Quark-Gluon Plasma Physics

3. Thermodynamics of the QGP

3.1 QGP thermodynamics and the MIT bag model

thermodynamics of
- relativistic Bose gas
- relativistic Fermi-gas

bag model of hadrons

constructing the phase diagram between pion gas and QGP

" realistic hadron gas and QGP

comments on 'Lattice QCD'

comments on phase transition and transition temperature

3.1.1 Thermodynamics of a relativistic Bose gas

probability density for occupation of state with relativistic energy E and degeneracy g

$$N(E) = \frac{g}{(2\pi)^3}(\exp(\frac{E-\mu}{T}) - 1)^{-1}$$

with energy $E^2 = p^2 + m^2$

(note: here and in the following ℏ=c=1)

and chemical potential μ controlling the average number of particles vs antiparticles

neglecting the particle mass (okay since in interesting region $E = 3T \gg m$)

and chemical potential (good as long as no additive quantum number)

Boson number density $n = \int N(E)d^3p = \frac{4\pi g}{(2\pi)^3}\int \frac{p^2 dp}{\exp(\frac{p}{T}) - 1}$

$$n = \frac{g}{\pi^2}T^3\zeta(3)$$

with Riemann Ӿ-function $\zeta(3) \approx 1.2$

Thermodynamics of a relativistic Bose gas

Boson energy density $$\epsilon = \int N(E) p d^3p = \frac{4\pi g}{(2\pi)^3} \int \frac{p^3 dp}{\exp(\frac{p}{T}) - 1}$$

$$\epsilon = \frac{3g}{\pi^2} T^4 \zeta(4)$$

with Riemann X-function $\zeta(4) = \frac{\pi^4}{90} \approx 1.08$

$$\boxed{\epsilon = \frac{\pi^2}{30} g T^4}$$

and we get the Energy per particle $\epsilon/n = 3T \frac{\zeta(4)}{\zeta(3)} \approx 2.7\,T$

Boson pressure $P = n^2 \partial \frac{\epsilon}{n} / \partial n \rightarrow \boxed{P = \frac{1}{3}\epsilon}$ free gas equation of state

Entropy density $d\sigma = d\epsilon/T$ and $d\epsilon = \text{const.}\, T^3 dT$

$$\sigma = \int d\sigma = \text{const.} \int T^2 dT = \frac{1}{3} \text{const.}\, T^3 \rightarrow \boxed{\sigma = \frac{1}{3} \frac{d\epsilon}{dT}}$$

$$\sigma = \frac{4\pi^2}{90} g\, T^3$$

Thermodynamics of a relativistic Bose gas

and the entropy per particle (boson) $\sigma/\mathrm{n} = 4\zeta(4)/\zeta(3) \approx 3.6$

cf. old Landau formula for pions: S = 3.6 dN/dy

3.1.2 Thermodynamics of a relativistic Fermi gas

probability density for occupation $\mathrm{N(E)} = \frac{\mathrm{g}}{(2\pi)^3}(\exp(\frac{\mathrm{E}-\mu}{\mathrm{T}})+1)^{-1}$

$\mathrm{E}^2 = \mathrm{p}^2 + \mathrm{m}^2$

but now I not generally 0

Fermion number density

$$\mathrm{n} = \frac{4\pi \mathrm{g}}{(2\pi)^3}\int \frac{\mathrm{p}^2 \mathrm{dp}}{\exp(\frac{\mathrm{p}-\mu}{\mathrm{T}})+1}$$

Energy density

$$\epsilon = \frac{4\pi \mathrm{g}}{(2\pi)^3}\int \frac{\mathrm{E}\mathrm{p}^2 \mathrm{dp}}{\exp(\frac{\mathrm{p}-\mu}{\mathrm{T}})+1}$$

cannot be solved analytically, only numerically

but there is analytic solution for sum of particle and antiparticle (e.g. quark and antiquark) (Chin, PLB 78 (1978) 552)

$$\epsilon_{\mathrm{q}} + \epsilon_{\bar{\mathrm{q}}} = \mathrm{g}(\frac{7\pi^2}{120}\mathrm{T}^4 + \frac{\mu^2}{4}\mathrm{T}^2 + \frac{\mu^4}{8\pi^2})$$

and

$$\mathrm{n_q} - \mathrm{n}_{\bar{\mathrm{q}}} = \mathrm{g}(\frac{\mu}{6}\mathrm{T}^2 + \frac{\mu^3}{6\pi^2})$$

specific example for fermions: quarks in QGP with no net baryon density (LHC)

$$\langle q \rangle = \langle \bar{q} \rangle \leftrightarrow \mu = 0$$

in that case quark number density $\mathrm{n_q} = \frac{\mathrm{g}}{\pi^2} \mathrm{T}^3 \mathrm{d}(3)$

Note: $\mathrm{d}(\alpha + 2) = \int \frac{\mathrm{x}^\alpha \mathrm{dx}}{\mathrm{e}^\mathrm{x} + 1}$ and $\mathrm{d}(3) \approx 0.9$

and quark and antiquark energy density $\epsilon_\mathrm{q} = \epsilon_{\bar{\mathrm{q}}} = \frac{3\mathrm{g}}{\pi^2} \mathrm{T}^4 \, \mathrm{d}(4) = \frac{7\pi^2}{240} \mathrm{g} \, \mathrm{T}^4$

with $\mathrm{d}(4) = \frac{7\pi^4}{720}$

the energy per quark is then $\epsilon/\mathrm{n} = 3\mathrm{T}\frac{\mathrm{d}(4)}{\mathrm{d}(3)} \approx 3.2\mathrm{T}$

Entropy density (computed as above for bosons)

$$\sigma = \frac{7\pi^2}{180} \mathrm{g} \, \mathrm{T}^3$$

and the entropy per fermion (quark) $\sigma/\mathrm{n} = 4\frac{\mathrm{d}(4)}{\mathrm{d}(3)} \approx 4.2$

Summary relativistic bosons and fermions (no chem.pot.)

- Energy density $\epsilon \propto \mathrm{T}^4$
- Pressure $\mathrm{P} = \frac{1}{3}\epsilon \propto \mathrm{T}^4$
- Entropy density $\sigma \propto \mathrm{T}^3$
- Particle number density $\mathrm{n} \propto \mathrm{T}^3$

➔ to obtain physical units of GeV/fm^3 or fm^{-3}, multiply with appropriate powers of ħc

- all are proportional to the number of degrees of freedom
- between bosons and fermions there is a factor 7/8

$$\epsilon_\mathrm{f} = \frac{7}{8}\epsilon_\mathrm{b}$$

due to the different factors
+1 in the fermion momentum integral and
-1 in the boson momentum integral

3.1.3 Short excursion: the bag model

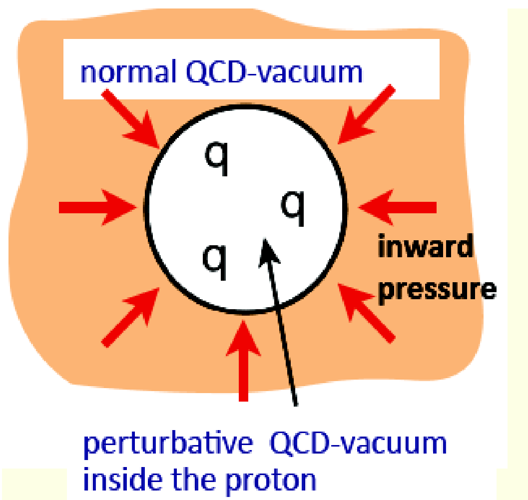
to deal with QCD in the nonperturbative regime (i.e. where α_S is not negligible) one needs to make models (alternative: lattice QCD see below)
for instance to treat the nucleon and its excitations

MIT bag model: build confinement and asymptotic freedom into simple phenomenological model

