2. Quantum Electrodynamics

Lagrangian for free spin ½ particle:

$$L(\vec{x},t) = i\overline{\psi}(\vec{x},t)\gamma^{\mu}\partial_{\mu}\psi(\vec{x},t) - m\overline{\psi}(\vec{x},t)\psi(\vec{x},t)$$

Applying the Euler-Lagrange formalism leads to the Dirac equation.

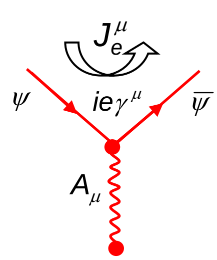
Demanding local phase invariance $\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x)$ leads to:

$$L(\vec{x},t) = \overline{\psi}(\vec{x},t)(i\gamma^{\mu}\partial_{\mu} - m)\psi(\vec{x},t) + e\overline{\psi}(\vec{x},t)\gamma^{\mu}\psi(\vec{x},t)A_{\mu}$$

To interpret the new field A_μ as photon field one has to introduce a term corresponding to the field energy:

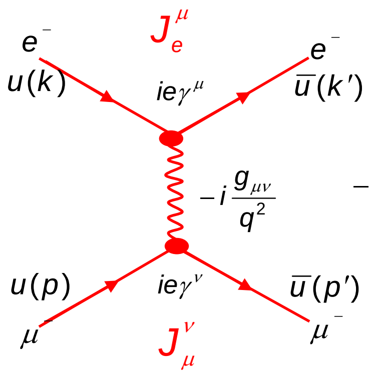
$$L = \overline{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi + e\overline{\psi}\gamma^{\mu}\psi A_{\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$$

$$L = \underbrace{\overline{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi}_{\text{free electron}} + \underbrace{e\overline{\psi}\gamma^{\mu}\psi A_{\mu}}_{\text{Interaction between electron and photon}} - \underbrace{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Photon field energy}}$$



Lagrangian defines the Feynman rules of a theory.

Feynman rules for scattering processes

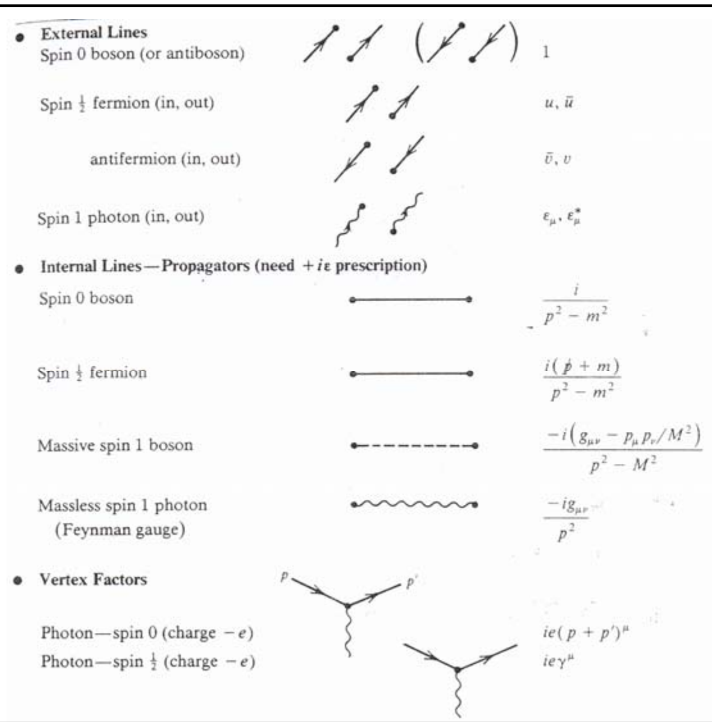


$$e^-\mu^- \rightarrow e^-\mu^-$$

$$-iM = \underbrace{\bar{u}(k')(ie\gamma^\mu)u(k)}_{ie\cdot J_e^\mu} \cdot \frac{-ig_{\mu\nu}}{q^2} \cdot \underbrace{\bar{u}(p')(ie\gamma^\nu)u(p)}_{ie\cdot J_\mu^\nu}$$

$$M = -\frac{e^2}{q^2}\bar{u}(k')\gamma_\mu u(k)\cdot\bar{u}(p')\gamma^\mu u(p)$$

