$\boxed{11-1}$ Lepton mixing in models with Majorana masses

a) The Majorana spinors:

$$\left(※\ \psi_L^c \equiv (\psi_L)^c = (\psi^c)_R \qquad \psi_R^c \equiv (\psi_R)^c = (\psi^c)_L \right)$$

$$\chi = \nu_L + (\nu_L)^c, \qquad \omega = \nu_R + (\nu_R)^c$$

$$\left(※\ \psi^c = C\bar{\psi}^T = C\gamma^0\psi^* \qquad \bar{\psi^c} = (\psi^c)^\dagger\gamma^0 = -\psi^T C^\dagger \qquad \to \bar{\psi^c}\psi^c = -\psi^T\gamma^0\psi^* \underset{T}{=} \bar{\psi}\psi \right)$$

$$\begin{cases} \bar{\omega}\omega = \bar{\nu}_R \nu_R^c + \bar{\nu}_R^c \nu_R \\ \bar{\chi}\omega + \bar{\omega}\chi = \bar{\nu}_L\nu_R + \bar{\nu}_L^c\nu_R^c + \bar{\nu}_R\nu_L + \bar{\nu}_R^c\nu_L^c = 2(\bar{\nu}_L\nu_R + \bar{\nu}_R\nu_L) \end{cases}$$

The mass terms in terms of the Majoran spinors:

$$\begin{aligned}
\mathcal{L}_{mass} &= -M_M(\bar{\nu}_R^c\nu_R + \bar{\nu}_R\nu_R^c) - M_D(\bar{\nu}_R\nu_L + \bar{\nu}_L\nu_R) \\
&= -M_M\bar{\omega}\omega - M_D\frac{1}{2}(\bar{\chi}\omega + \bar{\omega}\chi) \\
&= -(\bar{\chi}\ \bar{\omega})\begin{pmatrix} 0 & \frac{1}{2}M_D \\ \frac{1}{2}M_D & M_M \end{pmatrix}\begin{pmatrix}\chi \\ \omega\end{pmatrix} \\
&= -(\bar{\chi}\ \bar{\omega})\ \underbrace{V}_{V^{-1}}\underbrace{V^T \mathcal{M}\, V}_{\tilde{\mathcal{M}}}\underbrace{V^T\begin{pmatrix}\chi \\ \omega\end{pmatrix}}_{\text{mass eigen states}} = -(\bar{\eta}_1\ \bar{\eta}_2)\ \hat{\mathcal{M}}\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix}
\end{aligned}$$

(※ A symmetric matrix can be diagonalized by a orthogonal matrix.)

The eigenvalues of $\mathcal{M}$:

$$M_{1,2} = \frac{1}{2}\left(M_M \pm \sqrt{M_M^2 + M_D^2}\right) \qquad (\epsilon \ll 1) \qquad (M_1 \sim M_2 \sim M_M)$$

$$\xrightarrow[M_D \ll M_M]{} \frac{1}{2}M_M\left[1 \pm \left(1 + \frac{1}{2}\Big(\underbrace{\frac{M_D}{M_M}}_{\epsilon}\Big)^2\right)\right] = \begin{cases} M_M + \frac{1}{4}\epsilon^2 M_M \equiv M_2 \\ -\frac{1}{4}\epsilon^2 M_M \equiv -\epsilon^2 M_1 \end{cases}$$

$$\Rightarrow \hat{\mathcal{M}} = \begin{pmatrix} -\epsilon^2 M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix}$$

