IV.4. The mass matrix and the Cabibbo angles

So far we have treated diagonal Yukawa-terms. In general Isospin doublets need not to be mass eigenstates!

$$\underset{\text{(weak) isospin eigenstates}}{\psi'} = \underset{\text{unitary}}{V}\ \underset{\text{mass}}{\psi}$$

Quantum numbers (Flavour):

Families 1	2	3	T	T_3	Y	Q
$\begin{pmatrix} \psi_{\nu_{eL}} \\ \psi_{e_L} \end{pmatrix}$	$\begin{pmatrix} \psi_{\nu_{\mu L}} \\ \psi_{\mu_L} \end{pmatrix}$	$\begin{pmatrix} \psi_{\nu_{\tau L}} \\ \psi_{\tau_L} \end{pmatrix}$	$1/2$ $1/2$	$1/2$ $-1/2$	$-1/2$ $-1/2$	0 -1
ψ_{e_R}	ψ_{μ_R}	ψ_{τ_R}	0	0	-1	-1
$\begin{pmatrix} u_L \\ d'_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s'_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b'_L \end{pmatrix}$	$1/2$ $1/2$	$1/2$ $-1/2$	$1/6$ $1/6$	$2/3$ $-1/3$
u_R	c_R	t_R	0	0	$2/3$	$2/3$
d'_R	s'_R	b'_R	0	0	$-1/3$	$-1/3$

Isoscalars: $\Psi_L = \begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix}$, e.g.: $\psi_{1L} = \psi_{\nu_{eL}}$, $\psi_{2L} = \psi_{e_L}$ leptons

$\psi_{1L} = u_L$, $\psi_{2L} = d'_L$ quarks

$$(1) \qquad \Phi^+ \Psi_L = \phi_1^+ \psi_{1L} + \phi_2^+ \psi_{2L}$$

$$(2) \qquad \Phi^T \varepsilon \Psi_L = \phi_1 \psi_{2L} - \phi_2 \psi_{1L}$$

with $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(i) Isospin trafo's

(1), (2) are invariant under SU(2) Isospin rotations

$\mathcal{U}$: $\phi \to \mathcal{U}\phi$

$\Psi_L \to \mathcal{U}\Psi_L$, $\varepsilon \mathcal{U}^T \varepsilon = \mathcal{U}^+$

$\psi_R \to \psi_R$ as $\boxed{\varepsilon^T \vec{\sigma}^T \varepsilon = -\vec{\sigma}}$

$\vec{\sigma}$: p. 2-6a, 4-4

(1): $\phi^+ \Psi_L \to \phi^+ \mathcal{U}^+ \mathcal{U} \Psi_L = \phi^+ \Psi_L$

(2): $\phi^T \varepsilon \Psi_L \to \phi^T \mathcal{U}^T \varepsilon \mathcal{U} \Psi_L$

$$= \phi^T \underbrace{\varepsilon\, \varepsilon^T}_{\mathbb{1}_2} \overbrace{\mathcal{U} \varepsilon}^{\mathcal{U}^+} \mathcal{U} \Psi_L = \phi^T \varepsilon \Psi_L$$

(ii) Hypercharge $\boxed{Y_H = 1/2}$: $\Psi_L \to e^{i\omega Y_{\Psi_L}} \Psi_L$, $\phi \to e^{i\omega Y_H}$

Leptons: $\psi_R \to e^{i\omega Y_{\psi_R}} \psi_R$

$$\phi^+ \Psi_{L\,\text{leptons}} \to e^{-i\omega} \phi^+ \Psi_{L\,\text{leptons}}$$

$$\phi^T \varepsilon \Psi_{L\,\text{leptons}} \to \phi^T \varepsilon \Psi_{L\,\text{leptons}}$$

$\Rightarrow$ only $\boxed{\bar{\psi}_R \phi^+ \Psi_L}$ invariant under Hypercharge trafo's.