There are similar rules for other Feynman diagrams



- **External Lines**
  - Spin 0 boson (or antiboson) — 1
  - Spin $\frac{1}{2}$ fermion (in, out) — u, $\bar{u}$
  - antifermion (in, out) — $\bar{v}$, v
  - Spin 1 photon (in, out) — ε_μ, ε_μ^*
- **Internal Lines—Propagators (need $+i\varepsilon$ prescription)**
  - Spin 0 boson — $\frac{i}{p^2-m^2}$
  - Spin $\frac{1}{2}$ fermion — $\frac{i(\not{p}+m)}{p^2-m^2}$
  - Massive spin 1 boson — $\frac{-i(g_{\mu\nu}-p_\mu p_\nu/M^2)}{p^2-M^2}$
  - Massless spin 1 photon (Feynman gauge) — $\frac{-ig_{\mu\nu}}{p^2}$
- **Vertex Factors**
  - Photon—spin 0 (charge $-e$) — $ie(p+p')^\mu$
  - Photon—spin $\frac{1}{2}$ (charge $-e$) — $ie\gamma^\mu$

Halzen, Martin: Quarks&Leptons

3. Fermion-fermion scattering

3.1 Process $e^- \mu^- \to e^- \mu^-$

Sect. II.5 ➡
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{|\vec{p}_f|}{|\vec{p}_i|} \cdot |M_{fi}|^2$$

Sect. III.2 ➡
$$M = -\frac{e^2}{q^2} \underbrace{\bar{u}(k')\gamma_\mu u(k) \cdot \bar{u}(p')\gamma^\mu u(p)}_{\text{Spinors decribe a specific spin state of the fermions}}$$

For non-polarized ingoing particles and for non-observation of final state spin one observes unpolarized cross sections $\Rightarrow$ need to **average over possible initial spin states** and **sum over all final spin states**.

$$\overline{|M|^2} = \frac{1}{(2s_e+1)(2s_\mu+1)} \cdot \sum_{s_e, s_\mu} \sum_{s'_e, s'_\mu} |M|^2$$

$$\boxed{\overline{|M|^2}} = \frac{1}{(2s_e+1)(2s_\mu+1)} \cdot \sum_{s_e, s_\mu} \sum_{s'_e, s'_\mu} |M|^2$$

$$= \frac{1}{4} \cdot \frac{e^4}{q^4} \sum_{s_e, s_\mu} \sum_{s'_e, s'_\mu} [\bar{u}(k')\gamma^\mu u(k)][\bar{u}(k')\gamma^\nu u(k)]^* \cdot$$
$$[\bar{u}(p')\gamma_\mu u(p)][\bar{u}(p')\gamma_\nu u(p)]^*$$

$$\boxed{= \frac{e^4}{q^4} L_e^{\mu\nu} \cdot L_{muon,\mu\nu}}$$

Electron tensor $L_e^{\mu\nu} = \frac{1}{2} \sum_{s_e, s'_e} [\bar{u}(k')\gamma^\mu u(k)][\bar{u}(k')\gamma^\nu u(k)]^*$

Muon tensor $L_{muon,\mu\nu} = \frac{1}{2} \sum_{s_\mu, s'_\mu} [\bar{u}(p')\gamma_\mu u(p)][\bar{u}(p')\gamma_\nu u(p)]^*$

Useful relations I:

Completeness relation:

$$\sum_{s=1,2} u_s(p)\bar{u}_s(p) = \not{p} + m \qquad \not{a} = \gamma^\mu a_\mu$$

$$\sum_{s=1,2} v_s(p)\bar{v}_s(p) = \not{p} - m$$

$$\underbrace{\sum_{s_i,s_f} \bar{u}_f\gamma^0 u_i \bar{u}_i \gamma^0 u_f}_{\parallel} = \sum_{s_f} \bar{u}_f \underbrace{\gamma^0(\not{p}_i + m)\gamma^0}_{\text{Matrix A}} u_f = \sum_{j,k=1}^{4}\left\{\sum_{s_f} (\bar{u}_f)_j \mathbf{A}_{jk} (u_f)_k\right\}$$

$$\sum_{s_i,s_f} \left|\bar{u}_f\gamma^0 u_i\right|^2 = \sum_{j,k=1}^{4} \mathbf{A}_{jk}(\not{p}_f + m)_{kj} = \sum_j \mathbf{B}_{jj} = \mathrm{Trace}\,\mathbf{B}$$

Matrix $\mathbf{B} = \mathbf{A}(\not{p}_f + m)$

Useful relations II:

$$\sum_{s_i,s_f} \left|\bar{u}_f\gamma^0 u_i\right|^2 = \mathrm{Trace}\left(\gamma^0(\not{p}_i + m)\gamma^0(\not{p}_f + m)\right)$$

Trace theorems:

$\mathrm{Trace}(\mathbf{I}) = 4$

$\mathrm{Trace}(\text{odd number of } \gamma^\mu) = 0$

$\mathrm{Trace}(\not{a}\not{b}) = 4(ab) = 4a_\mu b^\mu$

$\mathrm{Trace}(\not{a}\not{b}\not{c}\not{d}) = 4(ab)(cd) + 4(ad)(bc) - 4(ac)(bd)$

Electron tensor $L_e^{\mu\nu} = \frac{1}{2}\sum_{s_e, s_e'} [\bar{u}(k')\gamma^{\mu}u(k)][\bar{u}(k')\gamma^{\nu}u(k)]^*$

Muon tensor $L_{\mu\nu}^{\text{Muon}} = \frac{1}{2}\sum_{s_\mu, s_\mu'} [\bar{u}(p')\gamma_{\mu}u(p)][\bar{u}(p')\gamma_{\nu}u(p)]^*$

After a lengthy calculation

Berechnung von $L_e^{\mu\nu}$:

$$\underbrace{(\bar{u}(k')\gamma^\mu u(k))^*}_{} = \text{1x1 Matrix, deswegen } (\ )^* = (\ )^\dagger$$
$$= (\bar{u}\gamma^\mu u)^\dagger = (u^\dagger(k')\gamma^0\gamma^\mu u(k))^\dagger$$
$$= u^\dagger(k)\gamma^{\mu\dagger}\gamma^0 u(k') \leftarrow \gamma^{0\dagger} = \gamma^0$$
$$= u^\dagger(k)\gamma^0\gamma^\mu u(k')$$
$$= \underline{\bar{u}(k)\gamma^\mu u(k')}$$

$$L_e^{\mu\nu} = \frac{1}{2}\sum_{s,s'} \bar{u}(k')\gamma^\mu u(k) \cdot \bar{u}(k)\gamma^\nu u(k')$$

mit $(A\cdot B\cdot C)_{\alpha\beta} = A_{\alpha\gamma} B_{\gamma\delta} C_{\delta\beta}$

$$= \frac{1}{2}\sum_{s,s'} \bar{u}_\alpha(k')\gamma^\mu_{\alpha\beta}u_\beta(k)\bar{u}^{(k)}_\chi\gamma^\nu_{\chi\lambda}u_\lambda(k')$$
$$= \frac{1}{2}\sum_{s'} u_\lambda(k')\bar{u}_\alpha(k')\gamma^\mu_{\alpha\beta}\cdot\sum_s u_\beta(k)\bar{u}_\chi(k)\gamma^\nu_{\chi\lambda}$$

mit Vollständigk.-Relation $\sum_s u\bar{u} = \not{k} + m$

$$= \frac{1}{2}(\not{k}' + m)_{\lambda\alpha}\gamma^\mu_{\alpha\beta}(\not{k} + m)_{\beta\chi}\gamma^\nu_{\chi\lambda}$$
$$= \frac{1}{2}[(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu]_{\lambda\lambda} =$$
$$= \frac{1}{2}\mathrm{Sp}[(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu]$$

