4-1 Compton scattering

First of all, let me present the field operators in order to fix my notation:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s \left(a^s_{\vec p}\, u^s(p)\, e^{-ipx} + b^{s\dagger}_{\vec p}\, v^s(p)\, e^{ipx} \right)$$

$$\bar\psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s \left(a^{s\dagger}_{\vec p}\, \bar u^s(p)\, e^{ipx} + b^s_{\vec p}\, \bar v^s(p)\, e^{-ipx} \right)$$

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_r \left(a^r_{\vec p}\, \varepsilon^r_\mu(p)\, e^{-ipx} + a^{r\dagger}_{\vec p}\, \varepsilon^{r*}_\mu(p)\, e^{ipx} \right)$$

The Compton scattering process $e^-(p) + \gamma(k) \to e^-(p') + \gamma(k')$:

$$|i\rangle = |e^-_s(p);\, \gamma_r(k)\rangle = a^{s\dagger}_{\vec p}\, a^{r\dagger}_{\vec k}\, |0\rangle$$

$$|f\rangle = |e^-_{s'}(p');\, \gamma_{r'}(k')\rangle = a^{s'\dagger}_{\vec p'}\, a^{r'\dagger}_{\vec k'}\, |0\rangle \quad \Rightarrow \quad \langle f| = \langle 0|\, a^{r'}_{\vec k'}\, a^{s'}_{\vec p'}$$

a) $$\underbrace{\langle f|\, S^{(2)}\, |i\rangle}_{= iT^{(2)}} = \langle f|\, \frac{i^2}{2!} \int d^4x \int d^4y\; T\left[\mathcal{L}_I(x)\, \mathcal{L}_I(y) \right] |i\rangle$$

$$= \langle f|\, \frac{(ie)^2}{2!} \int d^4x \int d^4y\; T\left[\bar\psi(x)\gamma^\mu\psi(x)A_\mu(x)\;\; \bar\psi(y)\gamma^\nu\psi(y)A_\nu(y) \right] |i\rangle$$

First, let us consider the electron fields:

$$\langle e^-_{s'}(p')|\; T\left[(\bar\psi\gamma^\mu\psi)_x\, (\bar\psi\gamma^\nu\psi)_y \right] |e^-_s(p)\rangle$$

$$= \langle 0|\, a^{s'}_{\vec p'} :\, \bar\psi_x\, \gamma^\mu\, \psi_x\, \bar\psi_y\, \gamma^\nu\, \psi_y : a^{s\dagger}_{\vec p}\, |0\rangle$$

$$= e^{ip'x}\, \bar u^{s'}(p')\, \gamma^\mu\, iS_F(x-y)\, \gamma^\nu\, u^s(p)\, e^{-ipy}$$

$$\left(\text{※ } \psi(x)\, a^{s\dagger}_{\vec p}|0\rangle = \int \frac{d^3p'}{(2\pi)^3 2E_{p'}} \sum_{s'} \underbrace{a^{s'}_{\vec p'}\, u^{s'}(p')\, e^{-ip'x}\, a^{s\dagger}_{\vec p}}_{= (2\pi)^3 2E_p \delta^{ss'} \delta(\vec p - \vec p')} |0\rangle = e^{-ipx}\, u^s(p)\, |0\rangle \right)$$

Therefore,

$$\langle f|\, S^{(2)}\, |i\rangle = (ie)^2 \int d^4x\, d^4y\; \bar u^{s'}(p')\, \gamma^\mu\, iS_F(x-y)\, \gamma^\nu\, u^s(p)\, e^{ip'x - ipy}$$
$$\times \langle \gamma_{r'}(k')|\, : A_\mu(x)\, A_\nu(y) :\, |\gamma_r(k)\rangle$$

Note that we have dropped the factor $1/2!$ since there is the identical term that comes from interchanging x and y.

$$\left(\text{※ The Feynman propagator of the Dirac field: } \quad iS_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i\, e^{-ip(x-y)}}{\not p - m + i\epsilon} \right)$$

