Prof Dr. Ulrich Uwer/ Dr. T. Weigand 8. Juni 2010

The Standard Model of Particle Physics - SoSe 2010 Assignment 8

(Due: June 17, 2010)

Final Exam:

- **Time and Place:** Monday, July 26, 2010, 9:15 in Großer Hörsaal, Philosophenweg 12
- **Covered material:** Contents of the lectures - including derivations etc. - and of the examples sheets
- **Allowed Sources:** 1 double-sided sheet of paper, DIN A 4, of notes (handwritten or typed)

1 $SU(2)$ is not enough

a.) Consider the group $SU(2)$, generated by T_i, $i = 1, 2, 3$, with

$$[T_i, T_j] = i\,\epsilon_{ijk} T_k. \tag{1}$$

The fundamental representation of $SU(2)$ is given by the action of T_i on a complex 2-vector, i.e. the T_i are hermitian traceless 2×2 matrices subject to (1). These are the Pauli matrices, $T_i = \frac{1}{2}\sigma_i$.

Independent of the representation, a convenient basis of generators is T_3 together with $T_\pm = T_1 \pm iT_2$. Compute the commutation relations $[T_+, T_-]$ and $[T_3, T_\pm]$ with the help of (1).

Note: The result can be interpreted as follows: The commutator is the action of the Lie algebra on itself. If $[T_3, \mathcal{O}] = q\,\mathcal{O}$, q is interpreted as the charge of the generator $\mathcal{O}$ with respect to T_3. The choice of basis T_3, T_+, T_- corresponds to a decomposition into the maximal commuting abelian subalgebra (here the U(1) subgroup generated by T_3) together with the eigenstates under T_3.

b.) Even before the full Standard Model was formulated, experiments suggested the following structure of electro-weak interactions for the observed leptons (restricting to one family)

$$\mathcal{L} \supset g(J_\mu W^\mu + c.c) + e J_\mu^{e.m.} A^\mu \tag{2}$$

$$J = \bar{\nu}_e(x)\gamma_\mu(1-\gamma_5)e(x), \qquad J_\mu^{e.m.} = -\bar{e}(x)\gamma_\mu e(x) \tag{3}$$

in terms of the charged weak current J_μ and the electro-magnetic current $J_\mu^{e.m.}$. Since there are at least 3 vector bosons involved - $W, W^\dagger, A$ - the simplest guess is to combine them into $SU(2)$. Since W has charge $+1$ under the electromagnetic $U(1)_Q$, this would identify W with the boson associated with generator T_+ and A with T_3. Then $U(1)_Q$ would be embedded into $SU(2)$.

To test this we check if the conserved charges associated with the above currents close to form $SU(2)$. I.e. we interpret the Noether charges

$$T_+ = \frac{1}{2}\int d^3x J_0(x), \qquad T_- = \frac{1}{2}\int d^3x J_0^\dagger(x), \qquad Q = \int d^3x J_0^{e.m.}(x). \tag{4}$$

as the generators of the hypothesised underlying symmetry.

Use the canonical anti-commutation relations for a Dirac fermion field $\psi(x)$ to show that

$$[T_+, T_-] = 2T_3, \qquad T_3 = \frac{1}{4}\int d^3x[\nu_e^\dagger(1-\gamma_5)\nu_e - e^\dagger(1-\gamma_5)e]. \tag{5}$$

This shows explicitly that the current algebra of J_μ, $J_\mu^\dagger$ and $J_\mu^{e.m.}$ does not close. In particular it is not possible to describe weak and electro-magnetic interactions of the observed particles simply in terms of an $SU(2)$ theory.

c.) Give another, very simple argument (no computation) why the structure of the currents $J_\mu^{e.m.}$ and J_μ makes it impossible that they close within a simple group.

e.) One solution to the above problem that was put forward is to introduce heavy, yet to be observed particles and modify the currents such as to make them close within $SU(2)$. Alternatively, one can enlarge the gauge symmetry to $SU(2)\times U(1)_Y$. This route was of course taken in formulating the Standard Model and lead to the prediction of the Z boson.

Give the form of the interactions in terms of the weak current J_μ, $J_\mu^{e.m.}$ and the neutral current J_μ^n as stated in the lecture. How exactly is the form of these interactions in agreement with the result (5)?

2 Gauge field interactions

The physical combinations of gauge fields in the $SU(2) \times U(1)_Y$ theory are W_μ, A_μ, Z_μ.

a) Recall from the lecture the precise definition of these fields in terms of the gauge fields A^i_μ associated with the generators of $SU(2)$ and B_μ, the hypercharge gauge field. Explain the significance of W_μ, A_μ, Z_μ.

b) The pure $SU(2) \times U(1)_Y$ Yang-Mills Lagrangian reads

$$\mathcal{L}_{gauge} = -\frac{1}{4}\sum_{i=1,2,3} F^i_{\mu\nu}(F^i)^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \tag{6}$$

in terms of the basis A^i_μ and B_μ of gauge fields. Show that in terms of the fields W_μ, A_μ, Z_μ this takes the following form:

$$-\frac{1}{2}(F^W_{\mu\nu})^\dagger F^{W\mu\nu} - \frac{1}{4}(F^A_{\mu\nu})^2 - \frac{1}{4}(F^Z_{\mu\nu})^2 + \tag{7}$$

$$iW^\mu W^{\nu\dagger}(eF^A_{\mu\nu} + g\cos(\theta_W)F^Z_{\mu\nu}) + \frac{1}{2}g^2(W^2W^{\dagger 2} - (WW^\dagger)^2), \tag{8}$$

where

$$F^A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad F^Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu, \tag{9}$$

$$F^W_{\mu\nu} = d_\mu W_\nu - d_\nu W_\mu, \qquad d_\mu = \partial_\mu + ieA_\mu + ig\cos(\theta_W)Z_\mu \tag{10}$$

$$e = g\sin(\theta_W) \tag{11}$$

Extract from this all interactions between A_μ, Z_μ and W_μ. What is the fundemental difference between W and Z as far as interactions with A are concerned?

3 Symmetries of the Higgs potential

The Standard Model Higgs field $\Phi(x)$ is taken to be in a fundamental representation of $SU(2)$, $\Phi = (\Phi_1, \Phi_2)^T$, with $\Phi_1 = \varphi_1 + i\varphi_2$, $\Phi_2 = \varphi_3 + i\varphi_4$ and all φ_i real fields. It has hypercharge $Y = \frac{1}{2}$.

a) What is the electro-magnetic charge of the complex fields Φ_1 and Φ_2?

b) Argue that the Higgs potential

$$V(\Phi) = \frac{\lambda}{2}\left(\Phi^\dagger\Phi - \frac{v}{2}\right)^2 \tag{12}$$

is invariant under the gauge group $SU(2) \times U(1)_Y$.

c) Given a physical theory, one has to distinguish between the global symmetries and the gauge symmetries. For a theory with global symmetry $\tilde{G}$, in general only a subgroup $G \subset \tilde{G}$ may be gauged. Indeed, it turns out that the above Higgs potential V has a larger *global* symmetry group $\tilde{G}$ of which the *gauged* symmetry group $G = SU(2) \times U(1)_Y$ is only a subgroup.

Determine this symmetry group by rewriting $V(\Phi)$ in terms of the real fields φ_i. Determine the remaining global symmetry group after Φ acquires the VEV $\Phi = (0, \frac{v}{\sqrt{2}})$. How many would-be Goldstone modes are there from the point of view of $\tilde{G}$?