$\boxed{10-1}$ Quark interactions

The gauged kinetic terms for quarks :

$$\mathcal{L}_q = \sum_q \bar{q}^i \, i D_\mu \gamma^\mu q^i \qquad q^i = \begin{cases} L^i = \begin{array}{c} i=1 \qquad\quad 2 \qquad\quad 3 \\ \left[\begin{pmatrix} u' \\ d' \end{pmatrix}_L, \begin{pmatrix} c' \\ s' \end{pmatrix}_L, \begin{pmatrix} t' \\ b' \end{pmatrix}_L \right] \end{array} \\ R_+^i = [\, u'_R \,,\ c'_R \,,\ t'_R \,] \\ R_-^i = [\, d'_R \,,\ s'_R \,,\ b'_R \,] \end{cases}$$

The covariant derivative (See Ex. 8-1 d)):

$$D_\mu = \partial_\mu + i \frac{g}{\sqrt{2}} (T^+ W_\mu^+ + T^- W_\mu^-) + i \frac{g}{c_W} (T^3 - s_W^2 Q) Z_\mu + ie Q_q A_\mu$$

- The charged current interactions :

$$\mathcal{L}_{c.c.} = -\frac{g}{\sqrt{2}} \bar{L}^i \gamma^\mu (T^+ W_\mu^+ + T^- W_\mu^-) L^i$$

$$= -\frac{g}{\sqrt{2}} \left\{ (\bar{u}' \ \bar{c}' \ \bar{t}')_L \gamma^\mu \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L W_\mu^+ + h.c. \right\}$$

The EW eigenstates $\rightarrow$ The mass eigenstates

$$\begin{pmatrix} u \\ c \\ t \end{pmatrix}_L = V_+^\dagger \begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_L , \quad \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L = V_-^\dagger \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \qquad (※ \ V_+ V_+^\dagger = V_- V_-^\dagger = 1)$$

$$\rightarrow \mathcal{L}_{c.c.} = -\frac{g}{\sqrt{2}} \left\{ (\bar{u} \ \bar{c} \ \bar{t})_L \gamma^\mu \underbrace{V_+^\dagger V_-}_{\equiv V} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L W_\mu^+ + h.c. \right\} \qquad (V : \text{CKM matrix})$$

$$= -\frac{g}{2\sqrt{2}} \left\{ (\bar{u} \ \bar{c} \ \bar{t}) \gamma^\mu (1 - \gamma_5) V \begin{pmatrix} d \\ s \\ b \end{pmatrix} W_\mu^+ + h.c. \right\}$$

$\Rightarrow$ The CKM matrix leads to flavour-changing charged currents.

- The electro-magnetic current interactions :

$$\mathcal{L}_{EM} = -e \left[\bar{L}^i Q \gamma^\mu L^i + \bar{R}_+^i Q \gamma^\mu R_+^i + \bar{R}_-^i Q \gamma^\mu R_-^i \right] A_\mu$$

$$= -\frac{2}{3} e \, (\bar{u}' \bar{c}' \bar{t}')_L \gamma^\mu \begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_L A_\mu - \frac{2}{3} e \, (\bar{u}' \bar{c}' \bar{t}')_R \gamma^\mu \begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_R A_\mu$$

$$+ \frac{1}{3} e \, (\bar{d}' \bar{s}' \bar{b}')_L \gamma^\mu \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L A_\mu + \frac{1}{3} e \, (\bar{d}' \bar{s}' \bar{b}')_R \gamma^\mu \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_R A_\mu$$

$$※ \ \begin{pmatrix} u \\ c \\ t \end{pmatrix}_R = U_+^\dagger \begin{pmatrix} u \\ c \\ t \end{pmatrix}_R , \quad \begin{pmatrix} d \\ s \\ b \end{pmatrix}_R = U_-^\dagger \begin{pmatrix} d \\ s \\ b \end{pmatrix}_R$$

$$= -\frac{2}{3} e \, (\bar{u} \ \bar{c} \ \bar{t}) \gamma^\mu \begin{pmatrix} u \\ c \\ t \end{pmatrix} A_\mu + \frac{1}{3} e \, (\bar{d} \ \bar{s} \ \bar{b}) \gamma^\mu \begin{pmatrix} d \\ s \\ b \end{pmatrix} A_\mu$$

