$\boxed{3-2}$ The Dirac spinor

a) Under a Lorentz transformation as

$$x \to x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu},$$

a field $\psi(x)$ is transformed as

$$\psi(x) \to \psi'(x') = S(\Lambda)\,\psi(x) = S(\Lambda)\,\psi(\Lambda^{-1}x')$$

where
$$\begin{cases} S(\Lambda) = 1 & : \ \psi(x) \text{ is a scalar field.} \\ S(\Lambda) = \Lambda = \exp\left(-\frac{i}{4}\omega_{\mu\nu}M^{\mu\nu}\right) & : \ \psi(x) \text{ is a vector field. (See Ex 1-1 c))} \\ S(\Lambda) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) & : \ \psi(x) \text{ is a Dirac spinor field.} \end{cases}$$

Using the following relation : $[[A,B],C] = \{A,\{B,C\}\} - \{B,\{A,C\}\}$,

$$\begin{aligned} \frac{i}{4}[\sigma^{\mu\nu}, \gamma^{\rho}]\,\omega_{\mu\nu} &= \frac{i}{4}\left[\frac{i}{2}[\gamma^{\mu},\gamma^{\nu}], \gamma^{\rho}\right]\omega_{\mu\nu} \\ &= \frac{-1}{8}\Big[\{\gamma^{\mu}, \underbrace{\{\gamma^{\nu},\gamma^{\rho}\}}_{=2g^{\nu\rho}}\} - \{\gamma^{\nu}, \underbrace{\{\gamma^{\mu},\gamma^{\rho}\}}_{=2g^{\mu\rho}}\}\Big]\omega_{\mu\nu} \\ &= \frac{-1}{4}\left[2\gamma^{\mu}\omega_{\mu}{}^{\rho} - 2\gamma^{\nu}\omega^{\rho}{}_{\nu}\right] \\ &= \omega^{\rho}{}_{\nu}\gamma^{\nu} = -\frac{i}{4}\omega_{\mu\nu}(M^{\mu\nu})^{\rho}{}_{\sigma}\gamma^{\sigma} \quad \text{(See Ex. 1-1 c).)} \end{aligned}$$

$$\Rightarrow \left(1 + \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\gamma^{\rho}\left(1 - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) = \left(1 - \frac{i}{4}\omega_{\mu\nu}M^{\mu\nu}\right)^{\rho}{}_{\sigma}\gamma^{\sigma}$$

This is the infinitesimal version of $S^{-1}(\Lambda)\gamma^{\rho}S(\Lambda) = \Lambda^{\rho}{}_{\sigma}\gamma^{\sigma}$.

b)
$$\begin{aligned} [i\gamma^{\mu}\partial_{\mu} - m]\psi(x) \to\ & [i\gamma^{\mu}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_{\nu} - m]\,S(\Lambda)\,\psi(\Lambda^{-1}x) \\ &= S(\Lambda)S^{-1}(\Lambda)\,[i\gamma^{\mu}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_{\nu} - m]\,S(\Lambda)\psi(\Lambda^{-1}x) \\ &= S(\Lambda)\,[i\Lambda^{\mu}{}_{\rho}\gamma^{\rho}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_{\nu} - m]\,\psi(\Lambda^{-1}x) \\ &= S(\Lambda)\,[i\gamma^{\nu}\partial_{\nu} - m]\,\psi(\Lambda^{-1}x) = 0 \end{aligned}$$

c) Under an infinitesimal Lorentz transformation

$$\begin{aligned} \bar{\psi}(x) = \psi^{\dagger}(x)\gamma^0 \to \bar{\psi}'(x) &= \psi^{\dagger}(\Lambda^{-1}x)\left(1 + \frac{i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})^{\dagger}\right)\gamma^0 \\ &= \psi^{\dagger}(\Lambda^{-1}x)\gamma^0\left(1 + \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) \end{aligned} \qquad \left(\begin{aligned} \because\ & \gamma^{0\dagger} = \gamma^0 \\ & \gamma^{i\dagger} = -\gamma^i \\ & \{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu} \end{aligned}\right)$$

$$\Rightarrow \bar{\psi}'(x) = \bar{\psi}(\Lambda^{-1}x)\,S^{-1}(\Lambda)$$

- $S(x) \to S'(x) = \bar{\psi}(\Lambda^{-1}x)\underbrace{S^{-1}(\Lambda)S(\Lambda)}_{=1}\psi(\Lambda^{-1}x) = S(\Lambda^{-1}x)$: scalar
- $J^{\mu}(x) \to J'^{\mu}(x) = \bar{\psi}(\Lambda^{-1}x)\underbrace{S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda)}_{=\Lambda^{\mu}{}_{\nu}\gamma^{\nu}}\psi(\Lambda^{-1}x) = \Lambda^{\mu}{}_{\nu}J^{\nu}(\Lambda^{-1}x)$: vector

d) Using the Dirac eq.

