II. Pre-requisites

1. Relativistic kinematics
2. Wave description of free particles
3. Scattering matrix and transition amplitudes
4. Cross section and phase space
5. Decay width, lifetimes and Dalitz plots

1. Relativistic kinematics

1.1 Notations

- 4-vector
  - contra-variant form $x^\mu = (x^0, \vec{x}) = (t, \vec{x}) \quad p^\mu = (p^0, \vec{p}) = (E, \vec{p})$
  - covariant form $x_\mu = (x^0, -\vec{x}) = (t, -\vec{x}) \quad p_\mu = (p^0, -\vec{p}) = (E, -\vec{p})$
- Metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad x_\mu = g_{\mu\nu} x^\nu \quad x^\mu = g^{\mu\nu} x_\nu$$

- Derivative operator

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

- Scalar product

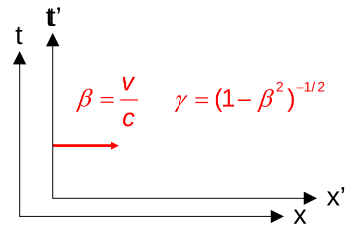
$$ab = a_\mu b^\mu = g_{\mu\nu} a^\nu b^\mu = (a^0 b^0 - \vec{a} \cdot \vec{b})$$

1.2 Lorentz invariants

<u>Lorentz transformation:</u>

moving particle with $p=(E,\vec{p})$

$$p' = \begin{cases} \begin{pmatrix} E' \\ \vec{p}_l' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E \\ \vec{p}_l \end{pmatrix} \\ \vec{p}_t' = \vec{p}_t \end{cases}$$



$\beta = \frac{v}{c}$ $\quad \gamma = (1-\beta^2)^{-1/2}$

w/r to rest frame: $\beta = \frac{|\vec{p}|}{E}$ $\quad \gamma = \frac{E}{m}$

Scalar products are invariant under Lorentz transformations: $a'b' = ab$

<u>Example 1:</u> invariant mass

$p^2 = p_\mu p^\mu = E^2 - \vec{p}^2 = m^2$

<u>Example 2:</u> center-of-mass energy of 2 particle collision

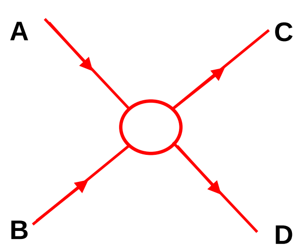
$s = (p_1+p_2)^2 = (E_1+E_2)^2 - (\vec{p}_1+\vec{p}_2)^2$

Lorentz scalars can only by functions of other Lorentz invariants (scalars).

Examples of Lorentz invariants: $\frac{1}{E}\frac{d\sigma}{d^3p}$ and $E \cdot \Gamma$ Lorentz invariant cross section / decay width

1.3 Mandelstam variables

$$A + B \to C + D$$



What are the Lorentz scalars the cross section can depend on ?

$p_i p_k$ with $p_{i,k\geq i} = p_A, p_B, p_C, p_D$

(unpolarized particles)

10 combinations

$p_i^2 = m_i^2$ — 4 constraints

4-mom. conservation: 4 constraints

➡ **2 independent products**

Instead of $p_i p_k$ use 2 out of the 3 **Mandelstam variables**

$s = (p_A + p_B)^2$

$t = (p_A - p_C)^2$

$u = (p_A - p_D)^2$

$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$

2. Wave description of free particles

2.1 Schrödinger Equation for non-relativistic free particles

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi$$

Solution for energy $E = \frac{p^2}{2m}$

$$\psi(\vec{r},t) = \frac{1}{\sqrt{V}}\exp\left[i(\vec{p}\vec{x} - Et)\right]$$

Continuity equation:

$$\rho = |\psi|^2$$

$$\vec{j} = \frac{1}{2im}\left(\psi^*(\nabla\psi) - (\nabla\psi^*)\psi\right)$$

$$\frac{\partial\rho}{\partial t} + \nabla\vec{j} = 0$$

Schrödinger Eq uses classical E-p relation $E^2=p^2/2m$ and the replacement $E \rightarrow i\frac{\partial}{\partial t}$ and $\vec{p} = -\vec{\nabla}$

2.2 Klein-Gordon Equation

Starts from relativistic energy relation $E^2=p^2+m^2$:

Describes relativistic Spin 0 particles

$$\frac{\partial^2}{\partial t^2}\phi - \nabla^2\phi + m^2\phi = 0$$

Solutions for energy values:

$$E_\pm = \pm\sqrt{p^2+m^2} \gtrless 0$$

$$\phi(\vec{r},t) = N\exp\left[i(\vec{p}\vec{x} - E_\pm t)\right]$$

negative E values cannot be ignored as otherwise solutions are incomplete

with $\rho = \left(i\phi^*\frac{\partial}{\partial t}\phi - i\phi\frac{\partial}{\partial t}\phi^*\right)$ and $\vec{j} = \left(-i\phi^*\vec{\nabla}\phi - i\phi\vec{\nabla}\phi^*\right)$

Continuity equation: $\frac{\partial\rho}{\partial t} + \nabla\vec{j} = 0$

For the solution: $\phi(\vec{r},t) = N \exp\left[i(\vec{p}\vec{x} - E_{\pm}t)\right]$

$$\vec{j} = \left(-i\phi^{*}\vec{\nabla}\phi - i\phi\vec{\nabla}\phi^{*}\right) \qquad \vec{j} = 2\vec{p}|N|^2$$

$$\rho = \left(i\phi^{*}\frac{\partial}{\partial t}\phi - i\phi\frac{\partial}{\partial t}\phi^{*}\right) \qquad \rho = 2E|N|^2$$

What are negative probabilities for the E < 0 solutions ?

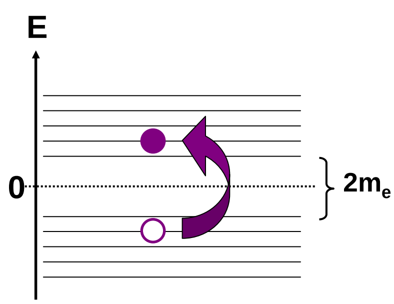
Normalization schemes:

N = 1/√(2EV) ⇒ 1 particle per unit volume V

N = 1/√V ⇒ 2E particles per unit volume V

2.3 Anti-particles

Dirac interpretation for fermions: **Vacuum = sea of occupied neg. E levels**



For fermions the negative energy levels are w/o influence as long as they are fully occupied

Missing e^- w/ negative energy corresponds to to a positron w/ E>0

<u>e^+e^- annihilation:</u>

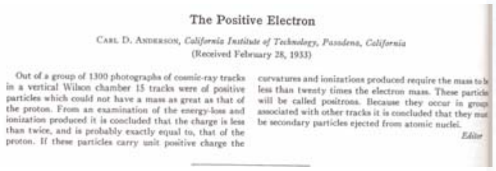
Free energy level in the sea. e^- drops into the hole and releases energy by photon emission: $E_\gamma > 2m_e$

<u>Photon conversion for $E_\gamma > 2m_e$</u>

Excitation of e^- from neg. energy level to pos. level: $\gamma \rightarrow e^+e^-$

Model predicts anti-particles (Discovery of positron by Anderson in 1933)

Discovery of positron



The Positive Electron

CARL D. ANDERSON, *California Institute of Technology, Pasadena, California*
(Received February 28, 1933)

Out of a group of 1300 photographs of cosmic-ray tracks in a vertical Wilson chamber 15 tracks were of positive particles which could not have a mass as great as that of the proton. From an examination of the energy-loss and ionization produced it is concluded that the charge is less than twice, and is probably exactly equal to, that of the proton. If these particles carry unit positive charge the curvatures and ionizations produced require the mass to be less than twenty times the electron mass. These particles will be called positrons. Because they occur in groups associated with other tracks it is concluded that they must be secondary particles ejected from atomic nuclei.

Editor

CARL D. ANDERSON

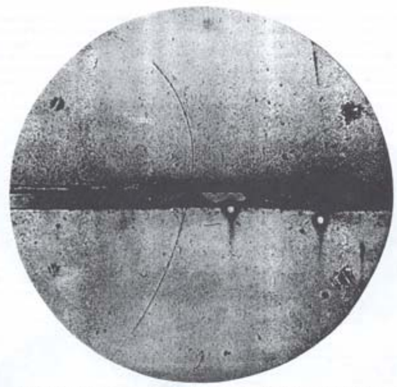


FIG. 1. A 63 million volt positron ($H\rho = 2.1 \times 10^5$ gauss-cm) passing through a 6 mm lead plate and emerging as a 23 million volt positron ($H\rho = 7.5 \times 10^4$ gauss-cm). The length of this latter path is at least ten times greater than the possible length of a proton path of this curvature.

