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The Standard Model of Particle Physics - SoSe 2010 Assignment 6

Note: There is no tutorial on June 3 due to Fronleichnam.

(Due: June 10, 2010)

1 Some group theory

Let G be a compact Lie group with hermitian generators T_a.

1.) From the general theory of Lie groups it follows that there exists an antisymmetric map

$$[T_a, T_b] = i f_{abc} T_c, \tag{1}$$

with summation over c understood, subject to the Jacobi identity

$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_b, T_a]] = 0. \tag{2}$$

Show that this implies the following property of the structure constants f_{abc}:

$$f_{ade} f_{bcd} + f_{bde} f_{cad} + f_{cde} f_{abd} = 0. \tag{3}$$

2.) To give a finite-dimensional unitary representation of G one has to find a set of $N \times N$ hermitian matrices T_a^R that satisfy the defining relation (1) and act by multiplication on an N-complex dimensional vector ψ_i, $i = 1, \ldots N$. N is the dimension of the representation. In this case, the bilinear $[,]$ in (1) is just the commutator of two matrices $N \times N$ matrices.

a.) Show that the Jacobi identity (2) is a triviality when expressed in terms of the representation T_a^R.

b.) Use a group theory book or any other source of your choice to revise the following definitions: irreducible representation - simple Lie algebra - semi-simple Lie algebra.

c.) For an irreducible unitary representation one can can normalise the generators T_a^R such that

$$\mathrm{tr}(T_a^R T_b^R) = C(R)\delta^{ab}, \tag{4}$$

where $C(R)$ is a constant for each representation. Use this to show that

$$f_{abc} = -\frac{i}{C(r)}\mathrm{tr}([T_a^R, T_b^R]\, T_c^R) \tag{5}$$

and deduce that f_{abc} is antisymmetric in all three indices.

2 Yang-Mills theory

Consider the gauge transformation

$$\psi_i(x) \to U_{ij}(x)\psi_j(x), \qquad U_{ij} = \exp(i\theta_a(x)T_a^R)_{ij}, \tag{6}$$

in some representation T_a^R of a simple Lie group G. In the following we omit the superscript R.

1.) The covariant derivative is a matrix valued operator defined via

$$D_\mu\psi(x) = (\partial_\mu \times 1 + igT_aA_{a\mu}(x))\psi(x). \tag{7}$$

Show that $D_\mu\psi(x)$ transforms under a gauge transformation as

$$D_\mu\psi(x) \to UD_\mu\psi(x) \tag{8}$$

provided the gauge field $\underline{A}_\mu \equiv (A_\mu)_a$ transforms as

$$\underline{T}\cdot(\underline{A})_\mu \to U\underline{T}\cdot(\underline{A})_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}, \tag{9}$$

where $\underline{T}\cdot(\underline{A})_\mu = T_aA_{a\mu}$ with summation over indices understood.
2.) The Yang-Mills field strength is defined as

$$\underline{T}\cdot(\underline{F})_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]. \tag{10}$$

Show that in components this is

$$(F_a)_{\mu\nu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - gf_{abc}(A_b)_\mu\,(A_c)_\nu. \tag{11}$$

Verify the transformation law

$$\underline{T}\cdot(\underline{F})_{\mu\nu} \to U\underline{T}\cdot(\underline{F})_{\mu\nu}U^{-1}. \tag{12}$$

Use this to show that

$$L_{YM} = -\frac{1}{4}\int d^4x(F_a)_{\mu\nu}(F_a)^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu + M)\psi \tag{13}$$

is gauge invariant.
Hint: Rewrite the gauge part as a trace and use the cyclic property of the trace.
3.) Derive the cubic and quartic self-interactions for the gauge fields from L_{YM}.
4.) Work out the special case where the Lie group is $G = U(1)$ and recover the known expressions for QED.

3 Propagator, gauge fixing and massive U(1)s

Consider the Lagrangian for an abelian gauge theory

$$L_{U(1)} = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}. \tag{14}$$

a.) Show that in momentum space it reads

$$L_{U(1)} = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\tilde{A}^\mu(-p)\mathcal{O}_{\mu\nu}(p)\tilde{A}^\nu(p), \qquad \mathcal{O}_{\mu\nu}(p) = -\Big(g_{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\Big)p^2. \tag{15}$$

b.) Consider the two operators

$$P^T_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \qquad P^L_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}. \tag{16}$$

Show that these are generalised projection operators in the sense that

$$P^T_{\mu\nu}\, P^{T\nu\lambda} = P^{T\nu}_{\mu}, \qquad P^L_{\mu\nu}\, P^{L\nu\lambda} = P^{L\nu}_{\mu}, \qquad P_T\, P_L = 0, \qquad P^T_{\mu\nu} + P^L_{\mu\nu} = g_{\mu\nu}. \tag{17}$$

Show that P_L does not couple to a conserved current J^μ.

c.) A general property of the propagator is that it is a Green's function, i.e. it is the inverse of the operators appearing in the quadratic terms in a lagrangian. In momentum space this means that

$$\tilde{\Delta}_{\mu\nu}(p)\,\mathcal{O}^{\nu\lambda}(p) = i\delta^\lambda_\mu. \tag{18}$$

For the Lagrangian (15) we have $\mathcal{O}^{\nu\lambda} = \mathcal{P}^{T\,\nu\lambda}$, which has no inverse. This is one way to see that a sensible definition of the quantum theory requires gauge fixing by adding the Lagrangian multiplier

$$L_{fix} = -\int d^4x \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{19}$$

Give the resulting operator $\mathcal{O}(p,\xi)$ in momentum space and show that the propagator is now well-defined and reads

$$\tilde{\Delta}_{\mu\nu}(p) = \Big(g_{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\Big)\frac{-i}{p^2+i\epsilon}. \tag{20}$$

d.) Now consider massive U(1) theory by adding a mass term,

$$L_{U(1),massive} = \int d^4x(-\frac{1}{4})F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\,M_A^2 A_\mu A^\mu. \tag{21}$$

Derive the propagator in momentum space. Show in particular that it contains a piece of the form $\frac{p_\mu p_\nu/M_A^2}{p^2-M_A^2+i\epsilon}$.

e.) Suppose now that the massive photon couples to a conserved current J_μ. Show that for large momenta, the relevant part of the propagator which couples to the conserved current scales like $\frac{1}{p^2}$ as is the case for a massless U(1) propagator. In other words, show that the piece $\frac{p_\mu p_\nu/M_A^2}{p^2-M_A^2+i\epsilon}$ does not couple to a conserved current. This is important because it ensures that the theory remains normalisable as this piece would scale like $\frac{1}{M_A^2}$ for large p^2 and destroy renormalisable.

Note: This completes the proof that a massive *abelian* theory is both unitary (see lecture) and renormalisable even though gauge invariance is broken. Note that renormalisability requires that J^μ is conserved and thus that there is still a global symmetry. Both arguments break down for non-abelian gauge theories.