Bem.: Über Indizes μ und ν bisher nicht sammier

$$= \frac{1}{2}\mathrm{Sp}[\not{k}'\gamma^\mu\not{k}\gamma^\nu + \not{k}'\gamma^\mu m\gamma^\nu + m\gamma^\mu\not{k}\gamma^\nu + m\gamma^\mu m\gamma^\nu]$$

Spur ungerader Zahl von γ's $= 0$: $\not{a} = a_\mu\gamma^\mu$!

$$= \frac{1}{2}\mathrm{Sp}[\not{k}'\gamma^\mu\not{k}\gamma^\nu + m^2\gamma^\mu\gamma^\nu]$$
$$= \frac{1}{2}\mathrm{Sp}[k'_\alpha\gamma^\alpha\gamma^\mu k_\beta\gamma^\beta\gamma^\nu + m^2\gamma^\mu\gamma^\nu]$$
$$= \frac{1}{2}k'_\alpha k_\beta\cdot\mathrm{Sp}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] + \frac{1}{2}m^2\,\mathrm{Sp}[\gamma^\mu\gamma^\nu]$$
$$= \frac{1}{2}k'_\alpha k_\beta\cdot 4\cdot[g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\mu\beta} - g^{\alpha\beta}g^{\mu\nu}] + \frac{1}{2}m^2\cdot 4g^{\mu\nu}$$
$$= 2\left[k'^\mu k^\nu + k'^\nu k^\mu - [k'\cdot k - m^2]g^{\mu\nu}\right]$$

also

$$\boxed{\begin{aligned} L_e^{\mu\nu} &= 2[k'^\mu k^\nu + k'^\nu k^\mu - [k'\cdot k - m^2]g^{\mu\nu}] \\ L_{\mu\nu}^{\text{Myon}} &= 2[p'_\mu p_\nu + p'_\nu\cdot p_\mu - [p'\cdot p - M^2]g_{\mu\nu}] \end{aligned}}$$

$$L_e^{\mu\nu} = \frac{1}{2}\sum_{s_e,s'_e}[\overline{u}(k')\gamma^{\mu}u(k)][\overline{u}(k')\gamma^{\nu}u(k)]^*$$

$$L_{\mu\nu}^{\text{Muon}} = \frac{1}{2}\sum_{s_\mu,s'_\mu}[\overline{u}(p')\gamma_{\mu}u(p)][\overline{u}(p')\gamma_{\nu}u(p)]^*$$

$$L_e^{\mu\nu} = 2\left(k'^{\mu}k^{\nu} + k'^{\nu}k^{\mu} - (k'\cdot k - m^2)g^{\mu\nu}\right)$$

$$L_{\mu\nu}^{\text{Muon}} = 2\left(p'_{\mu}p_{\nu} + p'_{\nu}p_{\mu} - (p'\cdot p - M^2)g^{\mu\nu}\right)$$

m electron mass
M muon mass

Spin averaged matrix element for $e^-\mu^- \to e^-\mu^-$

$$\overline{|M|^2} = \frac{1}{(2s_e+1)(2s_\mu+1)}\cdot\sum_{s_e,s_\mu}\sum_{s'_e,s'_\mu}|M|^2 \quad = \frac{e^4}{q^4}L_e^{\mu\nu}\cdot L_{\mu\nu}^{\text{Muon}}$$

$$= 8\frac{e^4}{q^4}\left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - m^2 p'\cdot p - M^2 k'\cdot k + 2m^2M^2\right]$$

exact 1st order result for $e^-\mu^- \to e^-\mu^-$

Relativistic limit → neglect masses m and M

$$\overline{|M|^2} = 8\frac{e^4}{(k-k')^4}\left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p')\right] = 2e^4\frac{s^2+u^2}{t^2}$$

