III. QED for "pedestrians"

1. Dirac equation for spin ½ particles
2. Feynman rules
3. Fermion-fermion scattering
4. Higher orders

1. Dirac Equation for spin ½ particles

Idea: Avoid negative energy values using a linear ansatz

" E = p + m "

$E \rightarrow i\frac{\partial}{\partial t}$

$\vec{p} = -\vec{\nabla}$

$$i\frac{\partial}{\partial t}\psi = -i\left(\alpha_1\frac{\partial}{\partial x_1}\psi + \alpha_2\frac{\partial}{\partial x_2}\psi + \alpha_3\frac{\partial}{\partial x_3}\psi\right) + \beta m\psi$$

$$E\psi = (\vec{\alpha}\cdot\vec{p} + \beta\cdot m)\psi$$

Coefficients α_i and β are determined demanding that the free particle solution satisfies the relativistic E-p relation: $E^2\psi = (\vec{p}^2 + m^2)\psi$

Cannot be solved by scalar coefficients:

→ 4×4 martices: $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ σ_i are Pauli matrices

Dirac Equation:

$$i\left(\beta\frac{\partial}{\partial t}\psi+\beta\vec{\alpha}\cdot\vec{\nabla}\psi\right)-m\cdot 1\cdot\psi=0$$

$$i\left(\gamma^{0}\frac{\partial}{\partial t}\psi+\vec{\gamma}\cdot\vec{\nabla}\psi\right)-m\cdot 1\cdot\psi=0$$

where $\gamma^{0}=\beta$ and $\gamma^{i}=\beta\alpha_i$, i = 1,2,3

$$i\gamma^{\mu}\partial_{\mu}\psi-m\psi=0$$

Solutions ψ are four-component spinors.

They describes the fundamental spin ½ particles:

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

Extremely compressed description

$$j=1\ldots4:\quad\sum_{k=1}^{4}\left(\sum_{\mu}i\cdot(\gamma^{\mu})_{jk}\frac{\partial}{\partial x^{\mu}}-m\delta_{jk}\right)\psi_k$$

1.1 γ Matrices

$$\gamma^{0}=\beta$$

$$\gamma^{i}=\beta\alpha_i\ ,\ \text{i}=1,2,3$$

$$\gamma^{5}=i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{2}\gamma^{3}$$

$$\alpha_i=\begin{pmatrix}0&\sigma_i\\ \sigma_i&0\end{pmatrix}\ \text{and}\ \beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\gamma^{0}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\gamma^{i}=\begin{pmatrix}0&\sigma_i\\ -\sigma_i&0\end{pmatrix},\quad \text{i}=1\ldots3$$

$$\gamma^{5}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

Rules

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2g^{\mu\nu},\quad \gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=0\ \text{ for }\mu\neq\nu$$

$$(\gamma^{\mu})^{+}=\gamma^{0}\gamma^{\mu}\gamma^{0},\quad(\gamma^{0})^{+}=\gamma^{0},\quad(\gamma^{k})^{+}=-\gamma^{k}$$

$$\gamma^{0}\gamma^{0}=\mathbf{I},\quad\gamma^{k}\gamma^{k}=-\mathbf{I},\quad k=1..3$$

$$\gamma^{5}\gamma^{\mu}+\gamma^{\mu}\gamma^{5}=0\,,\quad(\gamma^{5})^{+}=\gamma^{5}$$

1.2 Adjoint Equation (hermitian conjugated form)

Dirac eq.: $i\left(\gamma^0 \frac{\partial}{\partial t}\psi + \vec{\gamma}\cdot\vec{\nabla}\psi\right) - m\cdot 1\cdot\psi = 0$

(Dirac eq)$^+$: $i\left(\frac{\partial}{\partial t}\psi^+\gamma^0 + \vec{\nabla}\cdot\psi^+(-\vec{\gamma})\right) + m\cdot 1\cdot\psi^+ = 0$

hermitian conjugate

Introducing the adjoint spinor $\overline{\psi} = \psi^+\gamma^0$ allows to write the hermitian conjugate of the Dirac-Eq. in the covariant form:

$$\overline{\psi}(i\partial_\mu\gamma^\mu + m) = 0$$

$\rightarrow$ can be used to derive a continuity equation for a 4-vector current

1.3 Fermion currents and continuity equation

Define fermion current

$$j^\mu = (\overline{\psi}\gamma^\mu\psi)$$

$$\overline{\psi}\gamma^\mu\partial_\mu\psi + (\partial_\mu\overline{\psi})\gamma^\mu\psi = 0$$

$$\Rightarrow \quad \partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0$$

here: $\rho = j^0 > 0$

$j^0 = \overline{\psi}\gamma^0\psi = \psi^+\gamma^0\gamma^0\psi$

$= \psi^+\psi > 0$

Instead of the probability current the charge current is often used:

Electron current: $j_e^\mu = (-e)\cdot(\overline{\psi}\gamma^\mu\psi)$

Boson current: $j^\mu = (-e)\cdot 2p^\mu$ (reminder: Kl-G-Eq)

1.3 Free particle solutions for Dirac Eq.

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

Ansatz: $\psi(x) = u(p)\cdot \exp(\mp ipx)$ for $E = \pm\sqrt{p^2+m^2}$

With 4-comp spinor: $u(p) = \begin{pmatrix} \varphi(p) \\ \chi(p) \end{pmatrix}$ and φ, χ 2-comp. spinors