Quarks:

$$\phi^+ \Psi_{L\,\text{quarks}} \to e^{-i 1/3\, \omega} \phi^+ \Psi_{L\,\text{quarks}}$$

$$\phi^T \varepsilon \Psi_{L\,\text{quarks}} \to e^{i 2/3\, \omega} \phi^T \varepsilon \Psi_{L\,\text{quarks}}$$

$$\Rightarrow \quad \bar{u}_R \phi^T \begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \ \bar{c}_R \phi^T \varepsilon \begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \ \bar{t}_R \phi^T \varepsilon \begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \ \ldots,$$

$$\bar{d}_R \phi^+ \begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \ \bar{s}_R \phi^+ \ \ldots$$

In summary:

$$\mathcal{L}_y(x) = -\begin{pmatrix} \bar{\psi}_{e_R} \\ \bar{\psi}_{\mu_R} \\ \bar{\psi}_{\tau_R} \end{pmatrix} H'_\ell \begin{pmatrix} \phi^+ \psi_{e_L} \\ \phi^+ \psi_{\mu_L} \\ \phi^+ \psi_{\tau_L} \end{pmatrix}$$

$$+ \begin{pmatrix} \bar{u}_R \\ \bar{c}_R \\ \bar{t}_R \end{pmatrix} H'_q \begin{pmatrix} \phi^T \varepsilon \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \phi^T \varepsilon \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \phi^T \varepsilon \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix} - \begin{pmatrix} \bar{d}'_R \\ \bar{s}'_R \\ \bar{b}'_R \end{pmatrix} H_q \begin{pmatrix} \phi^+ \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \phi^+ \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \phi^+ \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix}$$

$$+ \text{h.c.}$$

Change of basis in field space: $\boxed{\mathcal{U}, \mathcal{U}', V \in U(3)}$

$$\psi_{R_{\ell/q}} \to \mathcal{U}_{\ell/q} \psi_{R_{\ell/q}}, \quad \psi'_{R_q} \to \mathcal{U}'_q \psi'_{R_q}$$

$$\psi_{L_{\ell/q}} \to V_{\ell/q} \psi_{L_{\ell/q}}$$

$$\Rightarrow \quad H_\ell \xrightarrow[\text{bi-unitary}]{} \mathcal{U}^+_\ell H_\ell V_\ell = \begin{pmatrix} h_e & & 0 \\ & h_\mu & \\ 0 & & h_\tau \end{pmatrix}$$

$$H'_q \to \mathcal{U}'^+_q H'_q V_q = \begin{pmatrix} h_u & & 0 \\ & h_c & \\ 0 & & h_t \end{pmatrix}$$

$$H_q \to \mathcal{U}_q^\dagger H_q V_q = V \begin{pmatrix} h_d & & 0 \\ & h_s & \\ 0 & & h_b \end{pmatrix} V^\dagger$$

[with $H_q = \tilde{\mathcal{U}} \begin{pmatrix} h_d & & 0 \\ & h_s & \\ 0 & & h_b \end{pmatrix} \tilde{V}$, $V = V_q^\dagger \tilde{V}^\dagger$, $\mathcal{U}_q^\dagger = V \tilde{\mathcal{U}}^\dagger$]

$\boxed{V \ : \text{Cabibbo-Kobayashi-Maskawa-Matrix}}$ CKM

- $V \in U(3)$ carries phase redundancy

$$V^\dagger \to \mathcal{U}_\varphi^\dagger V^\dagger \mathcal{U}_\Theta \quad \text{with} \quad \mathcal{U}_\varphi = \begin{pmatrix} e^{i\varphi_1} & & 0 \\ & e^{i\varphi_2} & \\ 0 & & e^{i3} \end{pmatrix}$$

$\mathcal{U}_\varphi \begin{pmatrix} h_d & & 0 \\ & h_s & \\ 0 & & h_b \end{pmatrix} \mathcal{U}_\varphi^\dagger = \begin{pmatrix} h_d & & 0 \\ & h_s & \\ 0 & & h_b \end{pmatrix}$ $\quad \psi \to \mathcal{U}_\Theta \psi$

5 phases (global phase drops out)

parameters: $9 - 5 = 4$

$$V = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix}$$

Cabibbo angles Θ_i: $\quad c_i = \cos \Theta_i$

$i = 1,2,3$: $\quad s_i = \sin \Theta_i \qquad \Theta_i \in]0, \pi/2]$

$\qquad \delta \in [0, 2\pi]$

$\boxed{\delta}$ CP-violation

- two families

$V \in U(2)$

phase redundancy: 3 phases (global phase drops out)

$$U_\varphi^\dagger V U_\Theta = \begin{pmatrix} e^{-i(\varphi_1-\theta_1)} V_{11} & e^{-i(\varphi_1-\theta_2)} V_{12} \\ e^{-i(\varphi_2-\theta_1)} V_{21} & e^{-i(\varphi_2-\theta_2)} V_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}$$

e.g. Nachtmann p.314

$\Rightarrow$ no CP-violation

total Yukawa Lagrangian: $\phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(\rho+v) \end{pmatrix}$

