III-2 The electromagnetic field

Maxwell's equations:

$$\partial_\mu F^{\mu\nu}(x) = j^\nu(x) \qquad (3.26a)$$

$$\varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu F_{\rho\sigma}(x) = 0 \qquad (3.26b)$$

with field strength

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \qquad (3.27)$$

and 4-vector potential $A_\mu(x)$.

(3.27) ~~trivially~~ satisfies (3.26b): $2\,\varepsilon^{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma = 0$

$j^\nu(x)$ is the 4-vector current density

$$j^\nu(x) = \begin{pmatrix} \rho(x) \\ \vec{j}(x) \end{pmatrix} \qquad (3.28)$$

and

$$F^{\mu\nu}(x) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \qquad (3.29)$$

Remarks:

(1) Inhomogeneous Maxwell eq. (3.26a):

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu = 0 \qquad (3.32)$$

$\Rightarrow$ conserved current!

(2) A^μ carries a redundancy, the gauge degrees of freedom: $F^{\mu\nu}$ is invariant under

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \alpha(x) \qquad (3.33)$$

$$\Rightarrow F^{\mu\nu} \to F^{\mu\nu} + [\partial^\mu, \partial^\nu]\alpha = F^{\mu\nu}$$

This redundancy can be removed by imposing a constraint on A_μ (gauge fixing condition):

Lorentz gauge (Landau): $\alpha(x): \partial_\mu \partial^\mu \alpha = -\partial_\mu A'^\mu$

$$A^{\mu'} \xrightarrow{\alpha} A^\mu = A^{\mu'} + \partial^\mu \alpha : \boxed{\partial_\mu A^\mu(x) = 0} \qquad (3.34)$$

$$\Rightarrow \partial_\mu F^{\mu\nu} = \Box A^\nu = j^\nu$$

consistent with (3.32). For $j^\nu = 0$, each component A^ν satisfies the KG-eq.

(3) We use Heaviside units (rational), that is removing factors of $\sqrt{4\pi}$ from the eq.: (3.26) reads with (3.28)/(3.29):

$$\vec{\nabla}\cdot\vec{E} = \rho \qquad \vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{j} \qquad (3.26a) \qquad (3.30)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad (3.26b)$$

Relation to cgs units: $\alpha = \frac{e_H^2}{4\pi}\frac{1}{\hbar c} \simeq \frac{1}{137}$

$$e_H = \sqrt{4\pi}\, e_{cgs}$$

$$\Rightarrow \boxed{e_{cgs}^2 = \frac{e_H^2}{4\pi}} \left(= \frac{e_{SI}^2}{4\pi\varepsilon_0}\right) \qquad (3.31)$$

$$\vec{E}_H = \frac{\vec{E}_{cgs}}{\sqrt{4\pi}}$$

$$\vec{B}_H = \frac{\vec{B}_{cgs}}{\sqrt{4\pi}}$$

$e_{cgs}^2 = (4.8\cdot 10^{-10})^2\ g\cdot cm^3/s^2 \qquad c = 3\cdot 10^{10}\ cm/s$

$\hbar_{cgs} = 1.05\cdot 10^{-27}\ erg\cdot s \qquad erg = g\ cm^2/s^2\ (= 10^{-7}\ J)$

Lagrangian density

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) - A_\mu(x) j^\mu(x) \qquad (3.32)$$

Quantisation of free field

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k_0} \left[e^{ikx} a_\mu^+(\vec{k}) + e^{-ikx} a_\mu(\vec{k}) \right] \qquad (3.33a)$$

with $k_0 = |\vec{k}|$, and the commutators

$$[a_\nu(\vec{k}'), a_\mu^+(\vec{k})] = -g_{\mu\nu} (2\pi)^3 2k_0 \, \delta(\vec{k}-\vec{k}')$$

$$[a_\nu(\vec{k}), a_\mu(\vec{k}')] = 0 = [a_\nu^+(\vec{k}), a_\mu^+(k')] \qquad (3.33b)$$

Remarks:

We have KG-eq.:

$$\Box A_\mu(x) = 0 \qquad (3.34)$$

but $\partial_\mu A_\mu(x) \simeq \int ik_\mu a_\mu^+ \ldots \neq 0$ $\quad \left(k_\mu [a_\nu(\vec{k}), a_\mu^+(\vec{k})] = -k_\nu (2\pi)^3 2k_0 \delta(\vec{k}-\vec{k}') \right)$

$$\Rightarrow \partial_\mu F^{\mu\nu}[A] \neq 0 \qquad (3.35)$$

Can we do better than (3.33)? No, it was not possible to construct A_{op} with $\partial_\mu A_{op,\mu} = 0$ +

(2) Fock space:

- vacuum $|0\rangle$ with $\langle 0|0\rangle = 1$

$$a_\nu(\vec{k})|0\rangle = 0 \qquad (3.36)$$

- one particle states

$$a_\nu^+(\vec{k})|0\rangle$$

with norm

$$\langle 0|a_\nu(\vec{k}')\, a_\nu^+(\vec{k})|0\rangle$$
$$= \langle 0|[a_\nu(\vec{k}'), a_\nu^+(\vec{k})]|0\rangle \qquad (3.37)$$
$$= -g_{\mu\nu}(2\pi)^3\, 2k_0\, \delta(\vec{k}-\vec{k}')$$

$\Rightarrow$ $\mu = \nu = i$: positive norm states

$\mu = \nu = 0$: <u>negative</u> norm states

$\Rightarrow$ no physical Hilbert space (no prob. interpret.)