A. Chodos, R.L. Jaffe, K. Johnson, C.B.Thorne, Phys. Rev. D10 (1974) 2599
T. DeGrand, R.L. Jaffe, K. Johnson, J. Kiskis, Phys. Rev. D12 (1975) 2060

hadrons considered as bags embedded into a non-perturbative QCD vacuum also called “physical vacuum” or “normal QCD vacuum”

space divided into 2 regions



Interior of bag: quarks have very small (current) masses, interaction weak

Exterior of bag: quarks are not allowed to propagate there, lower vacuum energy, no colored quarks or gluons but quark and gluon condensates

Hadrons in MIT bag model

Hadrons are considered drops of another, perturbative phase of QCD immersed into normal QCD vacuum

all non-perturbative physics included in one universal quantity, the bag constant B defined as the difference in energy density between perturbative and physical vacua:

$$\epsilon_{\mathrm{bag}} - \epsilon_{\mathrm{vac}} \equiv \mathrm{B} > 0$$

solve Dirac equation for massless quarks inside bag with volume V and surface S with special boundary conditions at the surface that

i) enforce confinement: quark current normal to bag surface = 0

ii) define a stability condition for bag: pressure of Dirac particles inside is balanced by difference in energy density inside and outside

$$\mathrm{H} = \mathrm{H}_{\mathrm{kin}} + \mathrm{H}_{\mathrm{spin-spin}} + \mathrm{B\,V}$$

- $\mathrm{H}_{\mathrm{kin}}$: kin. Energy of quarks confined in bag
- $\mathrm{H}_{\mathrm{spin-spin}}$: spin-spin interaction
- $\mathrm{B\,V}$: energy to make hole of volume V in phys. vacuum

Hadrons in MIT bag model

for (nearly) massless quarks $E_{kin} \propto 1/R$ → tries to extend bag
(spherical bag with radius R)

bag term $B \frac{4\pi}{3} R^3$ → tries to contract bag

⇒ equilibrium is reached

obtain e.g. for nucleon mass (spherical bag with 3 quarks in s-state)

$$E = 3\frac{\omega_{n,-1}}{R} + \frac{4\pi}{3} B R^3$$ with $\omega_{1,-1} = 2.04$ $\omega_{2,-1} = 5.40$ ...

and $$\frac{\partial E}{\partial R} = 0$$

internal energy determines the radius of the bag, if B is a universal constant

- determines masses and sizes of all hadrons
  rather successful with $B_{MIT} = 56\,\mathrm{MeV/fm^3}$
  baryon octet and decuplet as well as vector mesons well reproduced

note: often instead of B, $B^{1/4}$ in MeV is quoted $B_{MIT}^{1/4} = 146\,\mathrm{MeV}$

3.1.4 Thermodynamics of pion gas and QGP

pion gas: massless bosons with degeneracy $g_\pi = 3 \;\; \text{for} \;\; \pi^+, \pi^0, \pi^-$

energy density of pion gas $\epsilon_\pi = \frac{\pi^2}{30} g_\pi T^4 = 129\, T^4$ and pressure $P = \frac{1}{3}\epsilon = 43\, T^4$

after properly inserting missing powers of ℏc and using T in GeV

quark-gluon plasma:

gluons as massless bosons with degeneracy g_g = 2(spin) x 8(color) = 16

quarks massless fermions with degeneracy g_q = N_f x 2(spin) x 3(color) = 6 N_f

and same for antiquarks (here N_f is number of massless/light flavors)

additional contribution to energy density: to make quark-gluon gas, need to create cavity in vacuum

energy needed is given by the bag constant B "pressure of vacuum on color field"

analogy to Meissner effect: superconductor expells magnetic field

$\leftrightarrow$ QCD vacuum expels color field into bags

see: Baym and Chin
Phys. Lett. B62 (1976) 241

$\rightarrow \quad \epsilon = \epsilon_{\text{thermal}} + B$ and deriving pressure as above $P = \frac{1}{3}(\epsilon - 4B)$

bag model equation of state $\mathbf{P} = \mathbf{P}(\epsilon)$

$$\epsilon = \epsilon_{thermal} + B$$

let n be the quark number density, then the quark energy density $\epsilon_{thermal}$ is

$$\epsilon_{thermal} = c_1 n^{4/3}$$

note: the $n^{4/3}$ term arises because the particle density scales like T^3 while the quark energy density $\propto T^4$.
so the total bag energy density

$$\epsilon(n) = c_1 n^{4/3} + B$$

now we obtain the (quark) chemical potential from:

$$\mu = \frac{\partial \epsilon}{\partial n} = 4/3 c_1 n^{1/3}$$

then we get:

$$\mu n = 4/3 c_1 n^{4/3} = 4/3(\epsilon(n) - B)$$

from this we obtain the pressure P via:

$$P = \mu n - \epsilon(n)$$

so that

$$P = \mu n - \epsilon = 4/3(\epsilon - B) - \epsilon$$

$$P = 1/3(\epsilon - 4B)$$

this slide is for additional information not part of the standard course material

3.1.4 Thermodynamics of pion gas and QGP

What value to use for the bag constant?
from hadron phenomenology at T=0 and normal nuclear matter density

$$\mathrm{B} \approx 50 - 100\,\mathrm{MeV/fm^3}$$

but there are a number of problems with the MIT bag model

and there is good indication that B derived there is not the energy density of the QCD vacuum; conclusion: hadrons are not small drops of the new QCD phase but only a relatively small perturbation of the QCD vacuum

also B = B(T,n) (see e.g. E. Shuryak, Phys.Rept. 115 (1984) 151)
basic argument: at large T,n all non-perturbative phenomena suppressed

$\mathrm{B_{eff}} \approx 500 - 1000\,\mathrm{MeV/fm^3}$ vacuum energy density

energy density of quark-gluon gas

$$\epsilon_{\mathrm{qg}} = \frac{\pi^2}{30}(\mathrm{g_g} + \frac{7}{8}\mathrm{g_q})\mathrm{T}^4 + \mathrm{B} = \frac{\pi^2}{30}(16 + \frac{21}{2}\mathrm{N_f})\mathrm{T}^4 + \mathrm{B}$$

for N_f = 2 (u,d)

$$\epsilon_{\mathrm{qg}} = 1592\,\mathrm{T}^4 + 0.5 \quad (\frac{\mathrm{GeV}}{\mathrm{fm}^3})$$

Constructing the phase diagram

system always in phase with highest pressure

Gibbs conditions for critical point $$P_{QGP} = P_{piongas}$$

and $$\mu_{QGP} = \mu_{piongas} (= 0)$$

for N_f = 2 $$\frac{3\pi^2}{90} T_c^4 = \frac{\pi^2}{90} (16 + \frac{21}{2} N_f) T^4 - B$$

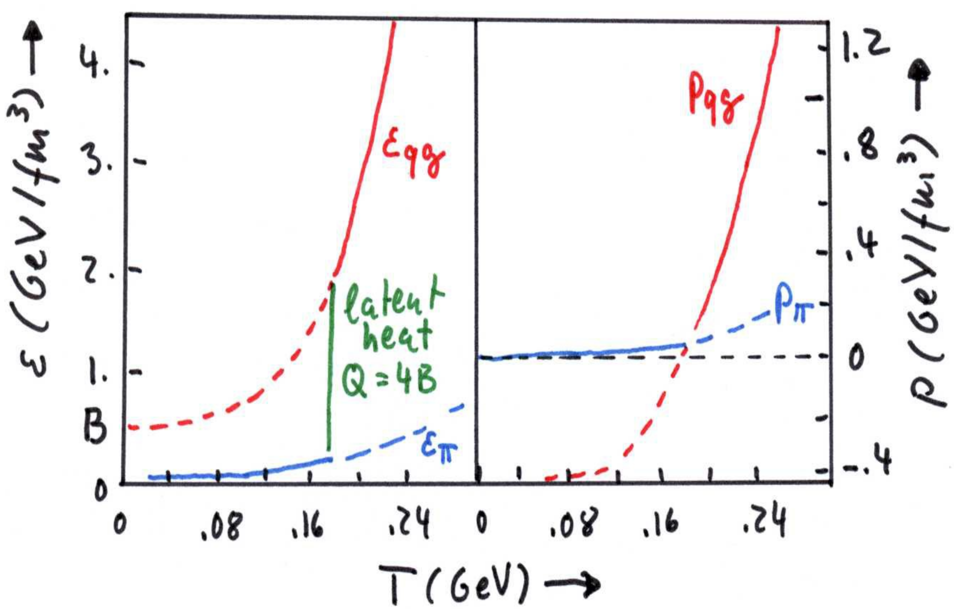
$$\frac{34\pi^2}{90} T_c^4 = B \qquad T_c = (\frac{90 \cdot 0.5\text{GeV} \cdot 0.197^3 \text{GeV}^3 \text{fm}^3}{34\pi^2 \text{fm}^3})^{1/4} = 0.18\,\text{GeV}$$