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \;\rightarrow\; (i\gamma^0\partial_0 - i\vec{\gamma}\cdot\vec{\partial} - m)\psi(x) = 0,$$

$$\begin{cases} \cdot\ \gamma^0(i\vec{\gamma}\cdot\vec{\partial} + m)\, u(p)\, e^{-ipx} = \gamma^0 \cdot i\gamma^0\partial_0\, u(p)\, e^{-ipx} = p^0 u(p)\, e^{-ipx} \\ \cdot\ \gamma^0(i\vec{\gamma}\cdot\vec{\partial} + m)\, v(p)\, e^{ipx} = -p^0 v(p)\, e^{ipx} \end{cases}$$

e) The Hamiltonian associated with $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$:

$$H = \int d^3x \left(\Pi(0,\vec{x})\, \dot{\psi}(0,\vec{x}) - \mathcal{L}|_{x=(0,\vec{x})} \right)$$

$$= \int d^3x \left(i\psi^\dagger\partial_0\psi - \psi^\dagger\gamma^0(i\gamma^0\partial_0 - i\vec{\gamma}\cdot\vec{\partial} - m)\psi \right)$$

$$= \int d^3x\, \bar{\psi}(i\vec{\gamma}\cdot\vec{\partial} + m)\psi$$

$$= \int d^3x\, \psi^\dagger\gamma^0(i\vec{\gamma}\cdot\vec{\partial} + m)\psi$$

Inserting $\psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s \left(e^{-ipx} u^s(p)\, \alpha^s(p) + e^{ipx} v^s(p)\, \beta^{s*}(p) \right)$,

$$H = \int d^3x \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s \left(e^{ipx} u^{s\dagger}(p)\, \alpha^{s*}(p) + e^{-ipx} v^{s\dagger}(p)\, \beta^s(p) \right)$$

$$\times \gamma^0(i\vec{\gamma}\cdot\vec{\partial} + m) \int \frac{d^3p'}{(2\pi)^3 2E_{p'}} \sum_{s'} \left(e^{-ip'x} u^{s'}(p')\, \alpha^{s'}(p') + e^{ip'x} v^{s'}(p')\, \beta^{s'*}(p') \right)$$

$$= \int d^3x \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s (\quad " \quad)$$

$$\times \int \frac{d^3p'}{(2\pi)^3 2E_{p'}} E_{p'} \sum_{s'} \left(e^{-ip'x} u^{s'}(p')\, \alpha^{s'}(p') - e^{ip'x} v^{s'}(p')\, \beta^{s'*}(p') \right)$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^3 2E_p} \frac{1}{2} \sum_{s,s'} \Big[e^{i(E_p - E_{p'})t} u^{s\dagger}(p)\, u^{s'}(p')\, \alpha^{s*}(p)\, \alpha^{s'}(p')\, \delta(\vec{p} - \vec{p}')$$

$$- e^{-i(E_p - E_{p'})t} v^{s\dagger}(p)\, v^{s'}(p')\, \beta^s(p)\, \beta^{s'*}(p')\, \delta(\vec{p} - \vec{p}')$$

$$- e^{i(E_p + E_{p'})t} u^{s\dagger}(p)\, v^{s'}(p')\, \alpha^{s*}(p)\, \beta^{s'*}(p')\, \delta(\vec{p} + \vec{p}')$$

$$+ e^{-i(E_p + E_{p'})t} v^{s\dagger}(p)\, u^{s'}(p')\, \beta^s(p)\, \alpha^{s'}(p')\, \delta(\vec{p} + \vec{p}') \Big]$$

$$\left(\begin{array}{l} \because\ u^s(p)\, u^{s'}(p) = 2E_p\, \delta^{ss'} \\ v^s(p)\, v^{s'}(p) = 2E_p\, \delta^{ss'} \\ u^{s\dagger}(\vec{p})\, v^{s'}(-\vec{p}) = v^{s\dagger}(\vec{p})\, u^{s'}(-\vec{p}) = 0 \end{array} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_s \left(\alpha^{s*}(\vec{p})\, \alpha^s(\vec{p}) - \beta^s(\vec{p})\, \beta^{s*}(\vec{p}) \right)$$

negative energy ...

