V Quantum Chromodynamics (QCD)

The theory of strong interactions provides the nuclear forces that keep nuclear cores together.

Peculiar properties: $\alpha_S = g_S^2/4\pi$

1) asympt. freedom: $\alpha_S(p^2 \to \infty) \to 0$ (Nobel Prize '04)
Gross, Wilczek, Politzer

2) confinement: $V_{q\bar{q}}(r) \sim r$
$q \equiv \bar{q}$ for large distances
Millenium Prize riddle
(Jaffe, Witten)

(3) self interaction of gauge fields

gauge fields are <u>colour</u> charged

Evidence for $\pm 1/3\, e$, $\pm 2/3\, e$ charged hadronic constituents with spin 1/2: quarks q
(J. Joyce)
Fin. wake

gluonic jets

'64 Gell-Mann, Zweig
Nobel Prize '69

5.1 The QCD - Lagrangian

Hadronic current:

$$j_\mu(x) = e \sum_q Q_q \, \bar{q}(x) \gamma^\mu q(x)$$

with $Q_{u,c,t} = 2/3$, $Q_{d,s,b} = -1/3$

Hadronic states are invariant under $SU(3)$ transformations in colour space:

$$q(x) \to \mathcal{U} \, q(x) \qquad \partial_\mu \mathcal{U} = 0$$

$$(q_\alpha(x) \to \mathcal{U}_{\alpha\beta} \, q_\beta(x) \qquad \alpha, \beta = 1,2,3)$$

with

$$\mathcal{U} \in \underset{\substack{\uparrow \\ \text{non-Abelian}}}{SU(3)} : \quad \mathcal{U}^+\mathcal{U} = \mathcal{U}^+\mathcal{U} = \mathbb{1}_3 , \quad \det \mathcal{U} = 1$$

Infinitesimal $\mathcal{U} = \mathbb{1}_3 + i \, \delta\varphi^a \lambda^a$, $a = 1, \ldots, 8$

λ^a : generators of $su(3)$ (Lie alg der $SU(3)$)

with $\mathrm{tr}\,\lambda = 0$, $\lambda^+ = \lambda$

[compare to σ^a : generators of $su(2)$ (weak interactions,

$$[\sigma^a, \sigma^b] = 2i\varepsilon^{abc}\sigma^c$$]

Quarks:

$$q(x) = \begin{pmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{pmatrix}$$

q_i Dirac spinors

$i = 1,2,3$ colour indices

Flavours:

$$q = \begin{array}{cccc} 1.5-4 & 1150-1350 & & \\ u & c & t^{170} & +2/3 \\ d & s & b & -1/3 \\ 4-8 & 80-130 & 4.1-4.4 & \end{array}$$

current quark masses

Lagrangian:

$$\mathcal{L}^{(0)}_{quark}(x) = \sum_q \bar{q}(x)\left(i\not\partial - m_q\right) q(x)$$

Bound states:

(1) Mesons: $q\bar{q}$

(2) Baryons: qqq

e.g.: $\pi^+ \sim u_1\bar{d}_1 + u_2\bar{d}_2 + u_3\bar{d}_3$

$P \sim \varepsilon_{\alpha\beta\gamma}\, u_\alpha u_\beta d_\gamma$

$\left(\to \varepsilon_{\alpha\beta\gamma}\, u_\alpha u_\beta d_\gamma \det U\right)$

gauge inv.

$$[\lambda^a, \lambda^b] = 2i f^{abc} \lambda^c$$

← structure constants

$$\mathrm{Tr}\,\lambda = 0$$

$$\mathrm{Tr}\,\lambda^a \lambda^b = 2\delta^{ab}$$

$$\{\lambda^a, \lambda^b\} = 4/3\, \delta^{ab} + 2 d^{abc} \lambda^c$$

$\sim \mathrm{Tr}\,\lambda^a\{\lambda^b \lambda^c\}$

see 4-4a

The Lagrangian $\mathcal{L}^{(0)}_{quark}(x)$ is invariant under

$$q \to \mathcal{U} q$$
$$\bar{q} \to \bar{q}\, \mathcal{U}^\dagger$$

$\mathcal{U}^\dagger \mathcal{U} = \mathbb{1}_3$

$$\mathcal{L}^{(0)}_{quark}(x) \to \sum_q \bar{q}\, \mathcal{U}^\dagger (i\not{\partial} - m)\, \mathcal{U} q = \mathcal{L}^{(0)}_{quark}(x)$$

Gauge principle (as in QED):
[and in electroweak theory]

We demand

$$\left.\begin{array}{l} q(x) \to \mathcal{U}(x)\, q(x) \\ \bar{q}(x) \to \bar{q}(x)\, \mathcal{U}^\dagger(x) \end{array}\right\} \mathcal{L}_{quark}(x) \to \mathcal{L}_{quark}(x)$$

but $\mathcal{L}^{(0)}_{quark} \to \mathcal{L}^{0}_{quark} + \sum_q \bar{q}\, (\mathcal{U}^\dagger \not{\partial} \mathcal{U})\, q$

$$\not{\partial} \delta^{\alpha\beta} \to \gamma^\mu D_\mu^{\alpha\beta}$$

with $D_\mu^{\alpha\beta} = \partial_\mu \delta^{\alpha\beta} + i g_s A_\mu^{\alpha\beta}$

see p. 3-10

SU(3):

$$[t^a, t^b] = i f^{abc} t^c$$

with

$$t^c = \frac{1}{2}\lambda^c$$

f^{abc} anti-symmetric

$$f^{123} = 1, \quad f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = 1/2$$

$$f^{458} = f^{678} = \sqrt{3}/2$$

and

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = 1/\sqrt{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

