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The Standard Model of Particle Physics - SoSe 2010 Assignment 2

(Due: April 29, 2010)

On this examples sheet we fill in some gaps of the derivation of the S-matrix as given in the lecture. The purpose of the following exercises is to review the relevant concepts. Please take this opportunity especially if you have not attended a QFT course before.

Note that for brevity we are ignoring most issues associated with the definition of the vaccum in the perturbed versus the unperturbed theory. A more thorough, but also more complicated presentation of the S-matrix can be found e.g. in Peskin/Schröder, Chapter 4.2, 4.3., 4.6.

Finally, what we actually need for practical purposes in this course is the definition of the S-matrix and its applications such as the one given in the lecture on the computation of scattering in Φ^4-theory.

1 Wick's theorem

a.) Recall the definition of the normal ordered product $:A:$ for an operator A as given in the lecture. Argue that

$$\langle :A: \rangle = 0. \tag{1}$$

b.) Verfiy that

$$\phi(x)\phi(y) =: \phi(x)\phi(y) : + [\phi^{(+)}(x), \phi^{(-)}(y)] \tag{2}$$

in terms of the positive and negative frequency solutions for a scalar field given in the lecture. Use this to show explicity that

$$T\Big(\phi(x)\phi(y)\Big) =: \phi(x)\phi(y) : + f(x,y), \tag{3}$$

where $f(x,y)$ is a c-number. Give the explicit form of $f(x,y)$ in terms of commutators of $\phi^{\pm}(x)$. Deduce from this that

$$T\Big(\phi(x)\phi(y)\Big) =: \phi(x)\phi(y) : + i\Delta_F(x-y). \tag{4}$$

c.) Write down Wick's theorem for four fields $T\Big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\Big)$.

2 Interaction picture

a.) Consider a free quantum system with Hamiltonian H_0, where the subscript reminds us that the system contains no interactions. Recall, e.g. by consulting your favourite quantum mechanics book, that one distinguishes the Schrödinger picture versus the Heisenberg picture as follows:

- In the Schrödinger picture, the operators A_S are time independent, while the states $|\psi_S(t)\rangle$ carry all information about the time evolution via the Schrödinger equation

$$i\frac{d}{dt}|\psi_S(t)\rangle = H_0\,|\psi_S(t)\rangle. \tag{5}$$

- In the Heisenberg picture the time evolution is carried by the operators $A_H(t)$. The states per se carry no time evolution; rather the state $|\psi_{H,t}\rangle$ is meant as the state created from the vacuum by the operator $\psi_H(t)$.

- Both pictures are related such that the time dependence of expectation values match, i.e. $\langle\psi_S(t)|A_S|\psi_S(t)\rangle = \langle\psi_{H,t}|A_H(t)|\psi_{H,t}\rangle$.

Suppose that at time t_0 the states in both pictures agree,

$$|\psi_S(t_0)\rangle = |\psi_{H,t_0}\rangle. \tag{6}$$

Convince yourself that the expectation values of the two pictures match as above if A_S and A_H are related as

$$A_H(t) = e^{iH_0t}A_S e^{-iH_0t}. \tag{7}$$

Give the law for the time evolution of $A_H(t)$.

b.) Go through the following points carefully to familiarize yourself with the concept of interaction picture operators:

- Consider the Lagrangian density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ describing the dynamics of a scalar field $\Phi(x)$ in Φ^4-theory,

$$\mathcal{L}_0 = -\frac{1}{2}\int \Phi(x)(\partial^2 + m^2)\Phi(x), \qquad \mathcal{L}_I = -\frac{\lambda}{4!}\Phi^4(x). \tag{8}$$

  The Hamiltonian $H = H_0 + H_I$ is a time independent quantity. It is defined for the the fields and their conjugate momenta at, say, time $t = 0$,

$$H_0 = \int d^3x\Big(\pi(0,\vec{x})\Phi(0,\vec{x}) - \mathcal{L}_0\Big), \qquad H_I = -\int d^3x\Big(\mathcal{L}_{\mathcal{I}}\Big). \tag{9}$$

- In the interaction picture, the interaction field $\Phi_I(t,\vec{x})$ is defined as

$$\Phi_I(t,\vec{x}) = e^{iH_0 t}\Phi_I(0,\vec{x})e^{-iH_0 t}. \tag{10}$$

  If we take $\Phi_I(0,\vec{x})$ to be the Schrödinger picture operator, $\Phi_I(t,\vec{x})$ can be viewed as a Heisenberg-picture operator of the *free* theory since only H_0 is used in its definition. This means that all previous results on quantisation of the free field equal time commutation relations and mode expansion directly carry over to $\Phi_I(t,\vec{x})$.