※ Check:

$$\begin{aligned}
V^T\mathcal{M}V &\hat{=} \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}\begin{pmatrix} 0 & M_D \\ M_D & M_M \end{pmatrix}\begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}\begin{pmatrix} -\epsilon M_D & M_D \\ M_D - \epsilon M_M & \epsilon M_D + M_M \end{pmatrix} \\
&= \begin{pmatrix} -\epsilon M_D - \epsilon M_D + \epsilon^2 M_M & \cancel{M_D} - \epsilon^2 M_D - \cancel{\epsilon M_M} \\ -\epsilon^2 M_D + \cancel{M_D} - \epsilon M_M & \epsilon M_D + \epsilon M_D + M_M \end{pmatrix} \\
&= \begin{pmatrix} -\epsilon^2 M_M & O(\epsilon^3) \\ O(\epsilon^3) & M_M + 2\epsilon^2 M_M \end{pmatrix} \sim \begin{pmatrix} -\epsilon^2 M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \tilde{\mathcal{M}}
\end{aligned}$$

(※ We can avoid the negative mass by a chiral trans. $\gamma_5\chi$ and $\gamma_5\omega$.)

The mass eigenstates:

$$\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix} = V^T\begin{pmatrix}\chi \\ \omega\end{pmatrix} = \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}\begin{pmatrix}\chi \\ \omega\end{pmatrix} = \begin{pmatrix}\chi - \epsilon\omega \\ \epsilon\chi + \omega\end{pmatrix} \sim \begin{pmatrix}\chi \\ \omega\end{pmatrix} \begin{matrix} \leftarrow \text{light state} \\ \leftarrow \text{heavy state} \end{matrix}$$

The order of magnitude of M_M:

$$M_{light} \sim \epsilon^2 M_M = \frac{M_D^2}{M_M}$$

$$\Rightarrow M_M = \frac{M_D^2}{M_{light}} \lesssim \frac{(\mathrm{TeV})^2}{\mathrm{eV}} = 10^{15}\ \mathrm{GeV}$$

b) Let us consider 3 families of ν_L and n families of sterile ($SU(2)\times U(1)$ singlet) ν_R:

$$\vec{\chi} = (\nu_L' + (\nu_L')^c)_i \qquad i = 1, 2, 3$$

$$\vec{\omega} = (\nu_R' + (\nu_R')^c)_\alpha \qquad \alpha = 1, 2, \cdots, n$$

The mass terms are

$$\mathcal{L}_{mass} = -(\vec{\overline{\chi}}^T \ \vec{\overline{\omega}}^T) \begin{pmatrix} 0 & \frac{1}{2}M_D \\ \frac{1}{2}M_D^T & M_M \end{pmatrix} \begin{pmatrix} \vec{\chi} \\ \vec{\omega} \end{pmatrix} \qquad \text{※ } M_D : 3\times n \text{ matrix}, \quad M_M : n\times n \text{ matrix}$$

Note that $M_M = M_M^T$ since $\overline{\nu}_{R\alpha}^c \nu_{R\beta} = \overline{\nu}_{R\beta}^c \nu_{R\alpha}$ due to $\omega_\alpha = \omega_\alpha^c$.

Similar to a), the mass terms in terms of the mass eigenstates are

$$\mathcal{L}_{mass} = -(\vec{\overline{\chi}}^T \ \vec{\overline{\omega}}^T) \underbrace{V V^\dagger \mathcal{M} V}_{\tilde{\mathcal{M}}} \underbrace{V^\dagger \begin{pmatrix} \vec{\chi} \\ \vec{\omega} \end{pmatrix}}_{\text{mass eigen states}} = -(\vec{\overline{\eta}}_1^T \ \vec{\overline{\eta}}_2^T)\, \hat{\mathcal{M}} \begin{pmatrix} \vec{\eta}_1 \\ \vec{\eta}_2 \end{pmatrix}$$

where

$$\hat{\mathcal{M}} = \begin{pmatrix} \epsilon^2 M_1 & \\ & M_2 \end{pmatrix}, \quad V = \begin{pmatrix} U_{11} & \epsilon U_{12} \\ \epsilon U_{21} & U_{22} \end{pmatrix}. \qquad \left(\text{※ } \epsilon = \frac{M_D}{M_M} \ll 1\right)$$

$$\begin{cases} \vec{\eta}_1 = (\nu_L + (\nu_L)^c)_i & \cdots\text{ light eigenstates} \\ \vec{\eta}_2 = (\nu_R + (\nu_R)^c)_\alpha & \cdots\text{ heavy eigenstates} \end{cases}$$