3-3 Spin and SU(2)

a) Under an infinitesimal rotation $\vec{x} \to \vec{x}'$ by an angle $\delta\theta$ around an axis in normalized direction $\vec{n}$

$$\vec{x}' = \vec{x} + \vec{n} \times \vec{x}\,\delta\theta = \vec{x} + \vec{n} \times \vec{x}\,\delta\theta ,$$

A wavefunction $\psi(\vec{x})$ transforms as

$$\begin{aligned}
\psi'(\vec{x}') = \psi(\vec{x}) &= \psi(\vec{x}' - \vec{n} \times \vec{x}'\,\delta\theta) \\
&\approx (1 - \delta\theta\, \vec{n} \times \vec{x}' \cdot \vec{\partial}')\,\psi(\vec{x}') \\
&= (1 - \delta\theta\, \vec{n} \cdot \underbrace{(\vec{x}' \times \vec{\partial}')}_{= i\vec{L}})\,\psi(\vec{x}') \\
&= (1 - i\,\delta\theta\, \vec{n} \cdot \vec{L})\,\psi(\vec{x}')
\end{aligned}$$

$$\Longrightarrow \psi'(\vec{x}') = e^{-i\theta \vec{n}\cdot\vec{L}}\,\psi(\vec{x}')$$

b) Note that $SU(2)$ and $SO(3)$ are locally isomorphic, which have the same Lie algebra.

c) Under a spatial rotation, a wavefunction of a spin-$\frac{1}{2}$ particle transforms as

$$\begin{aligned}
\chi'(\vec{x}', s) &= e^{-i\theta \vec{n}\cdot\vec{J}}\,\chi(\vec{x}', s) \qquad (\vec{J} = \vec{L} + \vec{S}) \\
&= e^{-i\theta \vec{n}\cdot\vec{L}}\, e^{-i\theta \vec{n}\cdot\vec{S}}\,\chi(\vec{x}', s) \\
&= e^{-\frac{i}{2}\theta \vec{n}\cdot\vec{\sigma}}\,\chi(R(\theta)\vec{x}', s) \qquad \text{since } \vec{S} = \tfrac{1}{2}\vec{\sigma} \text{ spin-}\tfrac{1}{2} \text{ operator}
\end{aligned}$$

d) A Lorentz transformation of the rotation around an axis $\vec{n}$:

$$\Lambda = \exp(-\tfrac{i}{4}\,\omega_{ij} M^{ij}) \qquad \text{with } \omega_{ij} = \theta\, n_k\, \varepsilon_{ijk}$$

The spinor transforms by

$$S(\Lambda) = \exp(-\tfrac{i}{4}\,\omega_{ij}\,\sigma^{ij}) = \exp(-\tfrac{i}{4}\,\theta\, n_k\, \varepsilon_{ijk}\,\sigma^{ij}) \qquad \left(\text{※ } \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \right)$$

Using the spin operator Σ defined as

$$\tfrac{1}{2}\epsilon_{ijk}\sigma^{ij} = \tfrac{1}{2}\epsilon_{ijk}\tfrac{i}{2}[\gamma^i, \gamma^j] = \tfrac{i}{2}\epsilon_{ijk}\gamma^i\gamma^j = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \equiv \Sigma_k ,$$

$$S(\Lambda) = \exp(-\tfrac{i}{2}\theta\, \vec{n}\cdot\vec{\Sigma}) = \begin{pmatrix} e^{-\frac{i}{2}\theta \vec{n}\cdot\vec{\sigma}} & 0 \\ 0 & e^{-\frac{i}{2}\theta \vec{n}\cdot\vec{\sigma}} \end{pmatrix}$$

$$\Longrightarrow S(\Lambda)\, u^s(p) = \sqrt{E+m} \begin{pmatrix} e^{-\frac{i}{2}\theta \vec{n}\cdot\vec{\sigma}}\,\chi_s \\ e^{-\frac{i}{2}\theta \vec{n}\cdot\vec{\sigma}}\,\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, e^{\frac{i}{2}\theta \vec{n}\cdot\vec{\sigma}}\, e^{-\frac{i}{2}\theta \vec{n}\cdot\vec{\sigma}}\,\chi_s \end{pmatrix}$$

$\Longrightarrow$ $\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ transform as spin-$\frac{1}{2}$ objects.

