$\boxed{2-1}$ Wick's theorem

a) e.g. $A = a(p_1)\, a^\dagger(p_2)\, a(p_3)$

The normal ordered product is defined as

$$:A: \;=\; a^\dagger(p_2)\, a(p_1)\, a(p_3)$$

Since $a(p)|0\rangle = 0$, $\langle 0| a^\dagger(p) = 0 \quad \Rightarrow \quad \langle :A: \rangle = 0$

b) $\phi(x)$ is decomposed into positive- and negative-frequency parts:

$$\phi(x) = \phi^+(x) + \phi^-(x),$$

where $\phi^+(x) = \int \frac{d^3p}{(2\pi)^3 2E_p}\, a(p)\, e^{-ipx}$, $\quad \phi^-(x) = \int \frac{d^3p}{(2\pi)^3 2E_p}\, a^\dagger(p)\, e^{ipx}$.

$$\phi(x)\phi(y) = \phi^+(x)\phi^+(y) + \underbrace{\phi^+(x)\phi^-(y)}_{= [\phi^+(x),\, \phi^-(y)] + \phi^-(y)\phi^+(x)} + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y)$$

$$= \;:\phi(x)\phi(y):\; + [\phi^+(x), \phi^-(y)]$$

$$T(\phi(x)\phi(y)) = \;:\phi(x)\phi(y):\; + \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0 \end{cases}$$

Recall the Feynman propagator in Ex. 1-4:

$$i\Delta_F(x-y) = \langle 0| T(\phi(x)\phi(y))|0\rangle$$
$$= \theta(x^0-y^0)\, \underline{\langle 0| \phi(x)\phi(y)|0\rangle} + \theta(y^0-x^0)\, \langle 0| \phi(y)\phi(x)|0\rangle$$

$$\hookrightarrow \langle 0| a(p_x)\, a^\dagger(p_y)|0\rangle$$
$$= \langle 0| [a(p_x), a^\dagger(p_y)]\, |0\rangle$$

Therefore,

$$T(\phi(x)\phi(y)) = \;:\phi(x)\phi(y):\; + i\Delta_F(x-y)$$

c) First, let us write down Wick's theorem for three fields :

$$T(\phi(x_1)\phi(x_2)\phi(x_3))$$

For $x_1^0 > x_2^0 > x_3^0$ and $x_2^0 > x_1^0 > x_3^0$,

$$T(\phi_1\phi_2\phi_3) = \{ :\phi_1\phi_2: + i\Delta_F(x_1-x_2)\} \underbrace{\phi_3}_{\phi_3^+ + \phi_3^-}$$

$$
\begin{aligned}
&:\phi_1\phi_2:\phi_3^- \\
&= \{ \phi_1^+\phi_2^+ + \phi_2^-\phi_1^+ + \phi_1^-\phi_2^+ + \phi_1^-\phi_2^- \}\phi_3^- \\
&= \phi_1^+\phi_3^-\phi_2^+ + \phi_1^+[\phi_2^+,\phi_3^-] \rightarrow \phi_3^-\phi_1^+\phi_2^+ + [\phi_1^+,\phi_3^-]\phi_2^+ + \phi_1^+[\phi_2^+,\phi_3^-] \\
&\quad + \phi_2^-\phi_3^-\phi_1^+ + \phi_2^-[\phi_1^+,\phi_3^-] \\
&\quad + \phi_1^-\phi_3^-\phi_2^+ + \phi_1^-[\phi_2^+,\phi_3^-] \\
&\quad + \phi_1^-\phi_2^-\phi_3^- \\
&= :\phi_1\phi_2\phi_3: + \phi_1[\phi_2^+,\phi_3^-] + \phi_2[\phi_1^+,\phi_3^-]
\end{aligned}
$$

$$\Rightarrow T(\phi_1\phi_2\phi_3) = :\phi_1\phi_2\phi_3: + i\Delta_F(x_1-x_2)\phi_3 + [\phi_2^+,\phi_3^-]\phi_1 + [\phi_1^+,\phi_3^-]\phi_2$$

For $x_2^0 > x_3^0 > x_1^0$ and $x_3^0 > x_2^0 > x_1^0$,

$$T(\phi_1\phi_2\phi_3) = :\phi_1\phi_2\phi_3: + i\Delta_F(x_2-x_3)\phi_1 + [\phi_3^+,\phi_1^-]\phi_2 + [\phi_2^+,\phi_1^-]\phi_3$$