※ S-, T-matrix, the invariant matrix element

$$S = \mathbb{1} + iT$$

$$\langle f|\, iT\, |i\rangle = (2\pi)^4\, \delta^{(4)}\left(\sum p_i - \sum p_f \right)\, i\mathcal{M}$$

b) Recall that

$$: A_\mu(x) A_\nu(y) : \; = A_\mu^+(x) A_\nu^+(y) + A_\nu^-(y) A_\mu^+(x) + A_\mu^-(x) A_\nu^+(y) + A_\mu^-(x) A_\nu^-(y) ,$$

where

$$A_\mu^+(x) \sim a_{\vec p}^r \, \varepsilon_\mu^r(p) \, e^{-ipx} \quad , \quad A_\mu^-(x) \sim a_{\vec p}^{r\dagger} \varepsilon_\mu^{r*}(p) \, e^{ipx} .$$

Therefore,

$$\langle \gamma_{r'}(k') | : A_\mu(x) A_\nu(y) : | \gamma_r(k) \rangle$$

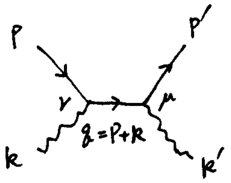
$$= \langle 0 | a_{\vec k'}^{r'} A_\nu^-(y) A_\mu^+(x) a_{\vec k}^{r\dagger} | 0 \rangle + \langle 0 | a_{\vec k'}^{r'} A_\mu^-(x) A_\nu^+(y) a_{\vec k}^{r\dagger} | 0 \rangle$$

$$= \underset{\text{(ii)}}{e^{ik'y} \varepsilon_\nu^{r'*}(k') \, \varepsilon_\mu^r(k) \, e^{-ikx}} + \underset{\text{(i)}}{e^{ik'x} \varepsilon_\mu^{r'*}(k') \, \varepsilon_\nu^r(k) \, e^{-iky}}$$

$$\left(\begin{aligned} * \; A_\mu^+(x) a_{\vec k}^{r\dagger} |0\rangle &= \int \frac{d^3k'}{(2\pi)^3 2E_{k'}} \sum_{r'} \underbrace{a_{\vec k'}^{r'} \varepsilon_\mu^{r'}(k') \, e^{-ik'x} \, a_{\vec k}^{r\dagger}}_{= (2\pi)^3 2E_k \delta^{rr'} \delta(\vec k - \vec k')} |0\rangle \\ &= e^{-ikx} \varepsilon_\mu^r(k) |0\rangle \end{aligned} \right)$$

For $\mathcal{M}_1$,

$$\langle f | S^{(2)} | i \rangle_1 = (ie)^2 \int d^4x \, d^4y \int \frac{d^4q}{(2\pi)^4} \bar u^{s'}(p') \gamma^\mu \frac{i e^{-iq(x-y)}}{\not q - m + i\epsilon} \gamma^\nu u^s(p) \, \varepsilon_\mu^{r'*}(k') \, \varepsilon_\nu^r(k)$$
$$\times e^{ix(p'+k')} e^{-iy(p+k)}$$

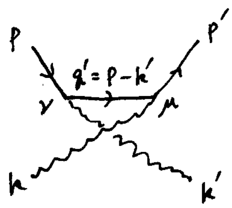


$$= (ie)^2 \int d^4q \, (2\pi)^4 \delta^{(4)}(p'+k'-q) \, \delta^{(4)}(q-p-k)$$
$$\times \bar u^{s'}(p') \gamma^\mu \frac{i}{\not q - m + i\epsilon} \gamma^\nu u^s(p) \, \varepsilon_\mu^{r'*}(k') \varepsilon_\nu^r(k)$$

$$= (2\pi)^4 \delta^{(4)}(p+k-p'-k') \, i \underbrace{\Big[(ie)^2 \bar u^{s'}(p') \gamma^\mu \frac{1}{\underset{\overset{\shortparallel}{\not p + \not k}}{\not q - m + i\epsilon}} \gamma^\nu u^s(p) \, \varepsilon_\mu^{r'*}(k') \varepsilon_\nu^r(k) \Big]}_{= \mathcal{M}_1}$$