3-4 Noether's theorem

a) $$\frac{\partial}{\partial t} Q(t) = \frac{\partial}{\partial t}\int d^3x\, J^0(t,\vec{x}) = \int d^3x\, \partial_0 J^0 \underset{\partial_\mu J^\mu = 0}{=} -\int d^3x\, \partial_i J^i \underset{\text{surface integral}}{=} 0$$

b) The Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

is invariant under $\psi(x) \to e^{i\alpha}\psi(x)$, $\bar{\psi}(x) \to \bar{\psi}(x)e^{-i\alpha}$.

By promoting $\alpha \to \alpha(x)$,

$$\delta_\alpha S = \int d^4x\, \bar{\psi} i\gamma^\mu\psi \cdot i\partial_\mu\alpha(x) \quad \Rightarrow \quad J^\mu = -\bar{\psi}\gamma^\mu\psi$$

The conserved charge is

$$\begin{aligned}
Q &= \int d^3x\, J^0 = -\int d^3x\, \bar{\psi}\gamma^0\psi = -\int d^3x\, \psi^\dagger\psi \\
&= -\int d^3x \int \frac{d^3p}{(2\pi)^3 2E_p}\sum_s\left(a_p^{s\dagger}u^{s\dagger}(p)\,e^{ipx} + b_p^s v^{s\dagger}(p)\,e^{-ipx}\right) \\
&\quad \times \int \frac{d^3p'}{(2\pi)^3 2E_{p'}}\sum_{s'}\left(a_{p'}^{s'}u^{s'}(p')\,e^{-ip'x} + b_{p'}^{s'\dagger}v^{s'}(p')\,e^{ip'x}\right) \\
&= -\int \frac{d^3p}{(2\pi)^3 2E_p}\frac{d^3p'}{2E_{p'}}\sum_{s,s'}\Big[\, a_p^{s\dagger}a_{p'}^{s'}\,u^{s\dagger}(p)\,u^{s'}(p')\,e^{i(E_p-E_{p'})t}\,\delta(\vec{p}-\vec{p}') \\
&\qquad + b_p^s b_{p'}^{s'\dagger}\,v^{s\dagger}(p)\,v^{s'}(p')\,e^{-i(E_p-E_{p'})t}\,\delta(\vec{p}-\vec{p}') \\
&\qquad + a_p^{s\dagger}b_{p'}^{s'\dagger}\,u^{s\dagger}(p)\,v^{s'}(p')\,e^{i(E_p+E_{p'})t}\,\delta(\vec{p}+\vec{p}') \\
&\qquad + b_p^s a_{p'}^{s'}\,v^{s\dagger}(p)\,u^{s'}(p')\,e^{-i(E_p+E_{p'})t}\,\delta(\vec{p}+\vec{p}')\Big] \\
&= -\int \frac{d^3p}{(2\pi)^3 2E_p}\sum_s\left(a_p^{s\dagger}a_p^s + b_p^s b_p^{s\dagger}\right) \\
&= -\int \frac{d^3p}{(2\pi)^3 2E_p}\sum_s\left(a_p^{s\dagger}a_p^s - b_p^{s\dagger}b_p^s\right)
\end{aligned}$$

(※ $u^{s\dagger}(p)\,u^{s'}(p) = 2E_p\,\delta^{ss'}$
$v^{s\dagger}(p)\,v^{s'}(p) = 2E_p\,\delta^{ss'}$
$u^{s\dagger}(\vec{p})\,v^{s'}(-\vec{p}) = v^{s\dagger}(\vec{p})\,u^{s'}(-\vec{p}) = 0$)

$$\Rightarrow \begin{cases} Q\,a_p^{s\dagger}|0\rangle = -\,a_p^{s\dagger}|0\rangle \\ Q\,b_p^{s\dagger}|0\rangle = +\,b_p^{s\dagger}|0\rangle \end{cases}$$

a particle and its anti-particle have opposite charge.

c) $$Q\psi(x) = -\int d^3y\, \psi^\dagger(y)\psi(y)\psi(x) = \int d^3y\, \underbrace{\psi^\dagger(y)\psi(x)}_{=\delta(\vec{x}-\vec{y}) - \psi(x)\psi^\dagger(y)}\psi(y) = \psi(x)(1+Q)$$

$$\Rightarrow [Q,\psi] = \psi$$

$$\begin{aligned}
e^{i\alpha Q}\psi e^{-i\alpha Q} &= (1 + i\alpha Q + \cdots)\psi(1 - i\alpha Q + \cdots) \\
&= \psi + i\alpha\underbrace{[Q,\psi]}_{=\psi} + \cdots \\
&= e^{i\alpha}\psi
\end{aligned}$$

$\Rightarrow$ The conserved charge is a generator of the original infinitesimal trans.