For $x_3^0 > x_1^0 > x_2^0$ and $x_1^0 > x_3^0 > x_2^0$,

$$T(\phi_1\phi_2\phi_3) = :\phi_1\phi_2\phi_3: + i\Delta_F(x_3-x_1)\phi_2 + [\phi_1^+,\phi_2^-]\phi_3 + [\phi_3^+,\phi_2^-]\phi_1$$

Therefore,

$$
\begin{aligned}
T(\phi_1\phi_2\phi_3) &= :\phi_1\phi_2\phi_3 + i\Delta_F^{12}\phi_3 + i\Delta_F^{23}\phi_1 + i\Delta_F^{31}\phi_2: \\
&= :\phi_1\phi_2\phi_3 + \wick{\c\phi_1\c\phi_2}\phi_3 + \phi_1\wick{\c\phi_2\c\phi_3} + \wick{\c\phi_1\phi_2\c\phi_3}:
\end{aligned}
$$

Wick's theorem for four fields :

$$
\begin{aligned}
T(\phi_1\phi_2\phi_3\phi_4) = :&\phi_1\phi_2\phi_3\phi_4 + \wick{\c\phi_1\c\phi_2}\phi_3\phi_4 + \wick{\c\phi_1\phi_2\c\phi_3}\phi_4 + \wick{\c\phi_1\phi_2\phi_3\c\phi_4} \\
&+ \phi_1\wick{\c\phi_2\c\phi_3}\phi_4 + \phi_1\wick{\c\phi_2\phi_3\c\phi_4} + \phi_1\phi_2\wick{\c\phi_3\c\phi_4} + \wick{\c\phi_1\c\phi_2}\wick{\c\phi_3\c\phi_4} \\
&+ \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}:
\end{aligned}
$$

Generally,

$$T(\phi_1\phi_2\cdots\phi_n) = :\phi_1\phi_2\cdots\phi_n + \text{all possible contractions}:$$

$\boxed{2-2}$ Interaction picture

a) Schrödinger eq. :

$$i\frac{d}{dt}|\psi_s(t)\rangle = H_0|\psi_s(t)\rangle \quad\Rightarrow\quad |\psi_s(t)\rangle = e^{-iH_0(t-t_0)}|\psi_s(t_0)\rangle$$

The relation between Schrödinger picture and Heisenberg picture :

$$\begin{aligned} A(t) &= \langle\psi_s(t)|A_s|\psi_s(t)\rangle \\ &= \langle\psi_s(t_0)|e^{iH_0(t-t_0)}A_s e^{-iH_0(t-t_0)}|\psi_s(t_0)\rangle \\ &= \langle\psi_{H,t_0}|A_H(t)|\psi_{H,t_0}\rangle \end{aligned}$$

$$\Longrightarrow\quad |\psi_s(t_0)\rangle = |\psi_{H,t_0}\rangle\,,\quad A_H(t) = e^{iH_0(t-t_0)}A_s e^{-iH_0(t-t_0)}$$

Heisenberg eq. :

$$\begin{aligned} \frac{d}{dt}A_H(t) &= iH_0 e^{iH_0(t-t_0)}A_s e^{-iH_0(t-t_0)} - e^{iH_0(t-t_0)}A_s e^{-iH_0(t-t_0)}iH_0 \\ &= i[H_0, A_H(t)] \end{aligned}$$

$$\Longrightarrow\quad i\frac{d}{dt}A_H(t) = [A_H(t), H_0]$$

c) Schrödinger eq. $(H = H_0 + H_I)$:

$$i\frac{d}{dt}|\psi_s(t)\rangle = H|\psi_s(t)\rangle$$

The relation between Schrödinger picture and interaction picture :

$$\begin{aligned} A(t) &= \langle\psi_s(t)|A_s|\psi_s(t)\rangle \\ &= \langle\psi_I(t)|e^{iH_0t}A_s e^{-iH_0t}|\psi_I(t)\rangle \\ &= \langle\psi_I(t)|A_I(t)|\psi_I(t)\rangle \end{aligned}$$

$$\Longrightarrow\quad |\psi_s(t)\rangle = e^{-iH_0t}|\psi_I(t)\rangle\,,\quad A_I(t) = e^{iH_0t}A_s e^{-iH_0t}$$