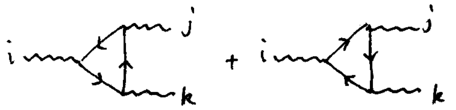
- The neutral EW current interactions :

$$\mathcal{L}_{N.C.} = -\frac{g}{2c_W} (\bar{u} \ \bar{c} \ \bar{t}) \gamma^\mu \frac{1}{2} \left(1 - \gamma_5 - \frac{8}{3} s_W^2\right) \begin{pmatrix} u \\ c \\ t \end{pmatrix} Z_\mu$$

$$- \frac{g}{2c_W} (\bar{d} \ \bar{s} \ \bar{b}) \gamma^\mu \frac{-1}{2} \left(1 - \gamma_5 - \frac{4}{3} s_W^2\right) \begin{pmatrix} d \\ s \\ b \end{pmatrix} Z_\mu$$

$\Rightarrow$ No flavour changing EM and neutral currents.

$\boxed{10\text{-}2}$ No gauge anomalies in the Standard Model



$$\sim \quad A_{ijk} = \sum_R \mathrm{Tr}\left[\{ T_i(R), T_j(R) \} T_k(R) \right]$$

- $SU(2)^3$

$$A_{ijk} = 0 \qquad \because\ T_i = \frac{1}{2}\sigma_i \ , \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad \text{and } \mathrm{tr}\,\sigma_i = 0$$

- $SU(2)^2 - U(1)$

$$A_{ijk} = \sum_R \mathrm{Tr}\left[\{T_i, T_j\} Y \right] = 2\frac{1}{2}\delta_{ij} \sum_{f_L} Y_{f_L} = 0$$

$$\because\ \sum_{f_L} Y_{f_L} = \underset{\substack{\uparrow \\ \text{lepton}}}{-\frac{1}{2}} + \underset{\substack{\uparrow \\ \text{color}}}{3}\ \underset{\nwarrow\ \text{quark}}{\frac{1}{6}} = 0$$

- $SU(2) - U(1)^2$

$$A_{ijk} = 0 \qquad \because\ \mathrm{tr}\,\sigma_i = 0$$

- $U(1)^3$

$$A_{ijk} = \sum_R \mathrm{Tr}[Y^3] = \underset{(\nu,\ell)_L}{2\left(-\frac{1}{2}\right)^3} - \underset{\ell_R}{(-1)^3} + 3\left\{ \underset{(u,d)_L}{2\left(\frac{1}{6}\right)^3} - \underset{u_R}{\left(\frac{2}{3}\right)^3} - \underset{d_R}{\left(-\frac{1}{3}\right)^3} \right\}$$

$$= -\frac{1}{4} + 1 + \left\{ \frac{1}{36} - \frac{8}{9} + \frac{1}{9} \right\}$$

$$= \frac{3}{4} - \frac{3}{4}$$

$$= 0$$

Note that $\left\{ \begin{array}{l} \text{left-handed fermions are weighted with } +1 \\ \text{right-} \quad \text{"} \qquad\qquad \text{"} \qquad \text{"} \quad -1 \end{array} \right.$

$\Longrightarrow$ The EW theory is anomaly free.

The cancellation of anomalies requires that leptons and quarks appear in complete multiplets with the structure of

$$[\ (\nu, \ell)_L,\ \ell_R,\ (u, d)_L,\ u_R,\ d_R\ ]$$

$\Longrightarrow$ SU(5) or SO(10) GUT ?

$\boxed{10\text{-}3}$ Global symmetries of the electro-weak lagrangian

※ $Q_L^i = \begin{pmatrix} u^i \\ d^i \end{pmatrix}_L$ $\quad L_L^i = \begin{pmatrix} \nu^i \\ e^i \end{pmatrix}_L$

The gauged kinetic terms for quarks and leptons :

$$\mathcal{L}_f = \sum_f \bar{\psi}_f^i \, i D_\mu \gamma^\mu \psi_f^i$$
$$= \bar{Q}_L^i \, i \not{D} \, Q_L^i + \bar{u}_R^i \, i \not{D} \, u_R^i + \bar{d}_R^i \, i \not{D} \, d_R^i + \bar{L}_L^i \, i \not{D} \, L_L^i + \bar{e}_R^i \, i \not{D} \, e_R^i$$

where a sum on the index i, which represents the generation, is implied in the lagrangian.