latent heat:

$$\epsilon_{qg} - \epsilon_{pion} (\text{at} T_c) = \frac{34\pi^2}{30} T_c^4 + B$$
$$= 1.54 + 0.5 = 2 \frac{\text{GeV}}{\text{fm}^3}$$

change in entropy density:

$$\sigma_{qg} - \sigma_{pion} = \frac{34 \cdot 4\pi^2}{90} T_c^3 = 11.4/\text{fm}^3$$



Now check the high baryon density limit

compute a T = 0 $\mu_{\cup}$point
cannot do this with pions alone, need nucleons

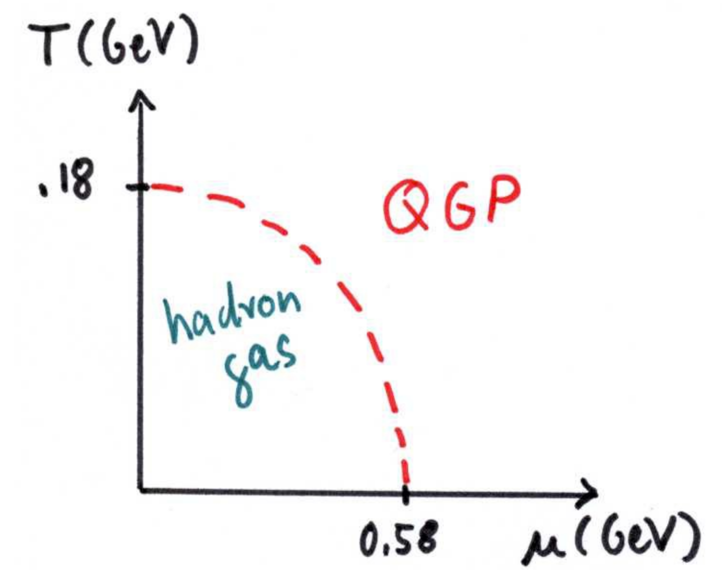
$$P_{\mathrm{pion}} = 0 \qquad P_{\mathrm{nucleon}} = \frac{g\mu^4}{3 \cdot 8\pi^2}$$ with g=4 (2(spin) x 2(isospin))

for the quark-gluon side at T=0

$$P_{q\bar{q}} = \frac{g\mu^4}{3 \cdot 8\pi^2} - B$$ with g=12 (quarks and antiquarks)

$$P_{\mathrm{nucleon}} = P_{q\bar{q}} \quad \rightarrow$$

$$\mu = \left(\frac{3\pi^2 \cdot 0.5\mathrm{GeV} \cdot 0.197^3\mathrm{GeV}^3\mathrm{fm}^3}{\mathrm{fm}^3}\right)^{1/4}$$
$$= 0.58\,\mathrm{GeV}$$

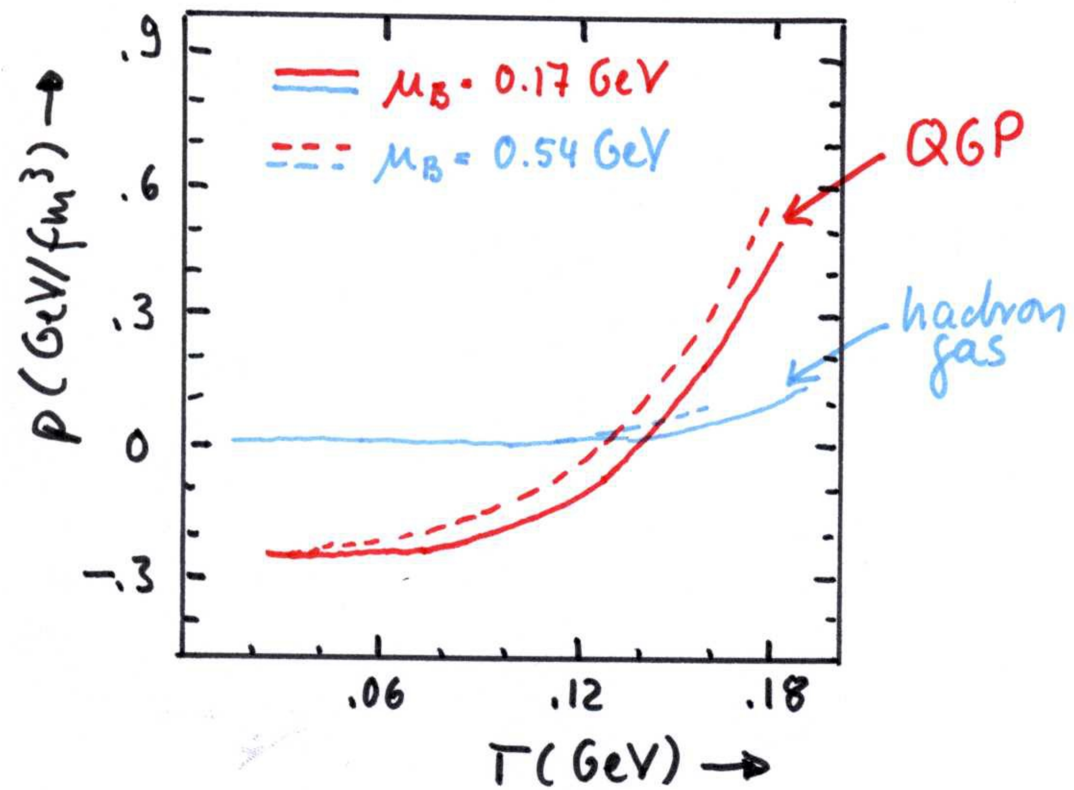


simple thermodynamics model gives first order phase transition,
<u>Caution</u>: this sets the scale, but there are a number of approximations
- pion gas is oversimplification for hadronic matter
- B = 0.5 GeV/fm^3 (should use B(T,n))