The electroweak eigenstates in terms of the mass eigenstates:

$$\begin{aligned} \vec{\nu}_L' &= \frac{1}{2}(1-\gamma_5)\vec{\chi} \\ &= \frac{1}{2}(1-\gamma_5)[U_{11}\vec{\eta}_1 + \epsilon U_{12}\vec{\eta}_2] \\ &= U_{11}\vec{\nu}_L + \epsilon U_{12}\vec{\nu}_R^c \end{aligned} \qquad \text{※ } \begin{pmatrix} \vec{\chi} \\ \vec{\omega} \end{pmatrix} = V \begin{pmatrix} \vec{\eta}_1 \\ \vec{\eta}_2 \end{pmatrix}$$

The charged current in terms of the mass eigenstates:

$$\begin{aligned} J_{cc}^\mu &= \vec{\overline{\nu}}_L'^T \gamma^\mu (1-\gamma_5) \vec{l}' \\ &= [\vec{\overline{\nu}}_L^T U_{11}^\dagger + \vec{\overline{\nu}}_R^{cT} \epsilon U_{12}^\dagger] \gamma^\mu (1-\gamma_5) U_e \vec{l} \qquad \text{※ } \vec{l} = U_e^\dagger \vec{l}' \\ &= \vec{\overline{\nu}}_L^T \gamma^\mu (1-\gamma_5) \underbrace{U_{11}^\dagger U_e}_{V_{PMNS}\ (3\times 3)} \vec{l} + \epsilon\, \vec{\overline{\nu}}_R^{cT} \gamma^\mu (1-\gamma_5) \underbrace{U_{12}^\dagger U_e}_{\tilde{V}\ (n\times 3)} \vec{l} \end{aligned}$$

↑ Pontecorvo-牧-中川-坂田

$$\sim \vec{\overline{\nu}}_L^T \gamma^\mu (1-\gamma_5) V_{PMNS} \vec{l}$$

The d.o.f of the mixing matrix for n families :

- a general unitary $n\times n$ matrix $U(n) \to n^2$
- mixing angles $O(n) \to {}_nC_2 = \frac{1}{2}n(n-1)$
- phases $\to n^2 - \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$

We can remove these phases by making phase rotations of fields ;

- physical phases $\to \frac{1}{2}n(n+1) - \begin{cases} n \leftarrow \vec{l}_L & \text{for Majorana } \nu \\ (2n-1) & \text{for Dirac } \nu \end{cases}$

$\vec{l}_L, \vec{\nu}_L$ overall phase

Note that Majorana mass term forbid phase shifts.

$$\Rightarrow V_{PMNS}\,(n=3) \text{ has } \begin{cases} 3 \text{ mixing angles } + 3 \text{ phases} & \text{for Majorana } \nu \\ \quad '' \quad\quad\quad + 1 \text{ phase} & \text{for Dirac } \nu \end{cases}$$

$3 \text{ phases} = \begin{cases} 1 \text{ CP phase} \\ 2 \text{ Majorana phase} \end{cases}$

$\sim V_{CKM}$

c) The neutral current in terms of the mass eigenstates :