$$\mathcal{L}_Y(x) = -\Bigg\{ \begin{pmatrix} \bar\psi_{e_R} \\ \bar\psi_{\mu R} \\ \bar\psi_{\tau R} \end{pmatrix} \begin{pmatrix} h_e & & 0 \\ & h_\mu & \\ 0 & & h_\tau \end{pmatrix} \begin{pmatrix} \psi_{eL} \\ \psi_{\mu L} \\ \psi_{\tau L} \end{pmatrix}$$

$$+ \begin{pmatrix} \bar u_R \\ \bar c_R \\ \bar t_R \end{pmatrix} \begin{pmatrix} h_u & & 0 \\ & h_c & \\ 0 & & h_t \end{pmatrix} \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix}$$

$$+ \begin{pmatrix} \bar d'_R \\ \bar s'_R \\ \bar b'_R \end{pmatrix} V \begin{pmatrix} h_d & & 0 \\ & h_s & \\ 0 & & h_b \end{pmatrix} V^+ \begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} + h.c. \Bigg\} \frac{v}{\sqrt{2}}(1+\rho/v)$$

$$= -\Big[m_e \bar\psi_e \psi_e + m_\mu \bar\psi_\mu \psi_\mu + m_\tau \bar\psi_\tau \psi_\tau$$

$$+ m_u \bar u u + m_c \bar c c + m_t \bar t t$$

$$+ \begin{pmatrix} \bar d' \\ \bar s' \\ \bar b' \end{pmatrix} V \begin{pmatrix} m_d & & 0 \\ & m_s & \\ 0 & & m_b \end{pmatrix} V^+ \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \Big] (1+\rho/v)$$

with $\boxed{m = h\, v/\sqrt{2}}$

charged quark current:

$$j^\mu_{cc} = \overline{\Psi}_q \, \gamma^\mu \left(T^1 + i T^2\right) \Psi_q$$

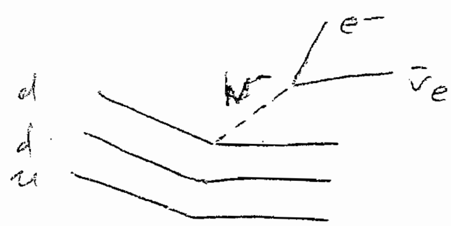
$$= \begin{pmatrix} \bar{u}_L \\ \bar{c}_L \\ \bar{t}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} = \begin{pmatrix} \bar{u}_L \\ \bar{c}_L \\ \bar{t}_L \end{pmatrix} V \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}$$

$\Rightarrow \quad d \to u + W^- \qquad V_{11}$

$s \to u + W^- \qquad V_{12}$

$b \to u + W^- \qquad V_{13}$

e.g. $n \longrightarrow p + W^-$



$$V = \begin{pmatrix} 0.97383^{+0.00024}_{-0.00023} & 0.2272 \pm 0.001 & (3.69 \pm 0.09)\cdot 10^{-3} \\ 0.2271^{+0.0010}_{-0.0010} & 0.9728 \pm 0.00024 & \left(42.21^{+0.10}_{-0.80}\right)\cdot 10^{-3} \\ \left(8.14^{+0.32}_{-0.64}\right)\cdot 10^{-3} & \left(41.61^{+0.12}_{-0.78}\right)\cdot 10^{-3} & 0.999100^{+0.00003}_{-0.0000} \end{pmatrix}$$

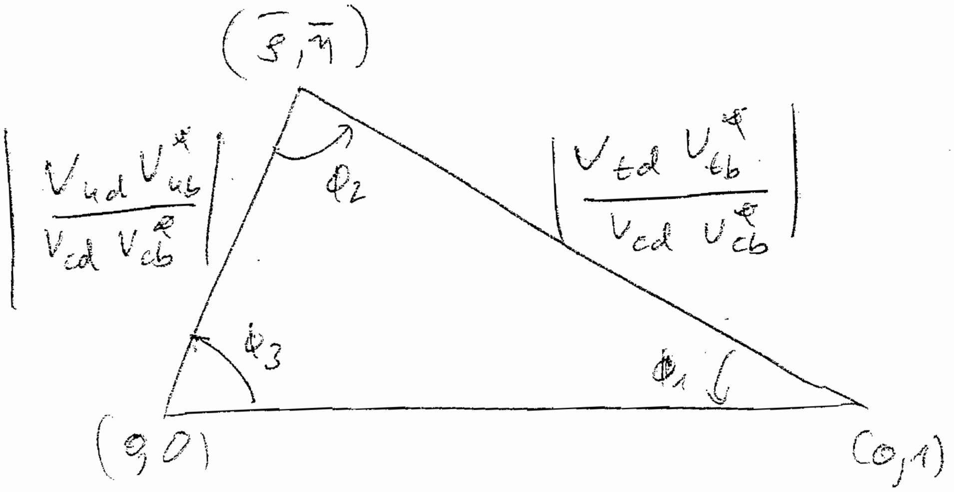
Unitarity triangle: $\sum_i V_{ij} V_{ik}^* = \delta_{jk}$

e.g.

