$\boxed{1-1}$ The Lorentz algebra

a) The Poincaré transformation (= Lorentz trans. + translation):

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$$\Rightarrow (x'-y')^\mu = \Lambda^\mu{}_\nu (x-y)^\nu$$

The distance in $\mathbb{R}^{1,3}$ is invariant under the Poincaré transformation:

$$\begin{aligned}(x-y)^2 &= (x'-y')^2 \\ &= (x'-y')^\mu (x'-y')_\mu \\ &= g_{\mu\nu} (x'-y')^\mu (x'-y')^\nu \\ &= g_{\mu\nu} \Lambda^\mu{}_\rho (x-y)^\rho \Lambda^\nu{}_\sigma (x-y)^\sigma \\ &= g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma (x-y)^\rho (x-y)^\sigma\end{aligned}$$

$$\Rightarrow g_{\rho\sigma} = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \quad \text{—①}$$

"Lorentz condition"

$$\underset{\times g^{\alpha\rho}}{\Longrightarrow} g^{\alpha\rho} g_{\rho\sigma} = g^{\alpha\rho} g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma$$

$$\delta^\alpha{}_\sigma = \Lambda_\nu{}^\alpha \Lambda^\nu{}_\sigma \quad \Rightarrow \Lambda_\nu{}^\alpha = (\Lambda^{-1})^\alpha{}_\nu \qquad \text{i.e. } \Lambda^T = \Lambda^{-1}$$

b) From (3), $\mathbb{1} = \Lambda^T \Lambda \quad \Rightarrow \quad 1 = (\det \Lambda^T)(\det \Lambda) = (\det \Lambda)^2$

$$\Rightarrow \det \Lambda = \pm 1$$

From ①, $\rho = \sigma = 0 \quad \Rightarrow \quad 1 = g_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0$

$$= (\Lambda^0{}_0)^2 - \sum_{i=1,2,3} (\Lambda^i{}_0)^2$$

$$\Rightarrow (\Lambda^0{}_0)^2 \geqq 1 \quad \Rightarrow \quad \Lambda^0{}_0 \geqq 1 \quad \text{or} \quad \Lambda^0{}_0 \leqq -1$$

The full Lorentz group breaks up into 4 disconnected subsets:

$$\begin{array}{ccccc} L_+^\uparrow & \overset{P}{\longleftrightarrow} & L_-^\uparrow = P L_+^\uparrow & \text{"orthochronous"} & (L^0{}_0 \geqq 1) \\ \updownarrow T & & \updownarrow T & & \\ L_+^\downarrow = T L_+^\uparrow & \overset{P}{\longleftrightarrow} & L_-^\downarrow = PT L_+^\uparrow & \text{"non orthochronous"} & (L^0{}_0 \leqq -1) \\ \text{"proper" } (\det L = +1) & & \text{"improper" } (\det L = -1) & & \end{array}$$

$$\begin{cases} P = \mathrm{diag}(1,-1,-1,-1) & : \text{parity} \\ T = \mathrm{diag}(-1,1,1,1) & : \text{time reversal} \\ PT = \mathrm{diag}(-1,-1,-1,-1) & \end{cases}$$

Note that only $L_+^\uparrow$ connects to the identity, while other three connect to the identity via the above discrete symmetry transformation.

c) The Lorentz transformation of the proper-orthochronous Lorentz group is generated by infinitesimal Lorentz transformations as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \qquad (\omega^\mu{}_\nu \ll 1)$$

From ①,

$$\begin{aligned} g_{\rho\sigma} &= g_{\mu\nu}(\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) \\ &= g_{\rho\sigma} + g_{\mu\sigma}\omega^\mu{}_\rho + g_{\rho\nu}\omega^\nu{}_\sigma + O(\omega^2) \end{aligned}$$

$$\Rightarrow \omega_{\rho\sigma} = -\omega_{\sigma\rho} \quad (\text{anti-symmetric tensor})$$

The dimension of the full Poincaré algebra in 4 dim. is $10 = \underset{\omega^\mu{}_\nu}{\underset{\uparrow}{6}} + \underset{a^\mu = \epsilon^\mu}{\underset{\uparrow}{4}}$.