- Let $|\Phi_I(t)\rangle$ denote a state created by $\Phi_I(t)$ by acting on the *unperturbed* vacuum $|0\rangle$:

$$|\Phi_I(t)\rangle = \Phi_I(t,\vec{x})|0\rangle. \tag{11}$$

  The relation between this state and the Schrödinger picture state $|\Phi_S(t)\rangle$ is seen as follows: At time $t = 0$, the Schrödinger picture and the Heisenberg picture state agree:

$$|\Phi_S(t)\rangle|_{t=0} = \Phi(0,\vec{x})|0\rangle. \tag{12}$$

  Thus

$$|\Phi_S(t)\rangle|_{t=0} = e^{-iH_0 t}e^{+iH_0 t}\Phi(0,\vec{x})e^{-iH_0 t}|0\rangle \tag{13}$$

$$\Rightarrow |\Phi_S(t)\rangle|_{t=0} = e^{-iH_0 t}\,|\Phi_I(t)\rangle. \tag{14}$$

c.) The time evolution for $|\Phi_S(t)\rangle$ involves the full Hamiltonian $H = H_0 + H_I$,

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\Phi_S(t)\rangle = H\,|\Phi_S(t)\rangle. \tag{15}$$

Deduce from this and from the relation (14) that

$$e^{-iH_0 t}\,i\frac{\mathrm{d}}{\mathrm{d}t}\,|\Phi_I(t)\rangle = H_I\,e^{-iH_0 t}\,|\Phi_I(t)\rangle \tag{16}$$

and thus

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\Phi_I(t)\rangle = \bar{H}_I(t)|\Phi_I(t)\rangle, \qquad \bar{H}_I(t) = e^{iH_0 t}H_I e^{-iH_0 t}. \tag{17}$$

This establishes that the time evolution of $\Phi_I(t,\vec{x})$ is governed by the interaction Hamiltonian evaluated now in terms of $\Phi_I(t,\vec{x})$. Since $\Phi_I(t,\vec{x})$ is essentially quantised as a free field, we can relabel $\Phi_I(t,\vec{x}) \to \Phi(x)$.

3 Time evolution and S-matrix

The equation (17) is our master equation for the evolution of a state in the interaction picture.

a.) This equation is integrated as follows:

$$|\Phi(t)\rangle = |\Phi_i\rangle + \frac{1}{i}\int_{-\infty}^{t} dt' \bar{H}_I(t')|\Phi(t')\rangle, \tag{18}$$

where $|\Phi_i\rangle$ is thie initial state at $t_i = -\infty$. Convince yourself that this evolution from $|\Phi_i\rangle$ at $t_i = -\infty$ up to $|\Phi(t)\rangle$ can be rewritten as

$$|\Phi(t)\rangle = U(t)|\Phi_i\rangle, \qquad U(t) = 1 + \int_{-\infty}^{t} dt' \bar{H}_I(t')U(t'), \tag{19}$$

and that the time evolution operator $U(t)$ satisfies the equation

$$i\frac{d}{dt}U(t) = \bar{H}_I(t)U(t). \tag{20}$$

b.) The equation (20) for $U(t)$ can be solved perturbatively as the series

$$U(t) = 1 + \frac{1}{i}\int_{-\infty}^{t} dt' \bar{H}_I(t') + \frac{1}{i^2}\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \bar{H}_I(t')\bar{H}_I(t'') + \ldots. \tag{21}$$

Can you explain the structure of this iteration?

c.) The final form for $U(t)$ is obtained by rewriting (21) in terms of the time-ordered product. In this process we change the integration range as follows:

$$U(t) = 1 + \frac{1}{i}\int_{-\infty}^{t} dt' \bar{H}_I(t') + \frac{1}{2!}\frac{1}{i^2}\int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt''\, T\Big(\bar{H}_I(t')\bar{H}_I(t'')\Big) + \tag{22}$$

$$+ \frac{1}{3!}\frac{1}{i^3}\int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' \int_{-\infty}^{t} dt'''\, T\Big(\bar{H}_I(t')\bar{H}_I(t'')\bar{H}_I(t''')\Big) + \ldots \tag{23}$$

Explain the appearance of this factor $n!$ at each order and the appearance of the time-ordered product.

d.) How is the S-matrix defined?