$$\boxed{V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0}$$

Jarlskog invariant:

$$\mathrm{Im}\left[V_{ij} V_{kl} V_{il}^* V_{kj}^*\right] = J \sum_{m,n} (\varepsilon_{ikm} \varepsilon_{jln})$$



neutral current

$$j^\mu_{nc\ quark} = \bar{\Psi}\,\gamma^\mu\left(T^3 - Q\sin^2\theta_W\right)\Psi$$

$$= \begin{pmatrix}\bar{u}\\ \bar{c}\\ \bar{t}\end{pmatrix}\gamma^\mu\left(\underset{\substack{\uparrow\\ T^3}}{\tfrac{1}{2}}\,\frac{1-\gamma_5}{2} - \underset{\substack{\uparrow\\ Q}}{\tfrac{2}{3}}\sin^2\theta_W\right)\begin{pmatrix}u\\ c\\ t\end{pmatrix}$$

$$+ \begin{pmatrix}\bar{d}\\ \bar{s}\\ \bar{b}\end{pmatrix}\gamma^\mu\left(\overset{\downarrow}{-\tfrac{1}{2}}\,\frac{1-\gamma_5}{2} + \overset{\substack{(-1/3)\\ \downarrow}}{1/3}\sin^2\theta_W\right)\begin{pmatrix}d\\ s\\ b\end{pmatrix}$$

$$\boxed{\text{no flavour changing neutral currents}}$$

Parameters (p. 5-14a)

(V) Cabibbo angles + phase

3 1

5.5* CP - Violation in the Standard model

CP :

(p. 2.32-2.34)

$$\psi' \to e^{i\varphi_4} \cdot \gamma_0 S(C)\, \overline{\psi}^T(-\vec{x}, t)$$

$j^\mu_{cc_{\text{leptons}}}$

charge conjug.

$$\overline{\psi}\gamma^\mu T^+ \psi \to -\overline{\psi}_{\text{lept}} \gamma^\mu T^- \psi_{\text{lept}}\, e^{i\chi}$$

with

$$\chi = \varphi_{\text{lept}} - \varphi_{\nu_{\text{lept}}}$$

$j^\mu_{cc_{\text{quarks}}}$:

$$\overline{\psi}_q \gamma^\mu T^+ \psi_q$$

$$\to -\overline{\psi}_q \gamma^\mu T^- \psi_q\, e^{i\chi}$$

iff

$$\boxed{\begin{pmatrix} e^{i\varphi_d} & & 0 \\ & e^{i\varphi_s} & \\ 0 & & e^{i\varphi_b} \end{pmatrix} V^T \begin{pmatrix} e^{-i\varphi_u} & & 0 \\ & e^{-i\varphi_c} & \\ 0 & & e^{-i\varphi_t} \end{pmatrix} = e^{i\chi} V^+}$$

$$\Rightarrow \quad \boxed{V = V^*} \quad \text{or} \quad \boxed{\delta = 0, \pi}$$

Remarks: (1) strong CP-problem : Θ-angle in QCD
U(1)-problem

$$\mathcal{L}_\Theta = \frac{\Theta g^2}{32\pi^2} \operatorname{tr} \underbrace{\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}}_{\tilde{F}^{\mu\nu}} F_{\mu\nu}$$

$$\left(\text{Euclidean} \quad \frac{g^2}{32\pi^2} \int \operatorname{tr} F_{\mu\nu} \tilde{F}_{\mu\nu} = n \in \mathbb{Z} \right)$$

$$|\Theta| < 10^{-9}$$

$$\mathcal{L}_{\text{'t Hooft}} \sim \left[\det_{s,t} \bar{\psi}_s (1-\gamma_5)/2\, \psi_t \right] \boxed{e^{i\Theta n}}$$

U(1)-Phase

(2) Neutrino masses:

$\boxed{\text{Neutrino oszillations}}$

$\Rightarrow$ missing neutrinos from the sun