- Rotations around the 3-axis :

$$\begin{pmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} \xrightarrow[\theta \ll 1]{} \begin{pmatrix} 1 & & & \\ & 1 & -\theta & \\ & \theta & 1 & \\ & & & 1 \end{pmatrix} = \delta^\mu{}_\nu + \underbrace{\begin{pmatrix} & & & \\ & & -\theta & \\ & \theta & & \\ & & & \end{pmatrix}}_{=\,\omega^\mu{}_\nu}$$

$$\Rightarrow \omega^1{}_2 = -\omega^2{}_1 = -\theta$$

$$\Rightarrow \omega_{12} = -\omega_{21} = \theta$$

- Boost along the 1-axis :

$$\begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta & & \\ \sinh\eta & \cosh\eta & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\xrightarrow[\eta \ll 1]{} \begin{pmatrix} 1 & \eta & & \\ \eta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \delta^\mu{}_\nu + \begin{pmatrix} & \eta & & \\ \eta & & & \\ & & & \\ & & & \end{pmatrix}$$

※ $\eta = \frac{1}{2}\ln\frac{1+\beta}{1-\beta}$ (rapidity)

$$\therefore \omega^0{}_1 = \omega^1{}_0 = \eta$$

$$\Rightarrow \omega_{01} = -\omega_{10} = \eta$$

Using the anti-symmetric nature, eq. (5) can be written as

$$\begin{aligned} \Lambda^\mu{}_\nu &= \delta^\mu{}_\nu + \omega^\mu{}_\nu \\ &= \delta^\mu{}_\nu + \frac{1}{2}(\omega^\mu{}_\nu - \omega_\nu{}^\mu) \\ &= \delta^\mu{}_\nu + \frac{1}{2}(\omega_{\alpha\beta}\, g^{\alpha\mu}\delta^\beta{}_\nu - \omega_{\alpha\beta}\, g^{\beta\mu}\delta^\alpha{}_\nu) \\ &= \delta^\mu{}_\nu - \frac{i}{4}\omega_{\alpha\beta}\,[\underbrace{2i(g^{\mu\alpha}\delta^\beta{}_\nu - g^{\mu\beta}\delta^\alpha{}_\nu)}_{=\,(M^{\alpha\beta})^\mu{}_\nu}] \end{aligned}$$

: a particular representation which acts on Lorentz 4-vectors.

Finite Lorentz transformations are

$$\Lambda^\mu{}_\nu = \exp\left(-\frac{i}{4}\omega_{\alpha\beta}M^{\alpha\beta}\right)^\mu{}_\nu .$$

The matrices $(M^{\alpha\beta})^\mu{}_\nu$ satisfy the Lorentz algebra

$$[M^{\mu\nu}, M^{\alpha\beta}] = 2i\left(g^{\mu\beta}M^{\nu\alpha} + g^{\nu\alpha}M^{\mu\beta} - g^{\mu\alpha}M^{\nu\beta} - g^{\nu\beta}M^{\mu\alpha}\right).$$

$\boxed{1-2}$ Action principle

a) The action :

$$S[\varphi] = \int d^4x \, \mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$$

The action principle :

$$0 = \delta S$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right)$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu (\delta\varphi)$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi$$

boundary term

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right\} \delta\varphi$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0$$: Euler-Lagrange eq.

b) The Lagrangian for the free scalar field :

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi(x) \partial^\mu \varphi(x) - \frac{1}{2} m^2 (\varphi(x))^2$$

This Lagrangian gives

$$-m^2 \varphi(x) - \partial_\mu \partial^\mu \varphi(x) = 0$$

$$\Rightarrow [\partial_\mu \partial^\mu + m^2] \varphi(x) = 0$$: Klein-Gordon eq.