By using the Mandelstam variables in the relativistic limit

$$s = (k+p)^2 = m^2 + M^2 + 2kp \approx 2kp \approx 2k'p'$$

$$t = (k-k')^2 = m^2 + M^2 - 2kk' \approx -2kk' \approx -2pp'$$

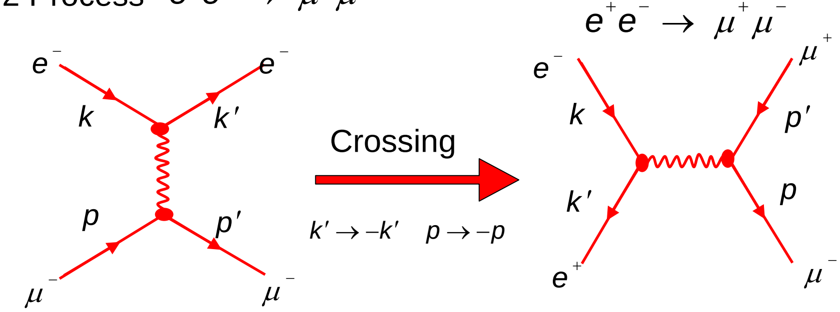
$$u = (k-p')^2 = m^2 + M^2 - 2kp' \approx -2kp' \approx -2k'p$$

Scattering cross section for any two non-identical spin ½ particles:

$$e^- \mu^- \to e^- \mu^-$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{|\vec{p}_f|}{|\vec{p}_i|} \cdot |M_{fi}|^2$$
$$= \frac{\alpha^2}{2s} \cdot \left(\frac{s^2 + u^2}{t^2} \right)$$

3.2 Process $e^+ e^- \to \mu^+ \mu^-$



$$t = (k - k')^2 \quad \to \quad \tilde{s} = (k - k')^2$$
$$s = (k + p)^2 \quad \to \quad \tilde{t} = (k - p)^2$$
$$u = (k - p')^2 \quad \to \quad \tilde{u} = (k - p')^2 = u$$

$$\overline{|M|^2}_{e^-\mu^- \to e^-\mu^-}(s,t,u) \quad = \quad \overline{|M|^2}_{e^+e^- \to \mu^+\mu^-}(\tilde{t},\tilde{s},\tilde{u})$$

$$\overline{|M|^2}_{e^-\mu^- \to e^-\mu^-}(s,t,u) = 2e^4 \frac{s^2 + u^2}{t^2} \quad \Rightarrow \quad \overline{|M|^2}_{e^+e^- \to \mu^+\mu^-}(\tilde{t},\tilde{s},\tilde{u}) = 2e^4 \frac{\tilde{t}^2 + \tilde{u}^2}{\tilde{s}^2}$$

Differential cross section for $e^+e^- \to \mu^+\mu^-$ (CMS)

Reminder:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{s}\cdot\frac{|\vec{p}_f|}{|\vec{p}_i|}\cdot|M_{fi}|^2$$

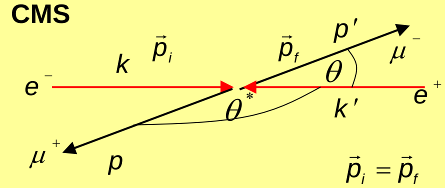
$$\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2}\cdot\frac{1}{s}\cdot\frac{t^2+u^2}{s^2}$$

$$= \frac{e^4}{64\pi^2}\cdot\frac{1}{s}\cdot(1+\cos^2\theta)$$

$e^2 = 4\pi\alpha$

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{CMS} = \frac{\alpha^2}{4s}\cdot(1+\cos^2\theta)}$$