Without the Yukawa couplings, $\mathcal{L}_f$ has (accidental) global symmetries,

$$Q_L^i \to U_{Q_L}^{ij} Q_L^j, \quad u_R^i \to U_{u_R}^{ij} u_R^j, \quad d_R^i \to U_{d_R}^{ij} d_R^j$$
$$L_L^i \to U_{L_L}^{ij} L_L^j, \quad e_R^i \to U_{e_R}^{ij} e_R^j$$

$\Longrightarrow$ The global family (flavor) symmetry group of $\mathcal{L}_f$ is $[U(3)]^5$.

(※ $\phi^c = i\sigma^2 \phi^*$)

The Yukawa couplings violate the global family symmetry,

$$\mathcal{L}_Y = - f_u^{ij} \bar{Q}_L^i \phi^c u_R^j - f_d^{ij} \bar{Q}_L^i \phi \, d_R^j - f_e^{ij} \bar{L}_L^i \phi \, e_R^j + \text{h.c.}$$

where f_u, f_d, f_e are 3×3 complex matrices in generation space.

A small subgroup of $[U(3)]^5$ is not violated :

- baryon number : $Q_L^i \to e^{i\theta} Q_L^i, \quad u_R^i \to e^{i\theta} u_R^i, \quad d_R^i \to e^{i\theta} d_R^i$
- lepton number : $L_L^i \to e^{i\theta'} L_L^i, \quad e_R^i \to e^{i\theta'} e_R^i$

Note that individual lepton numbers are conserved because the EW eigenstates are same as the mass eigenstates.

Thus, the Baryon number and the individual lepton numbers are accidental global symmetries of the SM.

※ The d.o.f. of the quark sector :

$$(\underset{\substack{\uparrow \\ f_u, f_d}}{2} \times \underset{\substack{\uparrow \\ \text{complex}}}{3^2 \times 2}) - (\underset{\substack{\uparrow \\ U_{Q_L}, U_{u_R}, U_{d_R}}}{3} \times 3^2 - \underset{\substack{\uparrow \\ \text{Baryon \#}}}{1}) = 10 = 6 \text{ (quark mass)} + \underset{\substack{\| \\ \text{3 mixing angles} \\ + \text{1 CP phase}}}{4} \text{ (CKM)}$$

※ The d.o.f. of the lepton sector :

$$(\underset{\substack{\uparrow \\ f_e}}{1} \times 3^2 \times \underset{\substack{\uparrow \\ \text{complex}}}{2}) - (\underset{\substack{\uparrow \\ U_{L_L}, U_{e_R}}}{2} \times 3^2 - \underset{\substack{\uparrow \\ \text{Lepton \#s}}}{3}) = 3 \text{ (lepton mass)}$$

$\boxed{10\text{-}4}$ Charge conjugation and CP violation

$$\mathcal{L}_{Dirac} = \bar\psi(x)\,(i\partial_\mu\gamma^\mu - m)\,\psi(x)$$

$$\delta S/\delta\bar\psi = 0 \to (i\partial_\mu\gamma^\mu - m)\psi(x) = 0 \quad \text{—①}$$

$$\delta S/\delta\psi = 0 \to -i\partial_\mu\bar\psi\gamma^\mu - m\bar\psi = 0 \to -\bar\psi(x)(i\overleftarrow{\partial}_\mu\gamma^\mu + m) = 0 \quad \text{—②}$$

a) $$② \xrightarrow{T} (i\partial_\mu\gamma^{\mu T} + m)\bar\psi^T(x) = 0$$

$$\xrightarrow{\times C} C(i\partial_\mu\gamma^{\mu T} + m)C^{-1}\underbrace{C\bar\psi^T(x)}_{=\psi^c(x)} = 0$$

If $C\gamma^{\mu T}C^{-1} = -\gamma^\mu$, $\psi^c(x)$ also satisfies the Dirac eq. $(i\partial_\mu\gamma^\mu - m)\psi^c(x) = 0$.