3.1.5 more realistic: replace pion gas by hadron gas

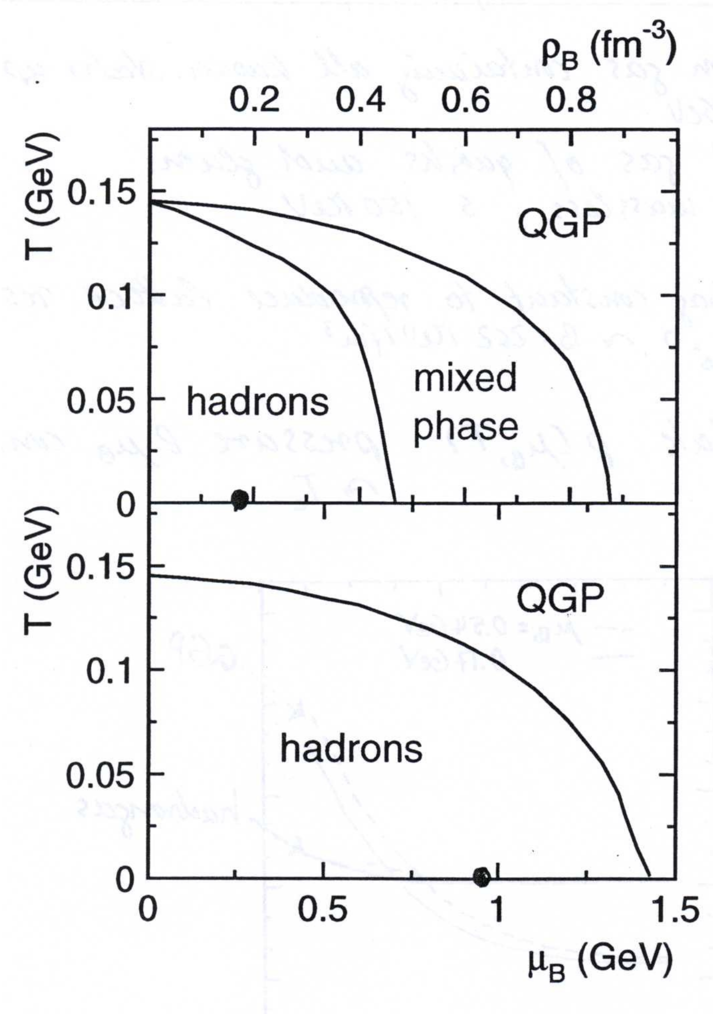
implement all know hadrons up to 2 GeV in mass
ideal gas of quarks and gluons, u,d massless, s 150 MeV
fix bag constant to match lattice QCD result (see below) at μ_b=0 ⟶ B=262 MeV/fm^3

compute P(μ_b,T) with
with pressure and μ_b continuous
to obtain T_c

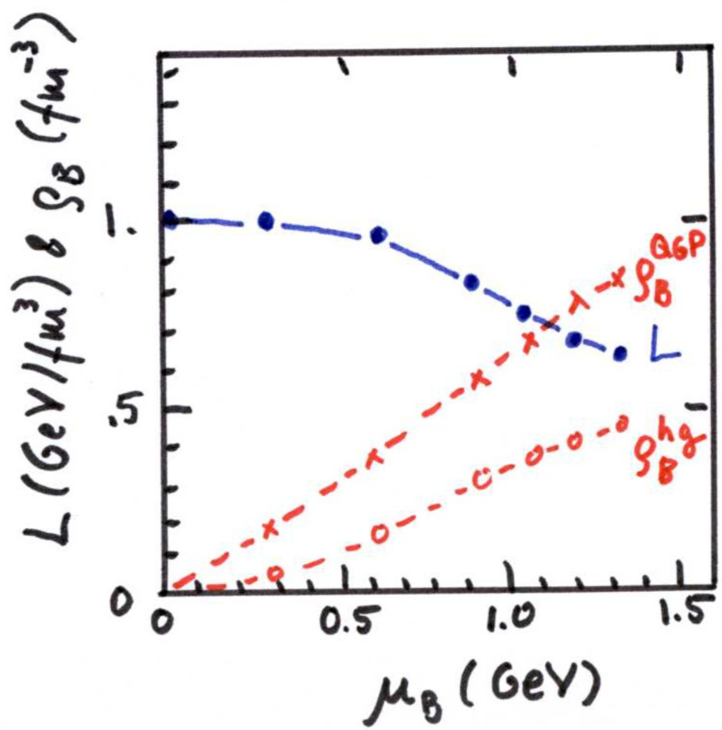


P. Braun-Munzinger, J. Stachel Nucl.Phys. A606 (1996) 320

Phase diagram constructed with hadron gas and QGP



P. Braun-Munzinger, J. Stachel Nucl.Phys. A606 (1996) 320



Note: chemical potential is continuous at phase transition but not the baryon density!

Speed of sound

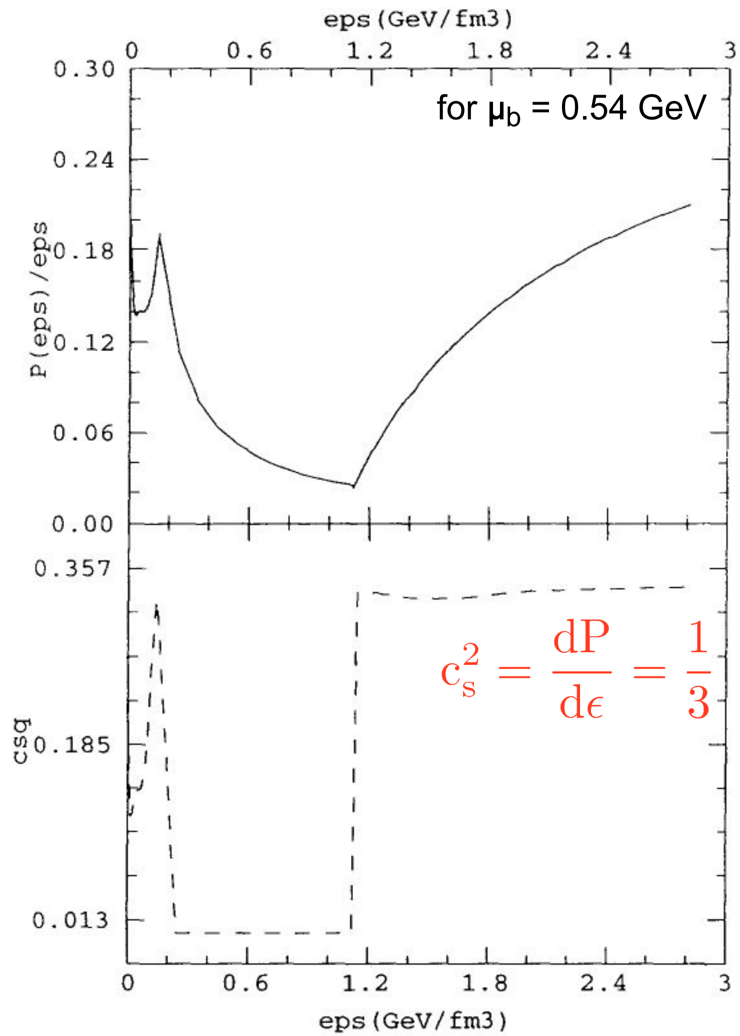
in relativistic gas without interactions,
speed of sound squared

$$c_s^2 = \frac{dP}{d\epsilon} = \frac{1}{3}$$

but in vicinity of phase transition strong
deviation of P from 1/3 J

there is always a minimum in speed of
sound

leading to a so-called 'softest point'



P. Braun-Munzinger, J. Stachel Nucl.Phys. A606 (1996) 320

3.2 Lattice QCD

QCD asymptotically free at large T and/or small distances
at low T and for finite size systems $\alpha_s = O(1)$

→ cannot use perturbation theory

instead define QCD at zero and finite temperature by putting gauge field on a space-time lattice and find solutions by solving path integrals numerically

↔ "lattice QCD"

formulated by K. Wilson in 1974 (Phys. Rev. D10 (1974) 2445)

