Prof Dr. Ulrich Uwer/ Dr. T. Weigand 13. April 2010

The Standard Model of Particle Physics - SoSe 2010 Assignment 1

(Due: April 22, 2010)

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Time: Thursdays, 14:15-16:00
Venue: Großer Hörsaal, Philosophenweg 12

First class: April 22, 2010

1 The Lorentz algebra

a.) Consider $\mathbb{R}^{1,3}$ as the space of 4-vectors endowed with metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Unless noted otherwise we use the notation x, y for 4-vectors: $x \equiv (x^\mu), \quad \mu = 0, \ldots 3$; $\vec{x}$ to denote their spacelike components $x^i, i = 1, 2, 3$; $xy = x^\mu y_\mu$ for the salar product in $\mathbb{R}^{1,3}$ and $\vec{x}\vec{y} = +\sum_i x^i y^i$.

Show that the rigid linear transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{1}$$

leaves the distance in $\mathbb{R}^{1,3}$ invariant,

$$(x' - y')^2 = (x - y)^2, \tag{2}$$

provided

$$\Lambda_\rho{}^\alpha \Lambda^\rho{}_\kappa = \delta^\alpha{}_\kappa, \qquad \text{i.e.} \qquad \Lambda^T = \Lambda^{-1}. \tag{3}$$

b.) Recall the following facts about the Lorentz group: From (3) it follows that

$$\det\Lambda = \pm 1. \tag{4}$$

The Lorentz group is not connected, but contains 4 connected components. These are distinguished by $\det\Lambda = 1$ vs. $\det\Lambda = -1$ together with $\Lambda^0{}_0 > 0$ vs. $\Lambda^0{}_0 < 0$.

The piece $\det\Lambda = 1$, $\Lambda^0{}_0 > 0$ is called connected to the identity because infinitesimal Lorentz transformations of this type can be written as

$$\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta + \omega^\alpha{}_\beta. \tag{5}$$

The other pieces can be obtained from the piece connected to the identity via the discrete symmetry transformations

$$\Lambda_P = \text{diag}(1,-1,-1,-1) \qquad \text{parity}, \qquad \Lambda_\text{T} = \text{diag}(-1,1,1,1) \qquad \text{time reversal.} \tag{6}$$

c.) Show that eq. (3) implies for infinitesimal Lorentz transformations of the type (5)

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{7}$$

Use this to argue that the full Poincaré algebra in 4 dimensions is 10-dimensional. Verify that rotations around the 3-axis by an angle θ are given by $\omega_{12} = \theta = -\omega_{21}$ (and all other components vanishing). What do transformations $\omega_{01} = \beta$ (and all other components vanishing) correspond to?

Finite Lorentz transformations are obtained from (5) by exponentiation. This can be written as

$$\Lambda^\rho{}_\tau = \exp\Big(-\frac{i}{4}(M^{\mu\nu})\omega_{\mu\nu}\Big)^\rho{}_\tau, \qquad (M^{\mu\nu})^\rho{}_\tau = 2i\Big(g^{\rho\mu}\delta^\nu{}_\tau - g^{\rho\nu}\delta^\mu{}_\tau\Big). \tag{8}$$

Fact: The matrices $(M^{\mu\nu})^\rho{}_\tau$ satisfy the Lorentz algebra

$$[(M^{\mu\nu}),(M^{\alpha\beta})] = 2i\Big(g^{\mu\beta}M^{\nu\alpha} + g^{\nu\alpha}M^{\mu\beta} - g^{\mu\alpha}M^{\nu\beta} - g^{\nu\beta}M^{\mu\alpha}\Big). \tag{9}$$

Optional: Verify these commutation relations.

2 Action Principle

a.) Consider a scalar field $\varphi(x)$ and its action

$$S[\varphi] = \int d^4x\,\mathcal{L}(\varphi(x),\partial_\mu\varphi(x)). \tag{10}$$

The variation of the action is defined as

$$\delta S = \int d^4x\Big(\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi)\Big). \tag{11}$$

Using integration by parts and neglecting boundary terms show that the action principle $\delta S = 0$ implies the Euler-Lagrange equations

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial\mathcal{L}}{\partial\varphi}. \tag{12}$$

b.) Write down the action for the free scalar field and derive the Klein-Gordon equation as the associated Euler-Lagrange equations.