b)
$$③\begin{cases}
\cdot\ C\gamma^{\mu T}C^{-1} = -\gamma^\mu \\
\cdot\ C[\gamma^\mu,\gamma^\nu]^T C^{-1} = C(\gamma^{\nu T}\overset{C^{-1}C}{\vee}\gamma^{\mu T} - \gamma^{\mu T}\overset{C^{-1}C}{\vee}\gamma^{\nu T})C^{-1} = \gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu = -[\gamma^\mu,\gamma^\nu] \\
\cdot\ C\gamma_5^T C^{-1} = C(i\gamma^0\gamma^1\gamma^2\gamma^3)^T C^{-1} = C(i\gamma^{3T}\gamma^{2T}\gamma^{1T}\gamma^{0T})C^{-1} = i\gamma^3\gamma^2\gamma^1\gamma^0 = \gamma_5 \\
\cdot\ C(\gamma^\mu\gamma_5)^T C^{-1} = C\gamma_5^T\gamma^{\mu T}C^{-1} = -\gamma_5\gamma^\mu = \gamma^\mu\gamma_5
\end{cases}$$

$$④\begin{cases}
\cdot\ (C^{-1})^T\gamma^\mu C^T = (C\gamma^{\mu T}C^{-1})^T = (-\gamma^\mu)^T = C^{-1}\gamma^\mu C & = -\gamma^{\mu T} \\
\cdot\ (C^{-1})^T[\gamma^\mu,\gamma^\nu]C^T = C^{-1}[\gamma^\mu,\gamma^\nu]C & = -[\gamma^\mu,\gamma^\nu]^T \\
\cdot\ (C^{-1})^T\gamma_5 C^T = C^{-1}\gamma_5 C & = \gamma_5^T \\
\cdot\ (C^{-1})^T(\gamma^\mu\gamma_5)C^T = C^{-1}(\gamma^\mu\gamma_5)C & = (\gamma^\mu\gamma_5)^T
\end{cases}$$

$$⑤\begin{cases}
\cdot\ \gamma^\mu C^T C^{-1} = (C^{-1}C)^T\gamma^\mu C^T C^{-1} = C^T\overbrace{(C^{-1})^T\gamma^\mu C^T}^{C^{-1}\gamma^\mu C}C^{-1} = C^T C^{-1}\gamma^\mu \\
\cdot\ [\gamma^\mu,\gamma^\nu]C^T C^{-1} = C^T C^{-1}[\gamma^\mu,\gamma^\nu] \\
\cdot\ \gamma_5 C^T C^{-1} = C^T C^{-1}\gamma_5 \\
\cdot\ (\gamma^\mu\gamma_5)C^T C^{-1} = C^T C^{-1}(\gamma^\mu\gamma_5)
\end{cases}$$

$$\Rightarrow [\Gamma^A, C^T C^{-1}] = 0 \qquad \Gamma^A \in \{\gamma^\mu, [\gamma^\mu,\gamma^\nu], \gamma, \gamma^\mu\gamma, \mathbb{1}\}$$

$$\Rightarrow C^T C^{-1} \propto \mathbf{1} \longrightarrow C^T = \eta C \quad (\eta:\ \text{non-zero constant})$$

$$④ \to (C^{-1})^T\Gamma^A C^T = C^{-1}\Gamma^A C = \begin{cases} -\Gamma^{AT} & \text{for } \Gamma^A = \gamma^\mu, [\gamma^\mu,\gamma^\nu] \quad \cdots\ 10 \\ +\Gamma^{AT} & \text{for } \Gamma^A = \gamma_5, \gamma^\mu\gamma_5, \mathbb{1} \quad \cdots\ 6 \end{cases}$$

$$\xrightarrow[C^T = \eta C]{\times C^T} \eta(\Gamma^A C) = \begin{cases} -(\Gamma^A C)^T & \cdots\ 10 \\ +(\Gamma^A C)^T & \cdots\ 6 \end{cases}$$

By requiring that $\Gamma^A C$ yields 10 symmetric and 6 antisymmetric matrices, $\eta = -1$.

$$\Rightarrow C^T = -C$$

c) $C\gamma^{\mu T}C^{-1} = -\gamma^\mu$

$$\xrightarrow{\dagger} (C^{-1})^\dagger \gamma^{\mu *} C^\dagger = -\gamma^{\mu\dagger}$$

$$\longrightarrow (C^{-1})^\dagger \gamma^{\mu T} C^\dagger = -\gamma^\mu \quad \text{since } \gamma^0 = (\gamma^0)^\dagger,\ \gamma^i = -(\gamma^i)^\dagger \ \text{(See Ex.3-1 a))}$$
$$\gamma_5 = (\gamma_5)^\dagger \quad (\ \text{''} \quad b))$$