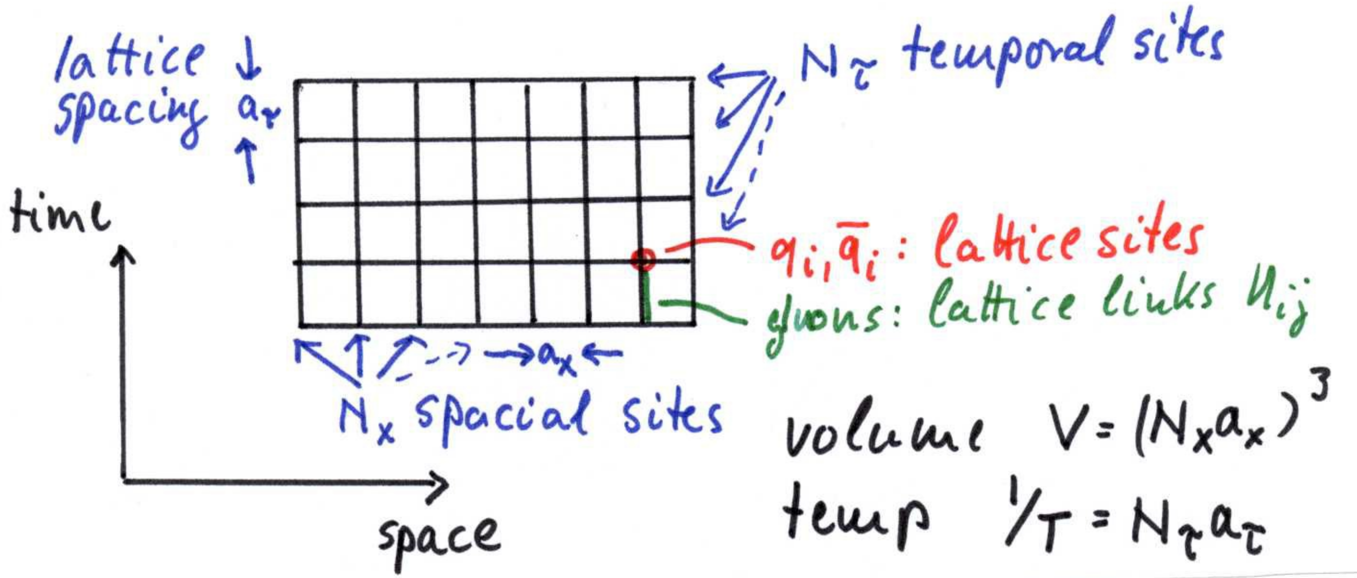
recent reviews:

A. Ukawa: J. Stat. Phys. 160 (2015) 1081, arXiv: 1501.04215 [hep-lat]
H.T. Ding, F. Karsch, S. Mukherjee, Int. J. Mod Phys. E24 (2015) 153007, arXiv: 1504.05274 [hep-lat]

Lattice QCD - schematic outline of basic (3) steps

i) use evolution in Euclidean time $\tau = \mathrm{it}$ instead of Minkowski time
to eliminate oscillations due to complex action

ii) replace Euclidean x,Πcontinuum by finite lattice



field theory with infinite number of degrees of freedom ⟶ finite many body problem
quantum field theory equivalent to classical statistical mechanics
with $\exp(-\mathrm{iHt}) \rightarrow \exp(-\mathrm{H}\tau) = \exp(-\mathrm{S})$

Powerful connection between quantum field theory and statistical mechanics
(already realized in 1960ies by K. Symanzik)

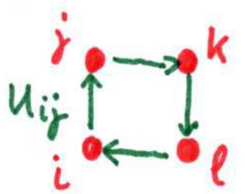
Lattice QCD basic steps

iii) evaluate partition function Z by using Feynman path integrals

$$Z = \mathrm{Tr}\,\exp(-H_{QCD}\tau)$$

$$\longrightarrow \quad Z = \int \Pi_{links} dU_{ij} \exp(-S(U))$$

action, given by sum over elementary plaquette



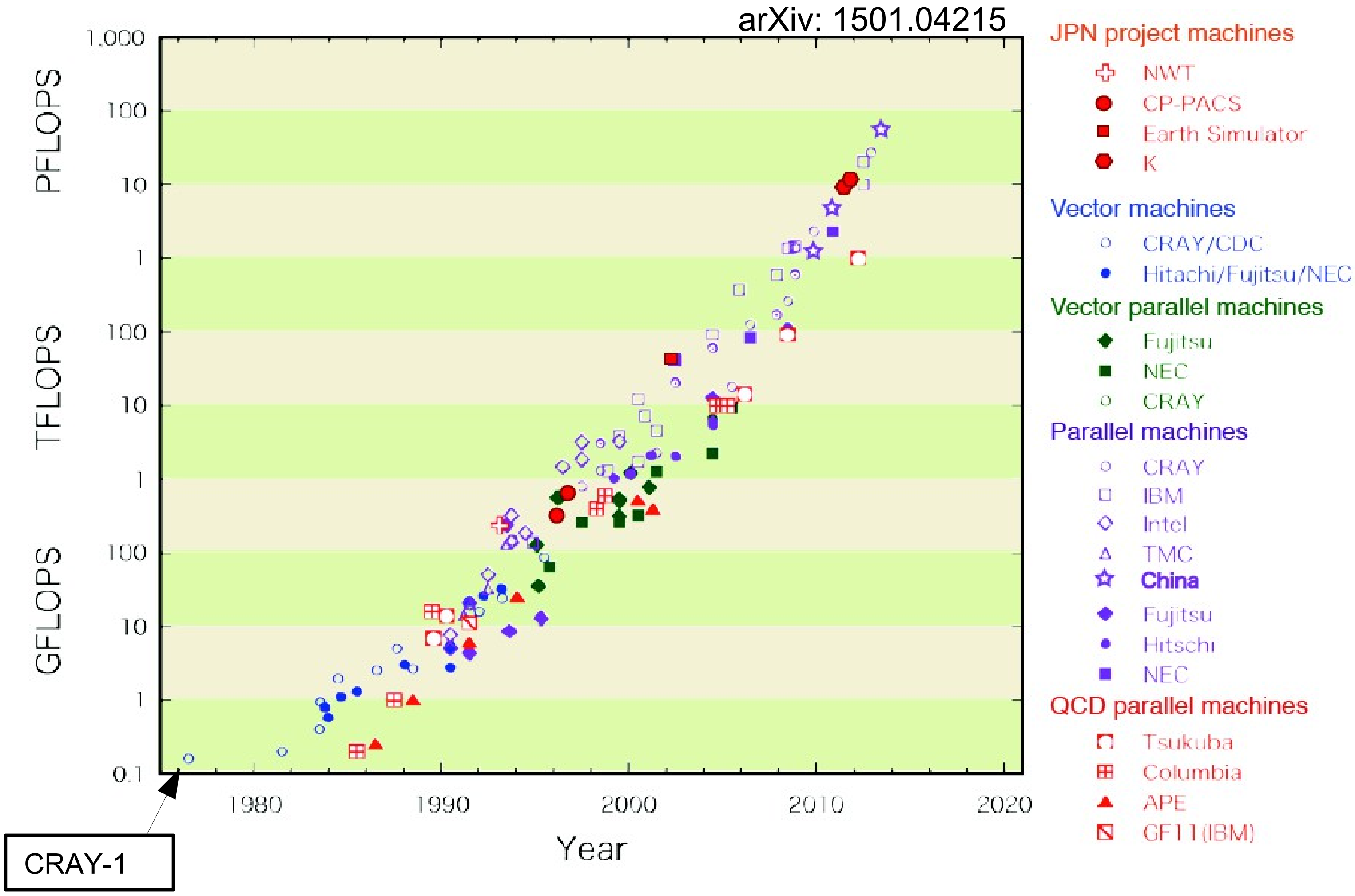
K. Wilson, Phys. Rev. D10 (1974) 2445

iv) lattices need to be big! e.g. 16^3 x 32 sites
have to sum over all color indices at each link → integral 10^7 dimensional
start with some values U_{ij} for all links, successively reassign new elements

to reduce computing time: use stochastic technique with clever weighting (exp(-S(U) favors small action)

have to sweep through entire lattice a few hundred times to evaluate thermodynamic quantities, baryon masses, wave functions