$\boxed{1-3}$ Momentum operator

a) The mode expansion for $\Phi(x)$ and $\Pi(x)$:

$$\Phi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \left(a(p)\, e^{-ipx} + a^\dagger(p)\, e^{ipx} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \left(a(p)\, e^{-iE_p t} + a^\dagger(-p)\, e^{iE_p t} \right) e^{i\vec{p}\cdot\vec{x}}$$

$$\Pi(x) = \dot{\Phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{-i}{2} \left(a(p)\, e^{-iE_p t} - a^\dagger(-p)\, e^{iE_p t} \right) e^{i\vec{p}\cdot\vec{x}}$$

The 4-momentum operator is defined as

$$P^\mu = (H, \vec{P}) = \int d^3x \left[\Pi(x)\, \partial^\mu \Phi(x) - g^{\mu 0} \mathcal{L} \right]$$

For $\mu = 0$:

$$P^0 = H = \int d^3x\, \mathcal{H}$$

$$= \frac{1}{2} \int d^3x \left[\Pi^2(x) + (\partial_i \Phi(x))^2 + m^2 \Phi^2(x) \right]$$

$$= \frac{1}{2} \int \frac{d^3p\, d^3p'}{(2\pi)^6} \Big[\frac{-1}{4} \left(a(p)\, e^{-iE_p t} - a^\dagger(-p)\, e^{iE_p t} \right)\left(a(p')\, e^{-iE_{p'} t} - a^\dagger(-p')\, e^{iE_{p'} t} \right)$$
$$+ \frac{-\vec{p}\cdot\vec{p}' + m^2}{4E_p E_{p'}} \left(a(p)\, e^{-iE_p t} + a^\dagger(-p)\, e^{iE_p t} \right)\left(a(p')\, e^{-iE_{p'} t} + a^\dagger(-p')\, e^{iE_{p'} t} \right) \Big]$$
$$\times \underbrace{\int d^3x\, e^{i(\vec{p}+\vec{p}')\cdot\vec{x}}}_{= (2\pi)^3 \delta^{(3)}(\vec{p}+\vec{p}')}$$

$(\because E_p^2 = \vec{p}^2 + m^2)$

$$= \frac{1}{8} \int \frac{d^3p}{(2\pi)^3} \Big[-\left(a(p)\, e^{-iE_p t} - a^\dagger(-p)\, e^{iE_p t} \right)\left(a(-p)\, e^{-iE_p t} - a^\dagger(p)\, e^{iE_p t} \right)$$
$$+ \left(a(p)\, e^{-iE_p t} + a^\dagger(-p)\, e^{iE_p t} \right)\left(a(-p)\, e^{-iE_p t} + a^\dagger(p)\, e^{iE_p t} \right) \Big]$$

$$= \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \left[a(p)\, a^\dagger(p) + a^\dagger(-p)\, a(-p) \right]$$

$$\downarrow p \to -p$$

$$= \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \left[a(p)\, a^\dagger(p) + a^\dagger(p)\, a(p) \right]$$

For $\mu = i = 1, 2, 3$:

$$P^i = -\int d^3x\, \Pi(x)\, \partial_i \Phi(x)$$

$$= -\int \frac{d^3p\, d^3p'}{(2\pi)^6} \Big[\frac{-i}{2} \left(a(p)\, e^{-iE_p t} - a^\dagger(-p)\, e^{iE_p t} \right) \frac{i\vec{p}'}{2E_{p'}} \left(a(p')\, e^{-iE_{p'} t} + a^\dagger(-p')\, e^{iE_{p'} t} \right)$$
$$\times \int d^3x\, e^{i(\vec{p}+\vec{p}')\cdot\vec{x}}$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_p} \vec{p} \left[a(p)\, a^\dagger(p) - a^\dagger(-p)\, a(-p) + a(p)\, a(-p)\, e^{-2iE_p t} - a^\dagger(-p)\, a(p)\, e^{2iE_p t} \right]$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_p} \vec{p} \left[a(p)\, a^\dagger(p) + a^\dagger(p)\, a(p) \right]$$