We can find

$$(C^{-1})^\dagger \Gamma^{AT} C^\dagger = \begin{cases} -\Gamma^A & \text{for } \Gamma^A = \gamma^\mu, [\gamma^\mu, \gamma^\nu] \\ +\Gamma^A & \text{for } \Gamma^A = \gamma_5, \gamma^\mu\gamma_5, \mathbb{1} \end{cases}$$

$$\xrightarrow{④} (C^{-1})^\dagger (C^{-1}\Gamma^A C) C^\dagger = \Gamma^A$$

$$\xrightarrow[\times CC^\dagger]{} \Gamma^A CC^\dagger = CC^\dagger \Gamma^A \rightarrow [\Gamma^A, CC^\dagger] = 0 \Rightarrow CC^\dagger \propto \mathbb{1}$$

We can choose $CC^\dagger = \mathbb{1}$, therefore $C^\dagger = C^{-1}$. (C is unitary.)

The representation $C = i\gamma^0\gamma^2$ satisfies $C^T = -C$ and $CC^\dagger = \mathbb{1}$:

- $C^T = (i\gamma^0\gamma^2)^T = i\gamma^{2T}\gamma^{0T} = i\gamma^2\gamma^0 = -i\gamma^0\gamma^2 = -C$ ※ γ^0, γ^2 : symmetric
- $CC^\dagger = i\gamma^0\gamma^2(-i\gamma^{2\dagger}\gamma^{0\dagger}) = \gamma^0\gamma^2(-\gamma^2)\gamma^0 = \mathbb{1}$ $\quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

d) The mode expansion of $\psi(x)$ and $\bar{\psi}(x)$:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s \left(a_p^s u^s(p) e^{-ipx} + b_p^{s\dagger} v^s(p) e^{ipx} \right)$$
$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s \left(a_p^{s\dagger} \bar{u}^s(p) e^{ipx} + b_p^s \bar{v}^s(p) e^{-ipx} \right)$$

$$\psi^c(x) = C\bar{\psi}(x)^T = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s \Big(a_p^{s\dagger} \underbrace{C\bar{u}^s(p)^T}_{=v^s(p)} e^{ipx} + b_p^s \underbrace{C\bar{v}^s(p)^T}_{=u^s(p)} e^{-ipx} \Big)$$
$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s \left(b_p^s u^s(p) e^{-ipx} + a_p^{s\dagger} v^s(p) e^{ipx} \right)$$

e) The charged current for quarks : CKM matrix

$$\mathcal{L}_{cc}^q = -\frac{g}{2\sqrt{2}} \left(\bar{u}_i \gamma^\mu (1-\gamma_5) V_{ij} d_j W_\mu^+ + \bar{d}_j \gamma^\mu (1-\gamma_5) V_{ij}^* u_i W_\mu^- \right)$$
$$\xrightarrow[CP]{} -\frac{g}{2\sqrt{2}} \left(\bar{d}_j \gamma^\mu (1-\gamma_5) V_{ij} u_i W_\mu^- + \bar{u}_i \gamma^\mu (1-\gamma_5) V_{ij}^* d_j W_\mu^+ \right)$$

If $V_{ij} \neq V_{ij}^*$, CP violation exists.

(※ When the number of generations $N_g \geq 3$, V_{ij} has a complex phase.)

P	C	CP
$V^\mu(x,t) \rightarrow V_\mu(-x,t)$	$V^\mu(x,t) \rightarrow -V^{\dagger\mu}(x,t)$	$V^\mu(x,t) \rightarrow -V_\mu^\dagger(-x,t)$
$\bar{\psi}_1\gamma^\mu\psi_2 \rightarrow \bar{\psi}_1\gamma_\mu\psi_2$	$\bar{\psi}_1\gamma^\mu\psi_2 \rightarrow -\bar{\psi}_2\gamma^\mu\psi_1$	$\bar{\psi}_1\gamma^\mu\psi_2 \rightarrow -\bar{\psi}_2\gamma_\mu\psi_1$
$\bar{\psi}_1\gamma^\mu\gamma_5\psi_2 \rightarrow -\bar{\psi}_1\gamma_\mu\gamma_5\psi_2$	$\bar{\psi}_1\gamma^\mu\gamma_5\psi_2 \rightarrow \bar{\psi}_2\gamma^\mu\gamma_5\psi_1$	$\bar{\psi}_1\gamma^\mu\gamma_5\psi_2 \rightarrow -\bar{\psi}_2\gamma_\mu\gamma_5\psi_1$