Huge increase in computing power since Wilson 1974



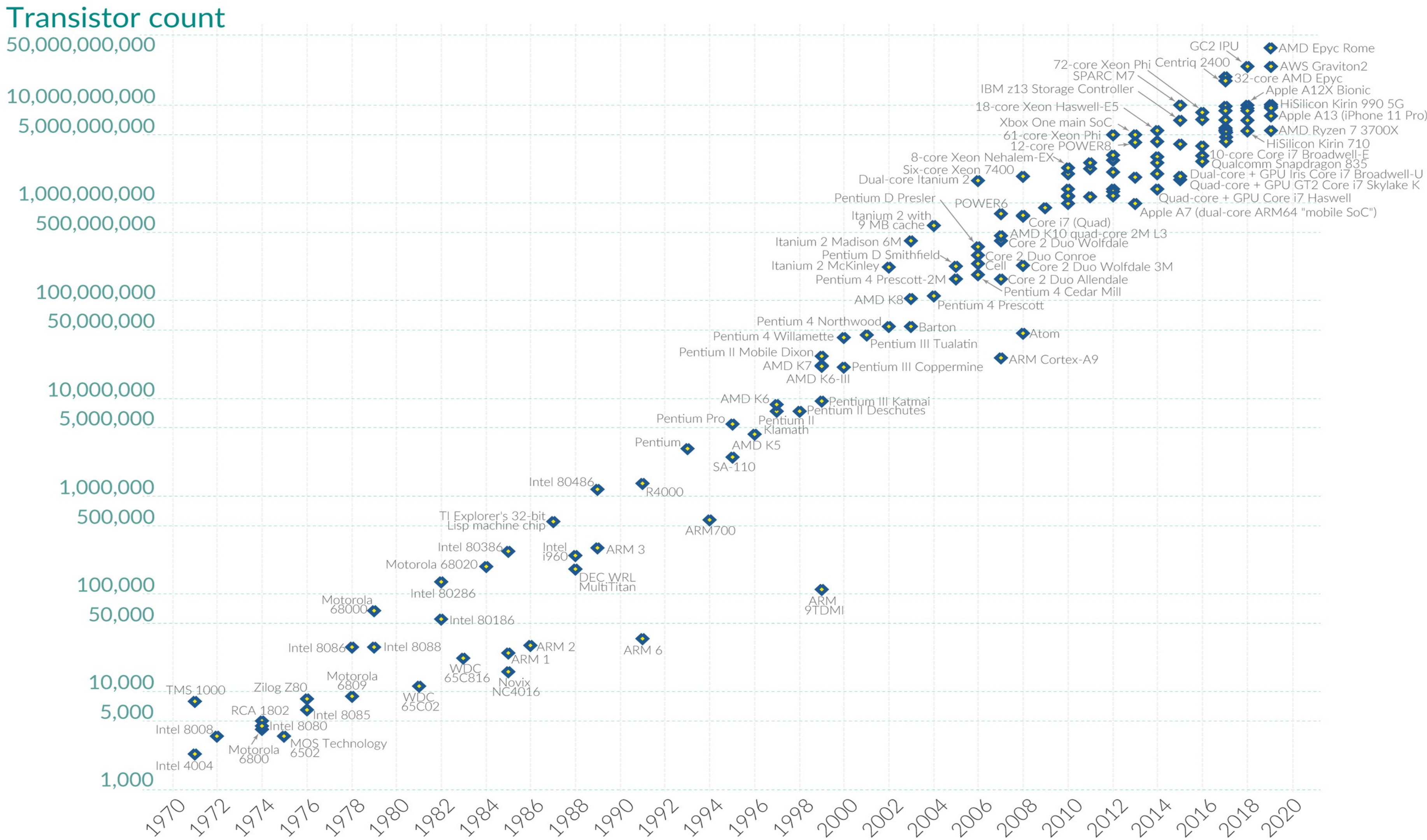
evolution of peak speed of supercomputers and improved algorithms:

→ now lattice QCD is a mature technique with increasingly reliable results

Moore's Law: The number of transistors on microchips doubles every two years

Our World in Data

Moore's law describes the empirical regularity that the number of transistors on integrated circuits doubles approximately every two years.
This advancement is important for other aspects of technological progress in computing – such as processing speed or the price of computers.

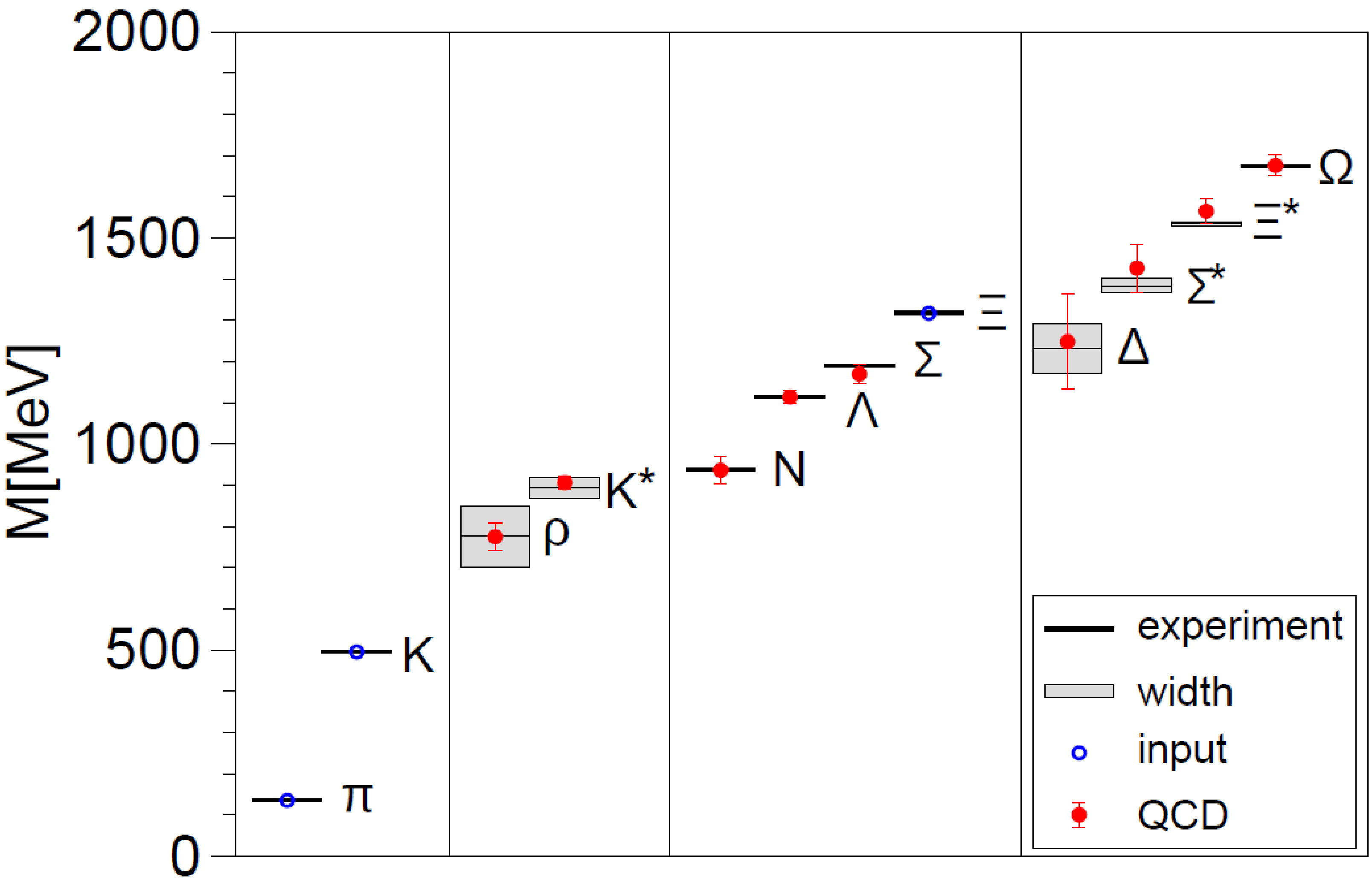


Data source: Wikipedia (wikipedia.org/wiki/Transistor_count)
OurWorldinData.org – Research and data to make progress against the world's largest problems.

Licensed under CC-BY by the authors Hannah Ritchie and Max Roser.