$$\left(\because \; \vec{p}\, a^\dagger(-\vec{p})\, a(-\vec{p}) \xrightarrow[\vec{p} \to -\vec{p}]{} -\vec{p}\, a^\dagger(\vec{p})\, a(\vec{p}) \qquad \vec{p}\, a(\vec{p})\, a(-\vec{p}) \longrightarrow -\vec{p}\, a(-\vec{p})\, a(\vec{p}) = -\vec{p}\, a(\vec{p})\, a(-\vec{p}) \right)$$

$$\Rightarrow P^\mu = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_p} p^\mu \left[a(p)\, a^\dagger(p) + a^\dagger(p)\, a(p) \right]$$

b)

$$[aa^\dagger + a^\dagger a,\ a^\dagger]$$
$$= aa^\dagger a^\dagger + \cancel{a^\dagger a a^\dagger} - \cancel{a^\dagger a a^\dagger} - a^\dagger a^\dagger a$$
$$= \{a^\dagger a + (2\pi)^3 2E\delta\} a^\dagger - a^\dagger a^\dagger a$$
$$= a^\dagger \{\cancel{a^\dagger a} + (2\pi)^3 2E\delta\} + (2\pi)^3 2E\delta\, a^\dagger - \cancel{a^\dagger a^\dagger} a$$
$$= 2 \times (2\pi)^3\, 2E\delta\, a^\dagger$$

※ canonical commutation relation:

$$[a(p), a^\dagger(p')] = (2\pi)^3 2E_p\, \delta^{(3)}(\vec{p}-\vec{p}')$$

$$\Rightarrow\ [P^\mu, a^\dagger(p)] = p^\mu a^\dagger(p)$$

$$[aa^\dagger + a^\dagger a,\ a]$$
$$= \cancel{aa^\dagger a} + a^\dagger a a - aaa^\dagger - \cancel{aa^\dagger a}$$
$$= a^\dagger a a - a\{a^\dagger a + (2\pi)^3 2E\delta\}$$
$$= \cancel{a^\dagger a a} - \{\cancel{a^\dagger a} + (2\pi)^3 2E\delta\} a - (2\pi)^3 2E\delta\, a$$
$$= -2 \times (2\pi)^3 2E\delta$$

$$\Rightarrow\ [P^\mu, a(p)] = -p^\mu a(p)$$

c)

$$P^\mu = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 2E_p}\, p^\mu\, [a(p)a^\dagger(p) + a^\dagger(p)a(p)]$$
$$a(p)a^\dagger(p) = a^\dagger(p)a(p) + (2\pi)^3 2E_p\, \delta^{(3)}(0)$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p}\, p^\mu \left[a^\dagger(p)a(p) + \frac{1}{2}(2\pi)^3 2E_p\, \delta^{(3)}(0)\right]$$

For $\mu = 1, 2, 3$:

$$P^i = \int \frac{d^3p}{(2\pi)^3 2E_p}\, p^\mu\, a^\dagger(p)a(p)$$

※ The 2nd term vanishes due to the symmetric integration.

For $\mu = 0$

$$P^0 = H = \int \frac{d^3p}{(2\pi)^3 2E_p}\, E_p\, a^\dagger(p)a(p) + \frac{1}{2}\int d^3p\, E_p\, \delta^{(3)}(0)$$

$\rightarrow \infty$

※ The 2nd term is the sum over all modes of the zero-point energies.