quark color charge: $t^a t^a = C_F \mathbb{1}$ with $C_F = 4/3$

gluon colour charge: $f^{abc} f^{abd} = C_A \delta^{cd}$ with $C_A = 3$

$$\left(= \mathrm{tr}_{adj.}\, t^c t^d = -C_A \delta^{cd}\right)$$

$$\left(\mathrm{tr}_{fund.}\, t^c t^d = 1/2\, \delta^{cd}\right)$$

Transformation of $A_\mu^{\alpha\beta} = A_\mu^a \underbrace{(\lambda^a/2)}_{t^a}{}^{\alpha\beta}$

$$A_\mu(x) \longrightarrow \mathcal{U}(x) A_\mu(x) \mathcal{U}^\dagger(x) - i/g_s \, \mathcal{U}(x) \partial_\mu \mathcal{U}^\dagger(x)$$

$$= -\frac{i}{g_s} \mathcal{U}(x) D_\mu \mathcal{U}^\dagger(x)$$

$$\Rightarrow \quad D_\mu(x) \rightarrow \mathcal{U}(x) D_\mu \mathcal{U}^\dagger(x)$$

As in QED we define the field strength

$$i g_s F_{\mu\nu} =: [D_\mu, D_\nu] = i g_s (\partial_\mu A_\nu - \partial_\nu A_\mu + i g_s [A_\mu, A_\nu])$$

with

$$F_{\mu\nu} \rightarrow \mathcal{U} F_{\mu\nu} \mathcal{U}^\dagger$$

and $\quad [A_\mu, A_\nu] = i f^{abc} A^b A^c \lambda^a$

or $\quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s f^{abc} A^b A^c$

Pure gauge theory: Yang-Mills

$$\mathcal{L}_{YM}(x) = -1/2 \operatorname{tr} F_{\mu\nu} F^{\mu\nu}$$
$$= -1/4 F_{\mu\nu}^a F^{a\,\mu\nu}$$

Full Lagrangian:

$$\mathcal{L}(x) = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_q \bar{q}(i\not{D} - m_q)\, q$$

with gauge symmetry $\mathcal{U} \in SU(3)$

$$q \to \mathcal{U}\, q$$

$$\bar{q} \to \bar{q}\, \mathcal{U}^+$$

$$A_\mu \to \mathcal{U} A_\mu \mathcal{U}^+ - i/g_s\, \mathcal{U}\, \partial_\mu \mathcal{U}^+$$

$$\Rightarrow \mathcal{L}(x) \to \mathcal{L}(x)$$

with $\quad -\frac{1}{2}\,\mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} \to -\frac{1}{2}\,\mathrm{tr}\, \mathcal{U} F_{\mu\nu} \mathcal{U}\, \mathcal{U}^+ F^{\mu\nu} \mathcal{U}^+$

$$= -\frac{1}{2}\,\mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}$$

Parameters: g_s or $\alpha_s = g_s^2/4\pi$ strong fine structure constant

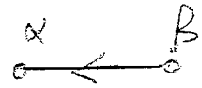
$m_{u,d,s,c,b,t}$ quark masses

gauge fixing (Lorentz)

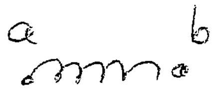
$$\int \mathcal{L}(x) \to \int \mathcal{L}(x) + \frac{1}{2\xi} \int \mathrm{tr}(\partial_\mu A^\mu)^2 + \underbrace{\int \bar{c}\, \partial_\mu D^\mu c}_{\text{ghosts}}$$

Feynman rules for QCD

gauge fixing: $\frac{1}{2\xi}\int(\partial_\nu A_\nu)^2$

quark propagator: 

: $i\,\frac{\not p + m}{p^2-m^2+i\varepsilon}\,\delta^{\alpha\beta}$

gluon propagator: 

: $-i\,\frac{1}{k^2+i\varepsilon}\left(g_{\mu\nu}\left[-\left(1-\frac{1}{\xi}\right)\frac{k_\mu k_\nu}{k^2+i\varepsilon}\right]\right)$

[ghost propagator: 

: $+i\,\frac{1}{p^2+i\varepsilon}$] ← gauge fixing

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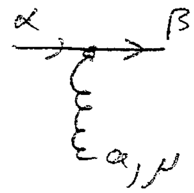
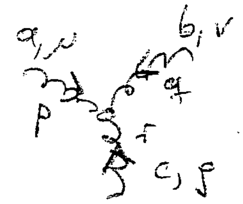
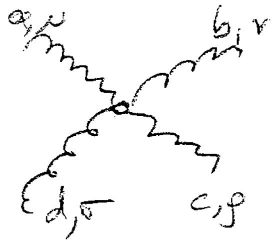
quark - gluon vertex: 

: $-i g_s (t^a)_{\alpha\beta}\,\gamma^\mu$

3 - gluon vertex: 

$$-g_s f^{abc}\left[g^{\mu\nu}(p-q)^\rho - g^{\nu\rho}(q-r)^\mu + g^{\rho\mu}(r-p)^\nu\right]$$

4 - gluon vertex: 

$$-ig_s^2\Big[f^{ead}f^{ebc}\left(g^{\mu\nu}g^{\sigma\rho}-g^{\mu\rho}g^{\nu\sigma}\right)$$
$$+f^{eac}f^{ebd}\left(g^{\mu\nu}g^{\sigma\rho}-g^{\mu\sigma}g^{\nu\rho}\right)$$
$$+f^{eab}f^{ecd}\left(g^{\mu\sigma}g^{\nu\rho}-g^{\mu\rho}g^{\nu\sigma}\right)\Big]$$