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = (\pm\gamma^\mu p_\mu - m)\begin{pmatrix} \varphi \\ \chi \end{pmatrix}\exp(\mp ipx) = 0$$

$$\gamma^\mu p_\mu = \gamma^0 p_0 - \vec{\gamma}\,\vec{p} = E\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma}\,\vec{p} \\ -\vec{\sigma}\,\vec{p} & 0 \end{pmatrix}$$

One obtains two coupled equations for the spinors φ and χ:

$$(E \mp m)\varphi - (\vec{\sigma}\,\vec{p})\chi = 0 \quad (*)$$
$$(E \pm m)\chi - (\vec{\sigma}\,\vec{p})\varphi = 0 \quad (**)$$

Solutions for positive energy: $E = +\sqrt{p^2+m^2}$ (upper sign)

In this case E+m ≥ 2m, while (E-m)→0 for non relativistic case ⇒ use eq. (**)

$$\chi = \frac{\vec{\sigma}\,\vec{p}}{E+m}\varphi \quad \text{with} \quad \vec{\sigma}\,\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

The spinor φ can be freely selected: $\varphi_1 = N\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\varphi_2 = N\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Eq. (*) is automatically fulfilled: $(\vec{\sigma}\,\vec{p})\chi = \frac{(\vec{\sigma}\,\vec{p})^2}{E+m}\varphi = \frac{\vec{p}^2}{E+m}\varphi = (E-m)\varphi$

Solutions for positive energy: $E = +\sqrt{p^2 + m^2}$

solution spin ↑

$$u_1(p) = N \cdot \begin{pmatrix} \varphi_1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi_1 \end{pmatrix} = N \cdot \begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E+m} \\ \dfrac{p_x + ip_y}{E+m} \end{pmatrix}$$

solution spin ↓

$$u_2(p) = N \cdot \begin{pmatrix} \varphi_2 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi_2 \end{pmatrix} = N \cdot \begin{pmatrix} 0 \\ 1 \\ \dfrac{p_x - ip_y}{E+m} \\ \dfrac{-p_z}{E+m} \end{pmatrix}$$

$N = \sqrt{E+m}$ (norm.)

$$u^+ u = 2E$$

1.4 Spin and helicity

- u_1 and u_2 are both solutions to the same energy value
- operator to distinguish the 2 solutions:

Helicity $$\frac{1}{2}\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|} = \frac{1}{2}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} & 0 \\ 0 & \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \end{pmatrix}$$

$$\left[\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}, H\right] = 0$$

$$H = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & m \end{pmatrix}$$

- u_1 and u_2 are eigenstates of $\frac{1}{2}\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$ with eigenvalues ±½
- u_1 and u_2 for highly relativistic particles $u_1 = \sqrt{E}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ and $u_2 = \sqrt{E}\cdot\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$

  $E \approx p_z$

  $p_x, p_y \approx 0$

1.5 Solutions for negative energies $E = -\sqrt{p^2 + m^2}$

$\Rightarrow \phi = \frac{\vec{\sigma}\cdot\vec{p}}{E - m}\chi$ and using $\chi_1 = N \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_2 = N \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

solution spin ↑

$$u_3(p) = N \cdot \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi_1 \\ \chi_1 \end{pmatrix} = N \cdot \begin{pmatrix} \frac{-p_z}{|E|+m} \\ \frac{-p_x - ip_y}{|E|+m} \\ 1 \\ 0 \end{pmatrix}$$

solution spin ↓

$$u_4(p) = N \cdot \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi_2 \\ \chi_2 \end{pmatrix} = N \cdot \begin{pmatrix} \frac{-p_x + ip_y}{|E|+m} \\ \frac{+p_z}{|E|+m} \\ 0 \\ 1 \end{pmatrix}$$

Particles: $E < 0, \vec{p}$

$u_3^{\uparrow}, u_4^{\downarrow}$

Anti-particles: $E > 0, -\vec{p}$

$v_2^{\downarrow}, v_1^{\uparrow}$

$$v_2(p) = N \cdot \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_1 \\ \chi_1 \end{pmatrix} = N \cdot \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

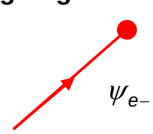
solution spin ↓

$$v_1(p) = N \cdot \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_2 \\ \chi_2 \end{pmatrix} = N \cdot \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

solution spin ↑ $N = \sqrt{E+m}$

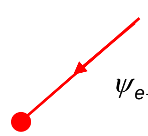
In- and out-going (anti)-particles

In-going electron:



$$\psi_{e-}(x) = u_{1,2}(p)\cdot\exp(-ip\cdot x) = u_{1,2}(p)\cdot\exp(-iEt)\exp(+i\vec{p}\cdot\vec{x})$$

Out-going positron:



$$\psi_{e+}(x) = v_{1,2}(p)\cdot\exp(+ip\cdot x) = v_{1,2}(p)\cdot\exp(+iEt)\exp(-i\vec{p}\cdot\vec{x})$$

To describe **out-going electrons** or **in-going positrons** the adjoint spinors $\overline{\psi}_{e-} = \psi_{e-}\gamma^0$ or $\overline{\psi}_{e+} = \psi_{e+}\gamma^0$ and $\overline{u}_{1,2}$ or $\overline{v}_{1,2}$ are used.