3 Momentum operator

a.) The 4-momentum operator of the free scalar field $\Phi(x)$ with conjugate momentum $\Pi(x)$ is defined as

$$P^\mu = \int d^3x' \, \Pi(t, \vec{x'}) \, \partial^\mu \, \Phi(t, \vec{x'}). \tag{13}$$

Use the mode expansion for $\Phi(t, \vec{x})$ and $\Pi(t, \vec{x})$ to recast this into the form

$$P^\mu = \frac{1}{2} \int d^3p \frac{1}{(2\pi)^3} \frac{1}{2E_p} p^\mu \Big(a(p) a^\dagger(p) + a^\dagger(p) a(p) \Big), \qquad E_p = +\sqrt{\vec{p}^2 + m^2} = p^0. \tag{14}$$

Hint: $\int d^3x \, e^{i(\vec{p}\cdot\vec{x})} = (2\pi)^3 \delta^{(3)}(\vec{p})$.

b.) Use the canonical commutation relations

$$[a(p), a(p')^\dagger] = (2\pi)^3 \, 2E_p \, \delta^{(3)}(\vec{p} - \vec{p}') \tag{15}$$

to show that

$$[P^\mu, a^\dagger(p)] = p^\mu a^\dagger(p), \qquad [P^\mu, a(p)] = -p^\mu a(p). \tag{16}$$

c.) Starting from (14), use the canonical commutation relations to bring P^μ into the form

$$P^\mu = \int d^3p \frac{1}{(2\pi)^3} \frac{1}{2E_p} \, p^\mu \, \Big(a^\dagger(p) \, a(p) + \frac{1}{2} (2\pi)^3 \, 2E_p \, \delta^{(3)}(0) \Big). \tag{17}$$

Interpret the last term as the divergent vacuum energy.
(Note: What counts in practice is only the energy difference between states. The divergent vacuum energy is renormalised away by using instead the normal ordered momentum operator $: P^\mu :$.)

4 The Feynman Propagator

The Feynman propagator of a scalar field is defined as

$$\begin{aligned} i\Delta_F(x-y) &= \langle 0 | T(\Phi(x)\Phi(y)) | 0 \rangle & (18) \\ &= \theta(x^0 - y^0) \langle 0 | \Phi(x)\Phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \Phi(y)\Phi(x) | 0 \rangle. & (19) \end{aligned}$$

a.) Show that

$$\langle 0 | \Phi(x)\Phi(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)}. \tag{20}$$

Hint: Plug in the mode expansion for $\Phi(x)$ and $\Phi(y)$ and use that $\langle 0|a(p)a^\dagger(p)|0\rangle = \langle 0|[a(p), a^\dagger(p)]|0\rangle$ as well as the commutation relations for the $a(p)$, $a^\dagger(p)$.

Conclude that

$$i\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\vec{p}(\vec{x}-\vec{y})}\Big(\theta(x^0-y^0)e^{-iE_p(x^0-y^0)} + \theta(y^0-x^0)e^{-iE_p(y^0-x^0)}\Big). \quad (21)$$

b.) Now consider the integral

$$I := \int dp^0 \frac{e^{-ip^0(x_0-y_0)}}{(p^0)^2 - \vec{p}^2 - m^2 + i\epsilon} \quad (22)$$

in the limit $\epsilon \to 0$ from above.

Complete the square to show that the integrand vanishes if

$$(p^0)^2 = \Big(\sqrt{\vec{p}^2 + m^2} - \frac{i}{2}\epsilon\Big)^2 + \frac{\epsilon^2}{4}. \quad (23)$$

Draw the two solutions for p^0 in the complex plane, with the real axis identified with the p^0-direction, neglecting the term quadratic in ϵ and taking $\epsilon > 0$. Use this pole structure to convert the integral into a contour integral in the complex plane and show that

$$I = \frac{2\pi}{i}\Big(\frac{1}{2E_p}\theta(x^0-y^0)e^{-iE_p(x^0-y^0)} + \frac{2\pi i}{2E_p}\theta(y^0-x^0)e^{+iE_p(x^0-y^0)}\Big), \quad (24)$$

where $E_p = +\sqrt{\vec{p}^2 + m^2}$.
Hint: The two cases appear because one has to close the contour either in the upper or the lower plane.

Use this result to deduce the famous representation of the Feynman propagator

$$i\Delta_F(x-y) = i\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}. \quad (25)$$