State-of-the-art light hadron spectrum from lattice QCD

S. Dürr, Z. Fodor et al. (Budapest-Marseille–Wuppertal Coll.) Science 322 (2008) 1225

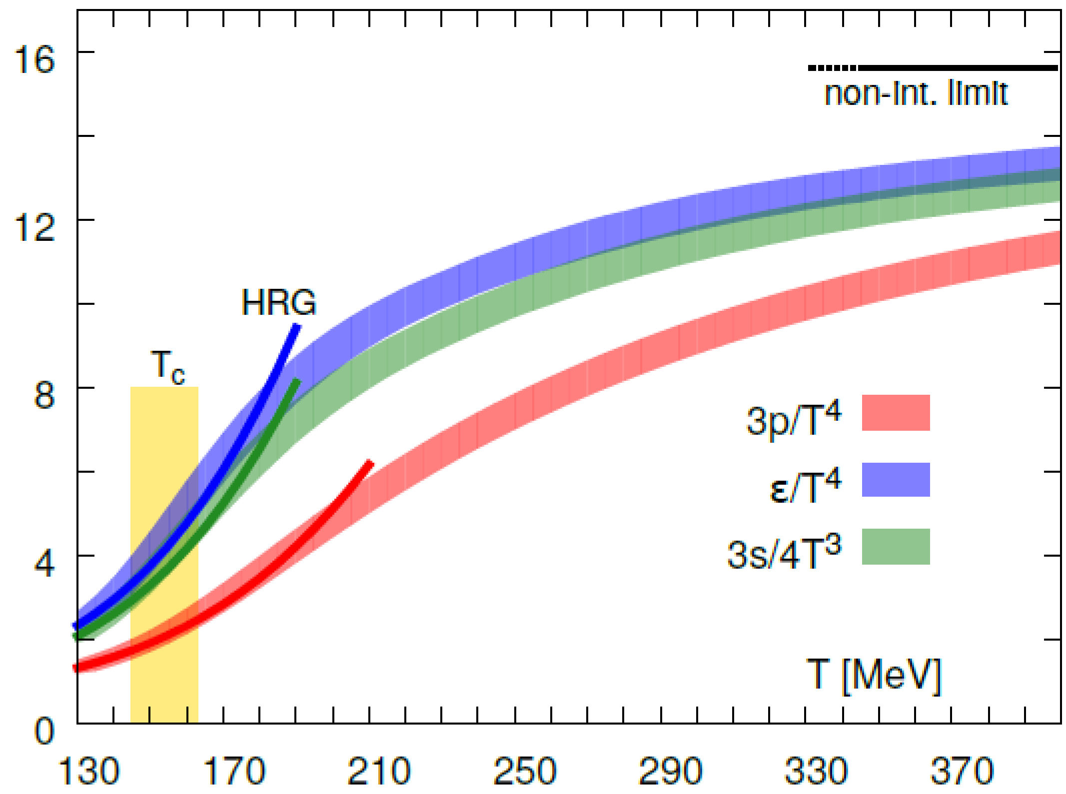


Equation of state in lattice QCD

consolidated results from different groups, extrapolated to continuum and chiral limit, for references see slide 28

rapid rise of energy density (normalized to T^4 rise for relativistic gas)
- signals rapid increase in degrees of freedom due to transition from hadrons to quarks and gluons

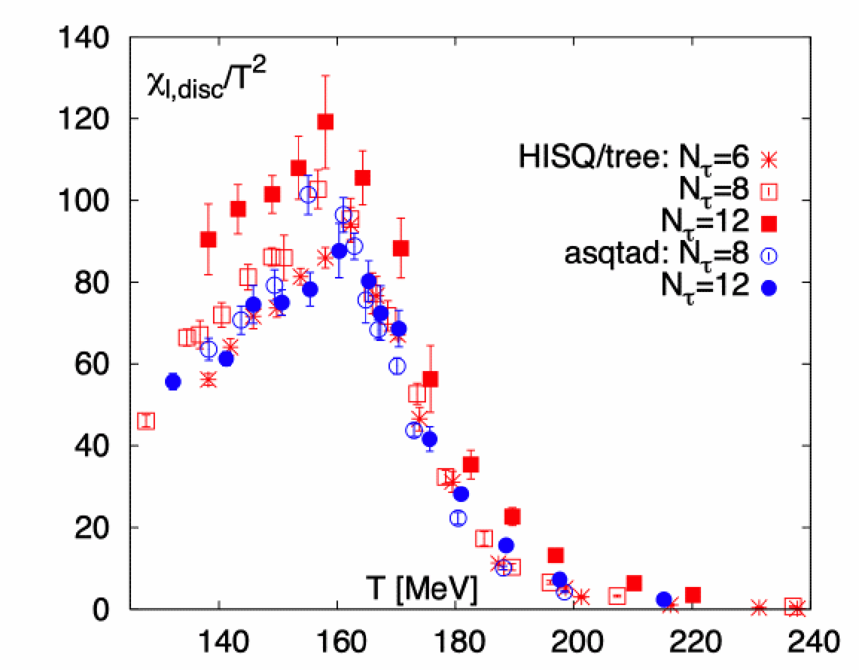
H.T. Ding. arXiv:1504.05274



What is most realistic value of the critical temperature?

order parameter: chiral condensate, smooth behavior across phase conversion
its susceptibility peaks at T_c

S.Borsanyi et al. Wuppertal-Budapest Coll., JHEP 1009 (2010) 073
A.Bazavov et al. HotQCD Coll., PRD 85 (2012) 054503



$$\langle \bar{\Psi}\Psi \rangle = \frac{T}{V}\frac{\partial \ln Z}{\partial m}$$

$$\chi_{\bar{\Psi}\Psi} = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial m^2}$$

T_c from peak in chiral susceptibility
= 154 ± 9 MeV for chiral restoration

latest updates on QGP transition temperature

HotQCD Collaboration, A. Bazavov *et al.*, "Chiral crossover in QCD at zero and non-zero chemical potentials," *Phys. Lett. B* **795** (2019) 15–21, arXiv:1812.08235 [hep-lat].

S. Borsanyi, Z. Fodor, J. N. Guenther, R. Kara, S. D. Katz, P. Parotto, A. Pasztor, C. Ratti, and K. K. Szabo, "QCD Crossover at Finite Chemical Potential from Lattice Simulations," *Phys. Rev. Lett.* **125** no. 5, (2020) 052001, arXiv:2002.02821 [hep-lat].