Kinematics for high-relativistic particles



$\vec{p}_i = \vec{p}_f$

$$s = (k+k')^2 \approx 4E_i^2$$

$$t = (k-p)^2 \approx -2kp \approx -2E_i^2(1-\cos\theta^*) \approx -\frac{s}{2}(1+\cos\theta)$$

$$u = (k-p')^2 \approx -2kp' \approx -2E_i^2(1-\cos\theta) \approx -\frac{s}{2}(1-\cos\theta)$$

← 1/s dependence from flux factor

$$\left.\frac{d\sigma}{d\Omega}\right|_{CMS} = \frac{\alpha^2}{4s}\cdot(1+\cos^2\theta)$$

$$\sigma_{tot} = \frac{4\pi\alpha^2}{3s} = \frac{86.86\,\text{nb}\,\text{GeV}^2}{s}$$

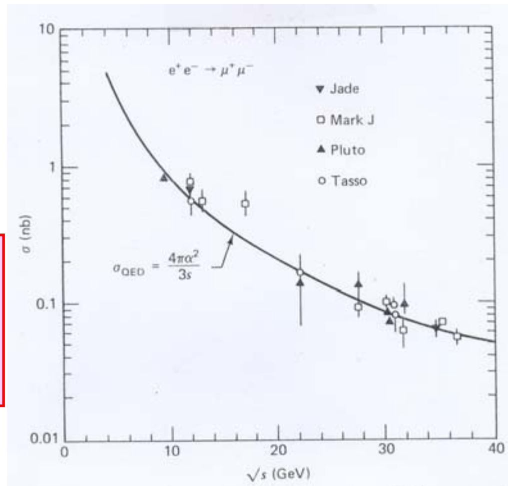


Fig. 6.6 The total cross section for $e^-e^+ \to \mu^-\mu^+$ measured at PETRA versus the center-of-mass energy.

3.3 Chirality, Helicity and angular distribution

Chirality operator:

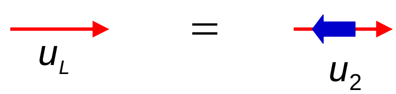
Projection of left- and right-handed components of spinor u

$$\boxed{\begin{aligned} u_L &= \frac{1}{2}(1-\gamma^5)u \\ u_R &= \frac{1}{2}(1+\gamma^5)u \end{aligned}} \qquad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Helicity operator:

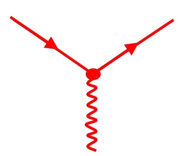
$$\boxed{H = \frac{1}{2}\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}} \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

In the **relativistic limit** (or for massless particles) the eigenstates of the helicity operator corresponds to the chirality states.



$$H = -\frac{1}{2}$$

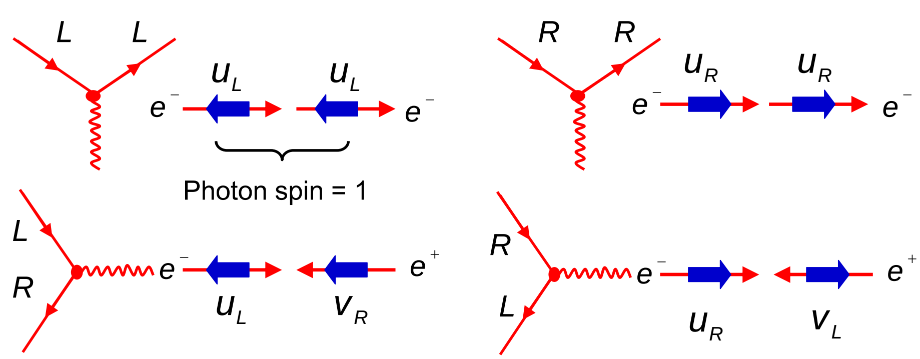
Decomposition of the fermion current:



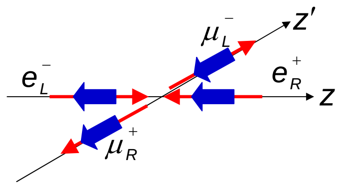
$$\begin{aligned} \bar{u}\gamma^\mu u &= (\bar{u}_R + \bar{u}_L)\gamma^\mu(u_R + u_L) \\ &= \bar{u}_R\gamma^\mu u_R + \bar{u}_L\gamma^\mu u_L \end{aligned}$$

Photon (vector $ie\gamma^\mu$) coupling:

Attention “arrows” correct only for massless electrons



Angular distribution $e^+e^- \to \mu^+\mu^-$



$\boxed{d^J_{m'm}}$

$$\text{Axis } z \xrightarrow{\text{rotation}} \text{Axis } z'$$

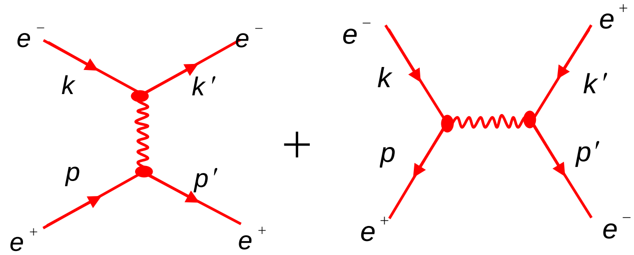
$$\left.\begin{matrix} J=1 \\ m_z=-1 \end{matrix}\right\} \xrightarrow{d^1_{-1-1}} \left\{\begin{matrix} J=1 \\ m_{z'}=-1 \end{matrix}\right.$$

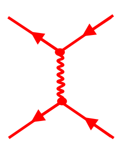
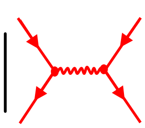
$$\left.\begin{matrix} J=1 \\ m_z=-1 \end{matrix}\right\} \xrightarrow{d^1_{+1-1}} \left\{\begin{matrix} J=1 \\ m_{z'}=+1 \end{matrix}\right.$$

Change of quantization axis

$$\boxed{\frac{d\sigma}{d\Omega} \sim \left(d^1_{-1-1}\right)^2 + \left(d^1_{+1-1}\right)^2 \sim \frac{1}{4}(1+\cos\theta)^2 + \frac{1}{4}(1-\cos\theta)^2 \sim 1+\cos^2\theta}$$

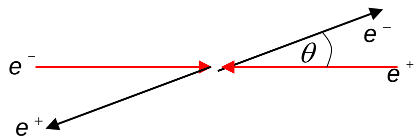
3.3 Bhabha scattering $e^+e^- \to e^+e^-$

$M =$ 

$\overline{|M|^2} = \underbrace{\left| \right.}_{e^-\mu^- \to e^-\mu^-}$  $\left.\right|^2 +$ interference $+ \underbrace{\left| \right.}_{e^+e^- \to \mu^+\mu^-}$  $\left.\right|^2$

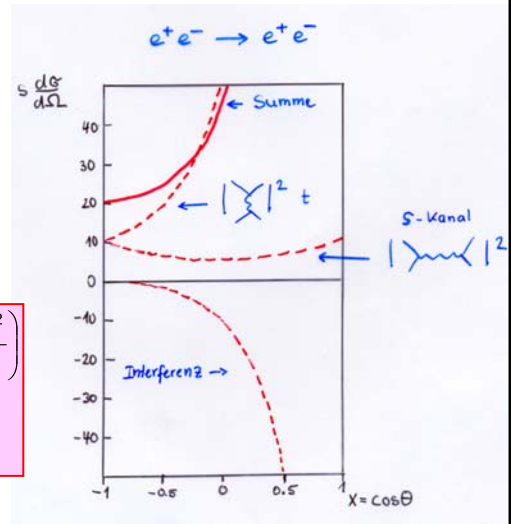
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left(\frac{s^2+u^2}{t^2} + \frac{2u^2}{ts} + \frac{t^2+u^2}{s^2}\right)}$$

CM system:



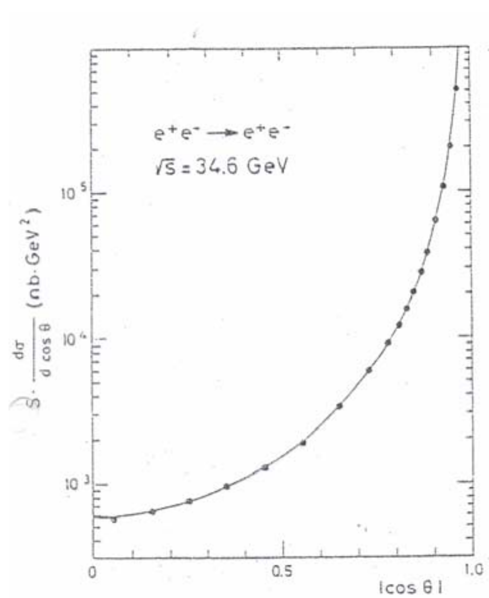
with x=cosθ

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left(\frac{4+(1+x)^2}{(1-x)^2} - \frac{(1+x)^2}{1-x} + \frac{1+x^2}{2}\right)$$
$$= \frac{\alpha^2}{2s}\left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2$$

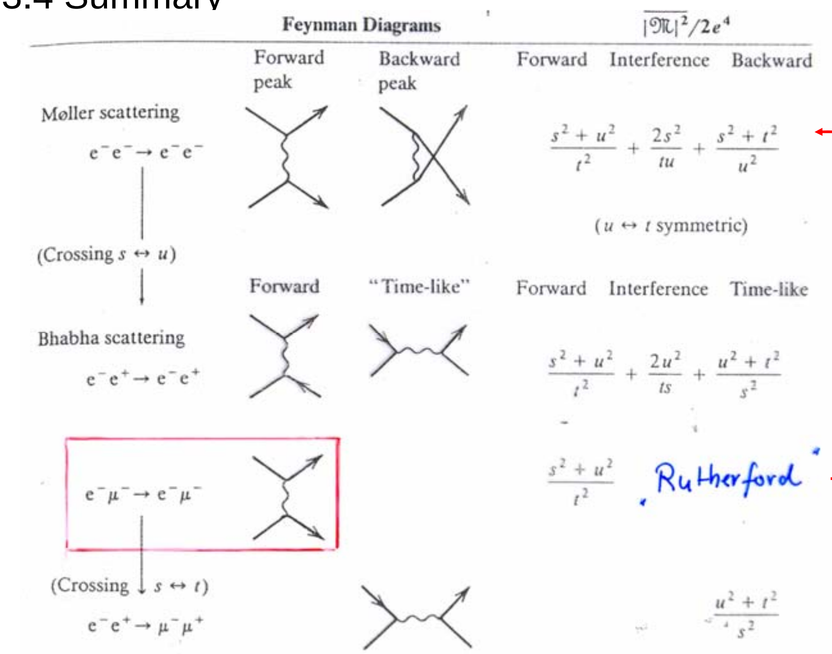


$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2$$

divergent for cosθ→1



3.4 Summary


Feynman Diagrams
$\overline{|\mathcal{M}|^2}/2e^4$
Forward peak
Backward peak
Forward
Interference
Backward
Møller scattering
$e^-e^- \to e^-e^-$
$\frac{s^2+u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2+t^2}{u^2}$
($u \leftrightarrow t$ symmetric)
(Crossing $s \leftrightarrow u$)
Forward
"Time-like"
Forward
Interference
Time-like
Bhabha scattering
$e^-e^+ \to e^-e^+$
$\frac{s^2+u^2}{t^2} + \frac{2u^2}{ts} + \frac{u^2+t^2}{s^2}$
$e^-\mu^- \to e^-\mu^-$
$\frac{s^2+u^2}{t^2}$
Rutherford
(Crossing $s \leftrightarrow t$)
$e^-e^+ \to \mu^-\mu^+$
$\frac{u^2+t^2}{s^2}$