$$\underbrace{i\frac{d}{dt}\left[e^{-iH_0t}|\psi_I(t)\rangle\right]}_{=\;\cancel{H_0 e^{-iH_0t}|\psi_I(t)\rangle} + e^{-iH_0t}\,i\frac{d}{dt}|\psi_I(t)\rangle} = \underbrace{H}_{=\;(\cancel{H_0}+H_I)}e^{-iH_0t}|\psi_I(t)\rangle$$

$$\Longrightarrow\quad i\frac{d}{dt}|\psi_I(t)\rangle = \underbrace{e^{iH_0t}H_I e^{-iH_0t}}_{=\;H_I(t)}|\psi_I(t)\rangle$$

2-3 Time evolution and S-matrix

a) Integration of eq. (17) :

$$i\left[\,|\Phi(t)\rangle - |\Phi(-\infty)\rangle\,\right] = \int_{-\infty}^{t} dt'\, \bar{H}_I(t')\,|\Phi(t')\rangle$$

$$\Rightarrow\ |\Phi(t)\rangle = |\Phi_i\rangle + \frac{1}{i}\int_{-\infty}^{t} dt'\, \bar{H}_I(t')\,|\Phi(t')\rangle$$

$$\xrightarrow[|\Phi(t)\rangle = U(t)|\Phi_i\rangle]{} \quad U(t)\,|\Phi_i\rangle = |\Phi_i\rangle + \frac{1}{i}\int_{-\infty}^{t} dt'\, \bar{H}_I(t')\, U(t')\,|\Phi_i\rangle$$

$$\Rightarrow\ U(t) = 1 + \frac{1}{i}\int_{-\infty}^{t} dt'\, \bar{H}_I(t')\, U(t')$$

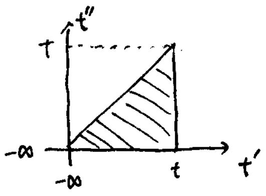
From eq. (17) , $\quad i\frac{d}{dt}U(t) = \bar{H}_I(t)\,U(t)$

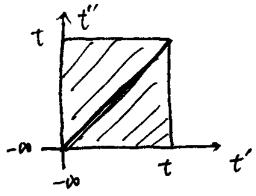
b) Eq. (19) for $U(t)$ can be solved perturbatively as the series :

$$\begin{aligned} U(t) &= 1 + \frac{1}{i}\int_{-\infty}^{t} dt'\, \bar{H}_I(t')\, U(t') \\ &= 1 + \frac{1}{i}\int_{-\infty}^{t} dt'\, \bar{H}_I(t')\left[1 + \frac{1}{i}\int_{-\infty}^{t'} dt''\, \bar{H}_I(t'')\, U(t'')\right] \\ &= \underbrace{1}_{\text{no interaction}} + \underbrace{\frac{1}{i}\int_{-\infty}^{t} dt'\, H_I(t')}_{\text{1st-order}} + \underbrace{\frac{1}{i^2}\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt''\, \bar{H}_I(t')\,\bar{H}_I(t'')}_{\text{2nd-order}} + \cdots \end{aligned}$$

c) We change the integration range as

$$\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt''\, \bar{H}_I(t')\,\bar{H}_I(t'') = \frac{1}{2}\int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt''\; T\left\{\bar{H}_I(t')\,\bar{H}_I(t'')\right\}$$





Therefore,

$$\begin{aligned} U(t) &= 1 + \frac{1}{i}\int_{-\infty}^{t} dt'\, \bar{H}_I(t') + \frac{1}{2!}\frac{1}{i^2}\int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt''\; T\left(\bar{H}_I(t')\,\bar{H}_I(t'')\right) \\ &\quad + \frac{1}{3!}\frac{1}{i^3}\int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' \int_{-\infty}^{t} dt'''\; T\left(\bar{H}_I(t')\,\bar{H}_I(t'')\,\bar{H}_I(t''')\right) + \cdots \\ &= T\left\{\exp\left[\frac{1}{i}\int_{-\infty}^{t} dt'\, \bar{H}_I(t')\right]\right\} \end{aligned}$$

d) The S-matrix is defined as

$$S = \lim_{t\to\infty} U(t) = T\left\{\exp\left[-i\int_{-\infty}^{\infty} dt\, \bar{H}_I(t)\right]\right\}$$

that describes scattering from $t_i = -\infty$ to $t_f = +\infty$.