T_{pc} = 156.5 ± 1.5 MeV

precision better than 1%

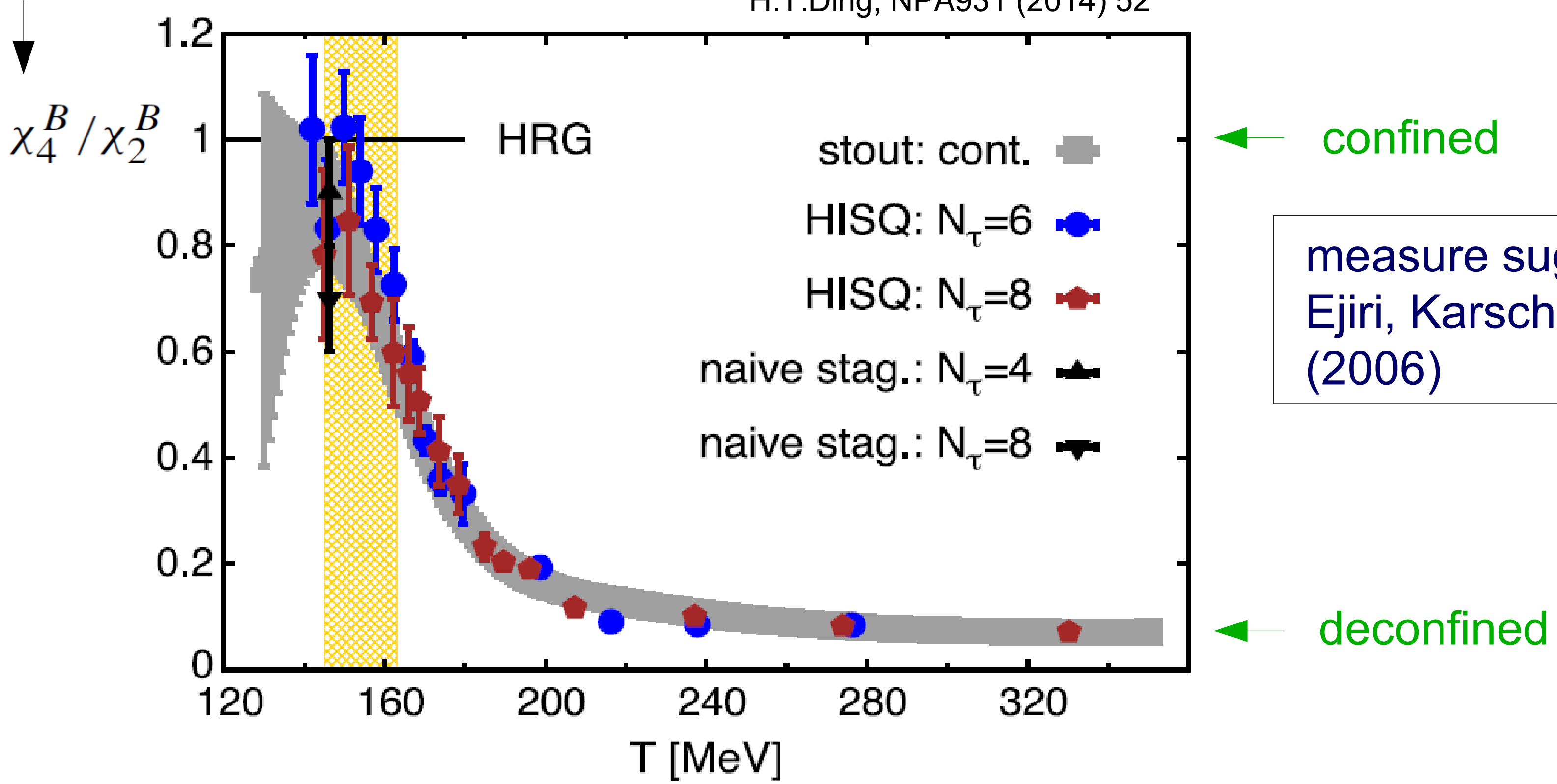
see below for comparison to experimental data

Measure of deconfinement in lQCD

$$\chi_n^B = \frac{d^n P/T^4}{d\mu_B^n}$$

$$\chi_4^B/\chi_2^B \propto \text{baryon number}^2$$

H.T.Ding, NPA931 (2014) 52



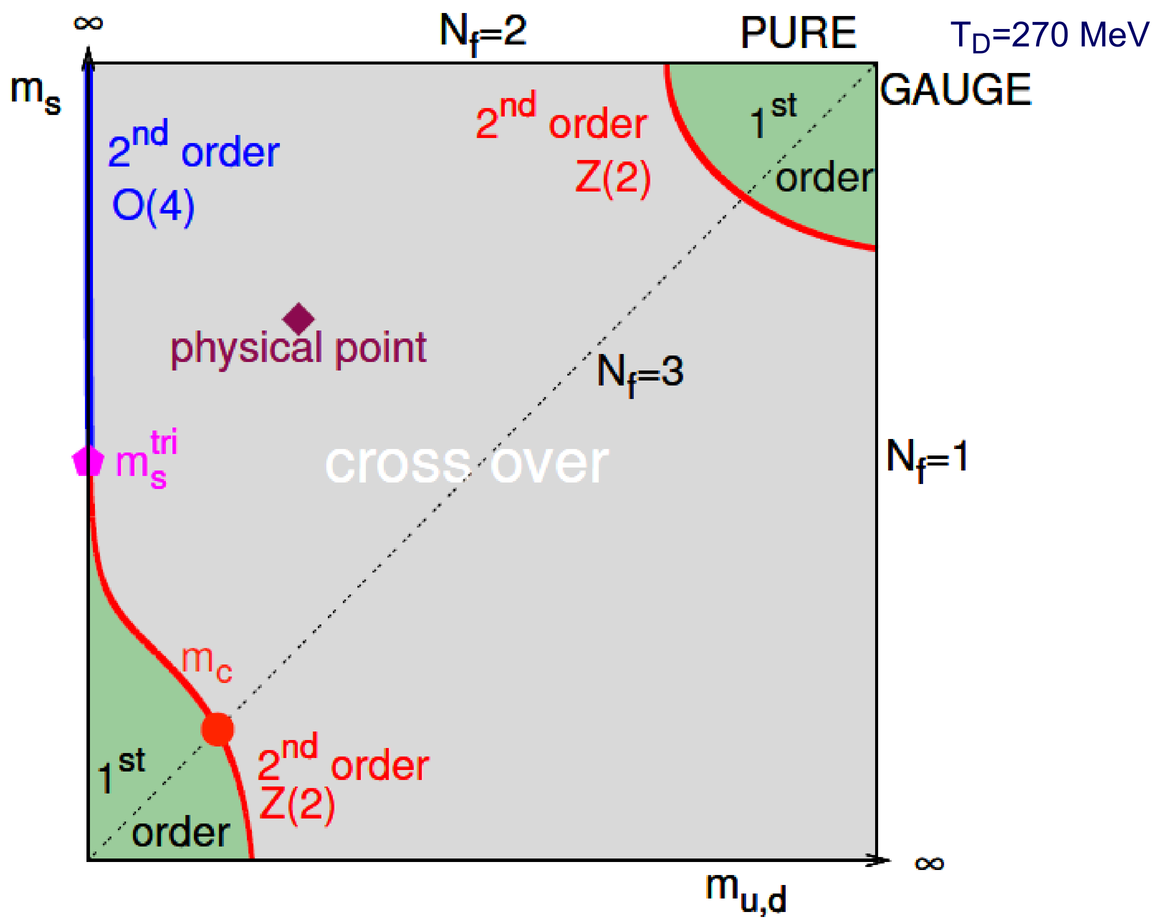
measure suggested by Ejiri, Karsch, Redlich (2006)

rapid drop suggests:
chiral cross over and deconfinement appear in the same narrow temperature range

Order of phase transition

present state-of-the art lattice QCD simulations give smooth cross over for realistic quark masses

critical role of strange quark mass



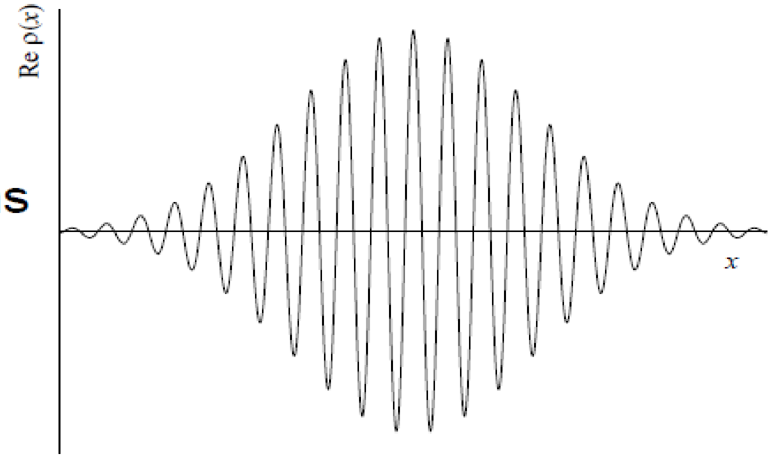
Phase diagram in 2+1 flavor QCD (arXiv: 1504.05273

Lattice QCD at finite baryon density

Lattice QCD at non-equal numbers of fermions and antifermions (non-zero baryon chemical potential) has a problem:
- for Fermions the partition function contains a Slater determinant
- for non-zero chemical potential this Slater determinant is complex
- straight forward importance sampling not possible
- oscillations i.e. the lattice QCD sign problem

$$\det M(\mu) = |\det M(\mu)| e^{i\theta}$$

dominant configurations in the path integral?



Use Taylor expansion to extrapolate into region of finite chemical potential
final results are shown when we compare the lattice predictions with experimental data later in the course