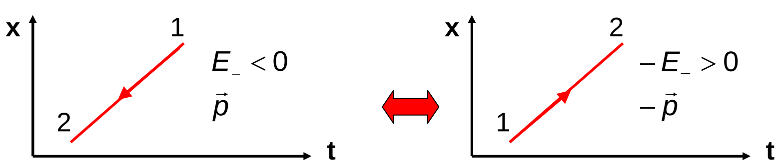
Feynman Stückelberg interpretation

Solutions with neg. energy propagate backwards in time:

$$E_+ = E: \quad \phi_+ = \frac{1}{\sqrt{2E}} \exp(i\vec{p}\vec{x} - iEt)$$

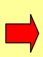
$$E_- = -E: \quad \phi_- = \frac{1}{\sqrt{2E}} \exp(i\vec{p}\vec{x} + iEt)$$

Solutions describe anti-particles propagating forward in time:



Neg. probability density

$$\left.\begin{aligned} \rho &= 2E|N|^2 \\ \vec{j} &= 2\vec{p}|N|^2 \end{aligned}\right\} \times q$$



$$J^0 = q \cdot 2E|N|^2$$
$$\vec{J} = q \cdot 2\vec{p}|N|^2$$

Charge density / currents

Example

Particle T^- with q= –e and energy E_ = –E < 0

$$J^0(T^-) = (-e)\cdot 2(-E)|N|^2 = (+e)\cdot 2(+E)|N|^2 = J^0(T^+)$$

$$\vec{J}(T^-) = (-e)\cdot 2\vec{p}|N|^2 = (+e)\cdot 2(-\vec{p})|N|^2 = \underbrace{\vec{J}(T^+)}$$

$$T^+ \text{ with } E(T^+) > 0,\ \vec{p}_{T^+} = -\vec{p}_{T^-}$$

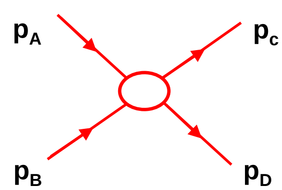
Description of creation and annihilation:

- Emission of anti-particle $\bar{T}$ with p^μ = (E, **p**) ⇔ absorption of particle T with p^μ = (-E, **-p**)
- Absorption of anti-particle $\bar{T}$ with p^μ = (E, **p**) ⇔ emission of T with p^μ = (-E, **-p**)

3. Scattering matrix and transition amplitude

Scattering process:

$$\pi p \rightarrow \pi p$$



Described through quantum numbers of initial and final state: $|i\rangle \rightarrow |i'\rangle$

Scattering operator (S matrix): $|i'\rangle = \mathbf{S}|i\rangle$

Measurement selects a specific state f. Probability to find f: $\langle f|i'\rangle = \langle f|\mathbf{S}|i\rangle = \mathbf{S}_{fi}$

As there is the probability that $|i'\rangle = |i\rangle$ it is useful to introduce the transition operator T $\quad \mathbf{S} = \mathbf{1} + \mathbf{T}$ with $T_{fi} = \langle f|\mathbf{T}|i\rangle$

Instead of T_{fi} , conventionally one uses the transition or scattering amplitude M_{fi}

$$T_{fi} = -i \cdot (2\pi)^4 \underbrace{N_A N_B N_C N_D}_{\text{normalization}} \underbrace{\delta^4(p_A + p_B - p_C - p_D)}_{\text{4-momentum conservation}} \cdot M_{fi}$$

normalization: $N_k = 1/\sqrt{V} \to$ 2E particles/V

4-momentum conservation

M_{fi}: Feynman rules for calculation

Transition rate per unit volume:

$$W_{fi} = \frac{|T_{fi}|^2}{T \cdot V} = \frac{1}{T \cdot V}(2\pi)^8 (N_A N_B N_C N_D)^2 [\delta^4(p_A + p_B - p_C - p_D)]^2 |M_{fi}|^2$$

$$[\delta^4(p_A + p_B - p_C - p_D)]^2 = \frac{VT}{(2\pi)^4}\delta^4(p_A + p_B - p_C - p_D)$$

$$= (2\pi)^4 \frac{1}{V^4}\delta^4(p_A + p_B - p_C - p_D)|M_{fi}|^2$$

Unitarity of S-Matrix

S matrix or S operator is unitary:

$$\mathbf{S}\mathbf{S}^+ = \mathbf{S}^+\mathbf{S} = \mathbf{1}$$

$$\mathbf{S} = \mathbf{S}^{-1}$$

Where the matrix elements of the adjoint operator are defined in the usual way

$$S_{fi}^* = S_{if}^+$$

One finds for the states |i'>

$$\langle i' | i' \rangle = \langle i | \mathbf{S}^+\mathbf{S} | i \rangle = \langle i | i \rangle$$

Unitarity:

Conservation of the probability in the scattering process:

What goes in, should also go out !!