$$\begin{aligned} J^\mu_{NC} &= \overline{\vec{\nu}}{}_L'^{T} \gamma^\mu (1-\gamma_5) \vec{\nu}_L' \\ &= [\overline{\vec{\nu}}{}_L^{T} U_{11}^\dagger + \overline{\vec{\nu}}{}_R^{cT} \epsilon U_{12}^\dagger] \gamma^\mu (1-\gamma_5) [U_{11}\vec{\nu}_L + \epsilon U_{12} \vec{\nu}_R^c] \\ &= \overline{\vec{\nu}}{}_L^{T} \gamma^\mu (1-\gamma_5) \vec{\nu}_L + \epsilon [\overline{\vec{\nu}}{}_L^{T} \gamma^\mu (1-\gamma_5) U_{11}^\dagger U_{12} \vec{\nu}_R^c \\ &\qquad + \overline{\vec{\nu}}{}_R^{cT} \gamma^\mu (1-\gamma_5) U_{12}^\dagger U_{11} \vec{\nu}_L] \\ &\quad + \epsilon^2 \overline{\vec{\nu}}{}_R^{cT} \gamma^\mu (1-\gamma_5) \vec{\nu}_R^c \end{aligned}$$

$\leftarrow \vec{\nu}_L / \vec{\nu}_R^c$ mixing terms

$\boxed{11-2}$ Symmetries of Majorana masses

a) See Ex. 10-3.

b) The d.o.f. of the lepton sector for Majorana ν :

$$\underbrace{(3^2 \times 2)}_{f_e \text{ complex}} + \underbrace{((3^2-3)\times 2)}_{f_\nu\ f^{ij} \text{ is symmetric}} - \underbrace{(2\times 3^2)}_{U_{L_L}, U_{e_R}} = 12$$

$= 6$ (lepton masses) $+ 6$ (PMNS)

11-3 Type II seesaw

a) See, e.g., 'Gravitation' (Misner, Thorne, Wheeler) P86 Ex. 3.12.

b) $$S^{(ij)} L^T_{(i} C^{-1} L_{j)} = \epsilon^{ik}\epsilon^{jl} S_{(kl)} L^T_{(i} C^{-1} L_{j)}$$

※ ϵ^{ij} : SU(2) invariant anti-symmetric tensor

$$\xrightarrow[\text{SU(2) trans.}]{} \epsilon^{ik}\epsilon^{jl} U_k^{\alpha} U_l^{\beta} S_{(\alpha\beta)} U_i^{\rho} L^T_{(\rho} C^{-1} U_j^{\sigma} L_{\sigma)}$$

$$= \epsilon^{ik} U_i^{\rho} U_k^{\alpha} \epsilon^{jl} U_j^{\sigma} U_l^{\beta} S_{(\alpha\beta)} L^T_{(\rho} C^{-1} L_{\sigma)}$$

$$= \epsilon^{\rho\alpha} \underbrace{\det U}_{=1} \cdot \epsilon^{\sigma\beta} \underbrace{\det U}_{=1} \cdot S_{(\alpha\beta)} L^T_{(\rho} C^{-1} L_{\sigma)}$$

$$= S^{(\rho\sigma)} L^T_{(\rho} C^{-1} L_{\sigma)} \qquad \text{SU(2) gauge inv.}$$

The Y charge of S :

$$\underset{Y=-\frac{1}{2}}{\uparrow} L^T_{(i} C^{-1} \underset{Y=-\frac{1}{2}}{\uparrow} L_{j)} S^{(ij)} \quad \Rightarrow \quad Y_S = +1$$

The Majorana mass term :

$$L^T_{(i} C^{-1} L_{j)} S^{(ij)} = \nu_L^T C^{-1} \nu_L S^{11} + e_L^T C^{-1} e_L S^{22} + \nu_L^T C^{-1} e_L S^{12} + e_L^T C^{-1} \nu_L S^{21}$$

$$S = \begin{pmatrix} \phi^0 & \phi^+ \\ \phi^+ & \phi^{++} \end{pmatrix}$$

when $\langle S^{11} \rangle = v \neq 0$,

$$v \cdot \nu_L^T C^{-1} \nu_L = - v \, \overline{\nu_L^c} \nu_L$$

※ The SU(2) triplet Higgs modifies the ρ parameter

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} \sim 1,$$

So the vacuum expectation value v should be small enough to explain the experimental data, compared with the VEU of the doublet Higgs.