*(P,\lambda) = -g_{\mu\nu} + \frac{P_\mu P_\nu}{M_A^2} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Therefore, the tensor structure is the projection onto the three spatial directions. These are the three polarization states of an on-shell massive vector boson.

From Ex. 6-3 c), the propagator of a massless vector boson ($\xi = 1$) is

$$\Delta_{\mu\nu}(P, \xi = 1) = \frac{-i}{P^2 + i\varepsilon}\, g_{\mu\nu}$$

Although the propagator has four components, corresponding to the transverse, longitudinal, and timelike polarizations, the unphysical longitudinal and timelike components cancel in computations due to the Ward identity, $P_\mu M^\mu = 0$.

$$\Longrightarrow \begin{cases} \text{massive } U(1) \text{ boson} & : \quad 3 \text{ d.o.f} \\ \text{massless } \quad " & : \quad 2 \text{ d.o.f} \end{cases}$$

7-2 The non-abelian Higgs mechanism

a) # of gauge fields = # of generators = the dimension of the adjoint rep.

$$\dim(G = U(4)) = 4^2 = 16$$
$$\dim(H = U(3)) = 3^2 = 9$$

From Goldstone's theorem, the # of Goldstone modes is

$$\dim G/H = \dim G - \dim H = 16 - 9 = 7$$

The d.o.f of the complex scalar field Φ is now 8, so the # of massive real d.o.f is $(8-7) = 1$.

b) The complex scalar field $\Phi(x)$ is parametrized by two real scalar fields, $\eta(x)$ and $f(x)$, as

$$\Phi(x) = e^{-i\eta^{\hat{a}}(x) T^{\hat{a}}} \frac{1}{\sqrt{2}}(\Phi_0 + f(x)) = U^{-1}(x) \frac{1}{\sqrt{2}}(\Phi_0 + f(x))$$

subject to $\mathrm{Im}(f^\dagger(x) T^a \Phi_0) = f^\dagger(x) T^a \Phi_0 - \Phi_0^\dagger T^a f(x) = 0$ —— ①

Let us take a gauge transformation $U(x)$ associated G/H :

$\left(※ \{T^a \in G\} = \{t^i \in H, T^{\hat{a}} \in G/H\} \right)$

- $\Phi(x) \to U(x)\Phi(x) = \frac{1}{\sqrt{2}}(\Phi_0 + f(x)) \equiv \Phi(x)$
- $A_\mu^a(x) T^a \to U(x) A_\mu^a(x) T^a U^{-1}(x) + \frac{i}{g}(\partial_\mu U(x)) U^{-1}(x) \equiv B_\mu^a(x) T^a$
  $= B_\mu^i(x) t^i + B_\mu^{\hat{a}}(x) T^{\hat{a}}$

c) Therefore, the covariant derivative on the Φ is

$$D_\mu \Phi \to (\partial_\mu + ig B_\mu^i t^i + ig B_\mu^{\hat{a}} T^{\hat{a}}) \frac{1}{\sqrt{2}}(\Phi_0 + f)$$
$$= \frac{1}{\sqrt{2}}(\partial_\mu f + ig B_\mu^i t^i f + ig B_\mu^{\hat{a}} T^{\hat{a}} (\Phi_0 + f)) \qquad \left(\because \begin{array}{l} \partial_\mu \Phi_0 = 0 \\ t^i \Phi_0 = 0 \end{array} \right)$$

Then the kinetic energy term for the Φ is (only quadratic term)

$$(D_\mu \Phi)^\dagger (D^\mu \Phi) \to \frac{1}{2}(\partial_\mu f)^\dagger (\partial^\mu f) + \frac{1}{2} g^2 B_\mu^{\hat{a}} B^{\hat{b}\mu} (\Phi_0^\dagger T^{\hat{a}})(T^{\hat{b}} \Phi_0)$$

- There is no $f - B_\mu^{\hat{a}}$ cross term because of the condition ① and $t^i \Phi_0 = 0$,

  $$\frac{1}{2}(\partial_\mu f)^\dagger (ig B^{\hat{a}\mu} T^{\hat{a}} \Phi_0) + \frac{1}{2}(-ig B_\mu^{\hat{a}} \Phi_0^\dagger T^{\hat{a}})(\partial^\mu f) = 0.$$

  The condition ① implies that $f(x)$ does not contain Goldstone modos.

  $\Rightarrow$ the unitary gauge

- There is no $B_\mu^i B^{i\mu}$ term. $\Rightarrow$ The gauge fields B_μ^i remain massless.
- The mass matrix for the fields $B_\mu^{\hat{a}}$ is $M_{\hat{a}\hat{b}}^2 = g^2 (\Phi_0^\dagger T^{\hat{a}})(T^{\hat{b}} \Phi_0)$.

Note that

- $\dim G/H$ Goldstone bosons $\eta^{\hat{a}}$ are eaten by gauge bosons $\Rightarrow$ $B_\mu^{\hat{a}}$: massive gauge bosons
- The # of massive scalars f depends on the representation of f.

d) $\left(\sum_{\hat{a}} w_{\hat{a}}^{\dagger} \Phi_0^{\dagger} T^{\hat{a}}\right)\left(\sum_{\hat{b}} T^{\hat{b}} \Phi_0 w_{\hat{b}}\right)$

$$= \sum_{\hat{a},\hat{b}} w_{\hat{a}}^{\dagger} \frac{1}{g^2} M^2_{\hat{a}\hat{b}} w_{\hat{b}}$$

hermitian real

$$(M^2_{\hat{a}\hat{b}} w_{\hat{b}} = \lambda w_{\hat{b}})$$

orthogonal eigenvectors

$$= \sum_{\hat{a},\hat{b}} w_{\hat{a}}^{\dagger} \frac{1}{g^2} \lambda w_{\hat{b}}$$

$$= \sum_{\hat{a}} \frac{1}{g^2} \lambda w_{\hat{a}}^{\dagger} w_{\hat{a}} \geq 0 \quad \Rightarrow \lambda \geq 0$$

If $\lambda = 0$, then $T^{\hat{a}} \Phi_0 w_{\hat{a}} = 0$. However, $T^{\hat{a}} \Phi_0 \neq 0$ since $T^{\hat{a}} \in G/H$.

Therefore, all eigenvalues λ are strictly positive, $\lambda > 0$, i.e. dim G/H gauge bosons receive positive masses.