1-4 The Feynman propagator

a) $\langle 0| \Phi(x) \Phi(y) |0\rangle$

$$= \langle 0| \int \frac{d^3p}{(2\pi)^3 2E_p} \{ a(p) e^{-ipx} + a^\dagger(p) e^{ipx} \} \int \frac{d^3p'}{(2\pi)^3 2E_{p'}} \{ a(p') e^{-ip'y} + a^\dagger(p') e^{ip'y} \} |0\rangle$$

$$= \langle 0| \int \frac{d^3p}{(2\pi)^3 2E_p} \int \frac{d^3p'}{(2\pi)^3 2E_{p'}} e^{-i(px-p'y)} a(p) a^\dagger(p') |0\rangle \qquad (\because a(p')|0\rangle = \langle 0|a^\dagger(p) = 0)$$

$$※\ \langle 0| a(p) a^\dagger(p') |0\rangle = \langle 0| [a(p), a^\dagger(p')] |0\rangle = (2\pi)^3 2E_p \delta^{(3)}(\vec{p}-\vec{p}') \underbrace{\langle 0|0\rangle}_{=1}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)}$$

The Feynman propagator of a scalar field is defined as

$$i\Delta_F(x-y) = \langle 0| T(\Phi(x)\Phi(y)) |0\rangle$$

$$= \theta(x^0-y^0) \langle 0| \Phi(x)\Phi(y) |0\rangle + \theta(y^0-x^0) \langle 0| \Phi(y)\Phi(x) |0\rangle$$

$$\left(※\ \int_{-\infty}^{\infty} d^3p\, e^{i\vec{p}\cdot\vec{x}} \xrightarrow[\vec{p}\to-\vec{p}]{} \int_{\infty}^{-\infty} d(-p)\, e^{-i\vec{p}\cdot\vec{x}} = \int_{-\infty}^{\infty} d^3p\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

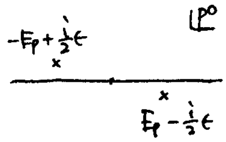
$$= \theta(x^0-y^0) \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)} + \theta(y^0-x^0) \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(y-x)}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \{ \theta(x^0-y^0) e^{-iE_p(x^0-y^0)} + \theta(y^0-x^0) e^{-iE_p(y^0-x^0)} \}$$

b) $$I = \int dp^0 \frac{e^{-ip^0(x_0-y_0)}}{(p^0)^2 - E_p^2 + i\epsilon} \qquad (E_p = +\sqrt{\vec{p}^2+m^2})$$

$$(p^0)^2 - (E_p^2 - i\epsilon) = (p^0)^2 - \left\{ (E_p - \tfrac{i}{2}\epsilon')^2 + \frac{\epsilon'^2}{4} \right\} \qquad (\epsilon' E_p = \epsilon)$$

$$= \int dp^0 \frac{e^{-ip^0(x_0-y_0)}}{(p^0 + (E_p - \frac{i}{2}\epsilon))(p^0 - (E_p - \frac{i}{2}\epsilon))}$$



$$= \theta(x^0-y^0) \int dp^0 \frac{e^{-ip^0(x_0-y_0)}}{(p^0 + (E_p - \frac{i}{2}\epsilon))(p^0 - (E_p - \frac{i}{2}\epsilon))}$$

$$+ \theta(y^0-x^0) \int dp^0 \frac{e^{+ip^0(y_0-x_0)}}{(p^0 + (E_p - \frac{i}{2}\epsilon))(p^0 - (E_p - \frac{i}{2}\epsilon))}$$

$$= \theta(x^0-y^0) \left(-2\pi i \frac{e^{-iE_p(x_0-y_0)}}{2E_p} \right) + \theta(y^0-x^0) \left(2\pi i \frac{e^{-iE_p(y_0-x_0)}}{-2E_p} \right) \qquad (\because \text{Cauchy's theorem})$$

$$= \frac{2\pi}{2iE_p} \{ \theta(x^0-y^0) e^{-iE_p(x^0-y^0)} + \theta(y^0-x^0) e^{-iE_p(y^0-x^0)} \}$$

Therefore,

$$i\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \times \frac{2iE_p}{2\pi} I = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}$$