For $\mathcal{M}_2$,

$$\langle f | S^{(2)} | i \rangle_2 = (ie)^2 \int d^4x \, d^4y \int \frac{d^4q'}{(2\pi)^4} \bar u^{s'}(p') \gamma^\mu \frac{i e^{-iq'(x-y)}}{\not q' - m + i\epsilon} \gamma^\nu u^s(p) \, \varepsilon_\nu^{r'*}(k') \, \varepsilon_\mu^r(k)$$
$$\times e^{+ix(p'-k)} e^{-iy(p-k')}$$

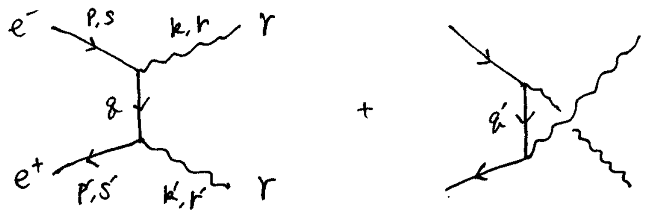


$$\sim \cdots \int d^4q' \, (2\pi)^4 \delta^{(4)}(p'-k-q') \, \delta^{(4)}(q'-p+k') \cdots$$

$$= (2\pi)^4 \delta^{(4)}(p+k-p'-k') \, i \underbrace{\Big[(ie)^2 \bar u^{s'}(p') \gamma^\mu \frac{1}{\underset{\overset{\shortparallel}{\not p - \not k'}}{\not q' - m + i\epsilon}} \gamma^\nu u^s(p) \, \varepsilon_\nu^{r'*}(k') \varepsilon_\mu^r(k) \Big]}_{= \mathcal{M}_2}$$

$\boxed{4\text{-}2}$ $e^+ + e^- \to \gamma + \gamma$

a)



(i) (ii)

b) $$i\mathcal{M}_i = \bar{v}_{s'}(p')\,(ie)\gamma^\mu \frac{i}{\not{q} - m_e + i\epsilon}(ie)\gamma^\nu\, u_s(p) \cdot \varepsilon_\nu^{r*}(k)\,\varepsilon_\mu^{r'*}(k')$$

$$i\mathcal{M}_{ii} = \bar{v}_{s'}(p')\,(ie)\gamma^\mu \frac{i}{\not{q}' - m_e + i\epsilon}(ie)\gamma^\nu\, u_s(p) \cdot \varepsilon_\mu^{r*}(k)\,\varepsilon_\nu^{r'*}(k')$$

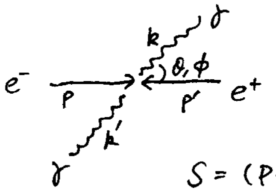
※ The cross section is given by

$$d\sigma = \underbrace{\frac{1}{2s}}_{\text{CM energy}}\ \underbrace{\frac{1}{4}}_{\text{spin average}}\ \underbrace{\sum_{s,s',r,r'}}_{\text{spin sum}} \underbrace{|\mathcal{M}|^2}_{i\mathcal{M} = i\mathcal{M}_i + i\mathcal{M}_{ii}}\ \underbrace{d\Phi_2}_{\text{phase space integral}}$$

$$d\Phi_2 = \frac{d^3k}{(2\pi)^3 2E_k}\frac{d^3k'}{(2\pi)^3 2E_{k'}}(2\pi)^4\delta^{(4)}(p+p'-k-k')$$
$$= \frac{1}{8\pi}\frac{d\cos\theta}{2}\frac{d\phi}{2\pi}.$$

※ Kinematics (in the $m_e = 0$ limit):

$$\begin{cases} p^\mu = E(1,0,0,1) \\ p'^\mu = E(1,0,0,-1) \\ k^\mu = E(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ k'^\mu = E(1, -\;\text{〃}, -\;\text{〃}, -\;\text{〃}) \end{cases}$$



$$s = (p+p')^2 = 4E^2$$

$$\sqrt{s} = 2E$$