Reminder: QED result for transition amplitude

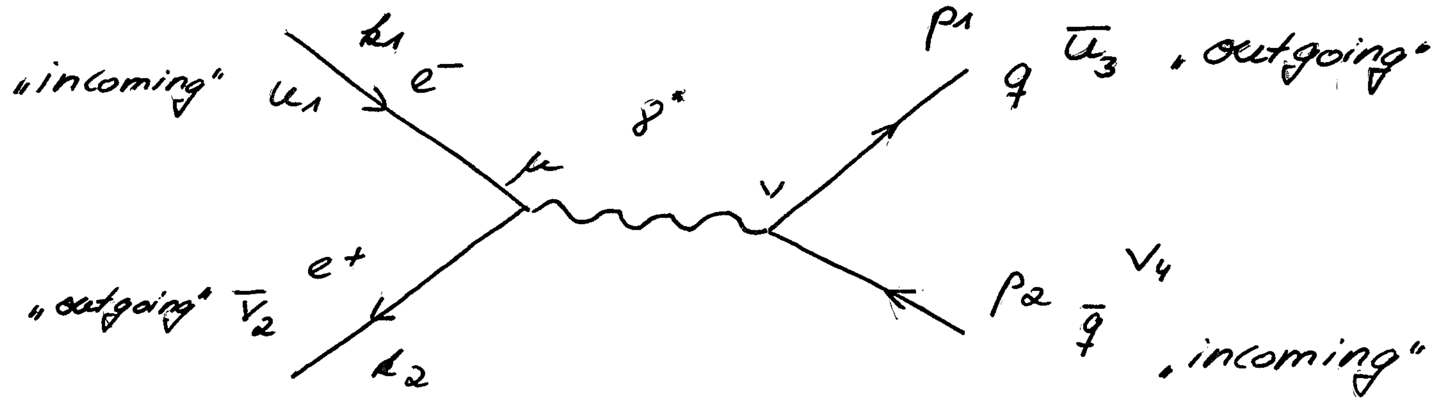
(matrix element ≡ transition amplitude)

particle spinor $u(p,s)$ p: 4 momentum

antiparticle spinor $v(p,s)$ s: spin

(slide 1)

Example: $e^+e^- \to q\bar{q}$



$$-i\mathcal{M} = [\bar{v}_2 (ieQ_e\gamma^\mu) u_1] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}_3 (ieQ_q\gamma^\nu) v_4]$$

q^2: 4 momentum transfer $q^2 = (k_1+k_2)^2 = (p_1+p_2)^2$

$$|\mathcal{M}|^2 = \sum_{\text{spin, color}} e^4 Q_e^2 Q_q^2 \frac{1}{(k_1+k_2)^4} (\bar{v}_4 \gamma_\nu u_3)(\bar{u}_3 \gamma_\mu v_4)(\bar{u}_1 \gamma^\nu v_2)(\bar{v}_2 \gamma^\mu u_1)$$

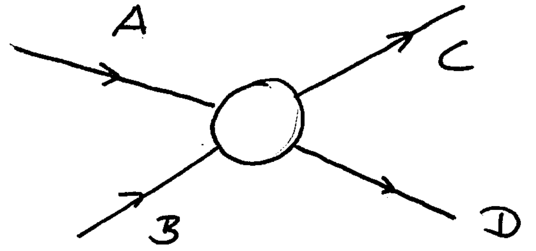
$$= 32 e^4 Q_e^2 Q_q^2 N_c \frac{1}{(k_1+k_2)^4} [(k_1p_1)(k_2p_2) + (k_1p_2)(k_2p_1)]$$

LI (only products of 4-momenta)

What is the differential cross-section for this reaction?

Kinematics:

kinematic of process described by 4 momenta p_A, p_B, p_C, p_D



← this is no Feynman diagram!

possible LI combinations can depend on

$p_A^2, p_B^2, p_C^2, p_D^2, p_Ap_B, p_Ap_C, p_Ap_D, p_Bp_C, p_Bp_D, p_Cp_D$

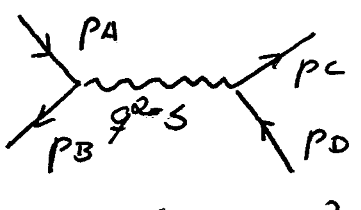
=> 10 combinations

$p_i^2 = m_i^2$ => 4 constraints

$p_A + p_B = p_C + p_D$ -> 4 constraints

=> 2 independent scalar products

Usually use 2 of the 3 Mandelstam variables s, t, u



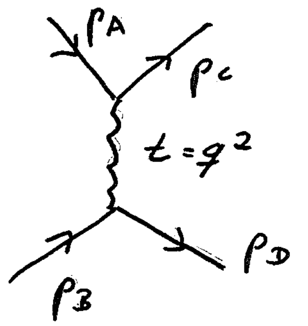
$s = (p_A + p_B)^2$

$= [(E_A^*, \vec{p}_A^{\,*}) + (E_B^*, -\vec{p}_A^{\,*})]^2$

in CMS

* indicate CMS

$= E_{CMS}^2$

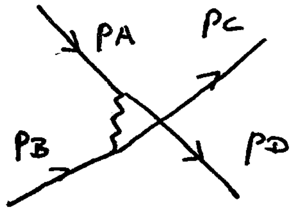


$t = (p_A - p_C)^2$

$= (p_D - p_B)^2$

u only important for undistinguishable particles in the final state

$s+t+u = m_A^2 + m_B^2 + m_C^2 + m_D^2$



$q^2 = u = (p_A - p_D)^2$

Express $|\mathcal{M}|^2$ in terms of Mandelstam variables

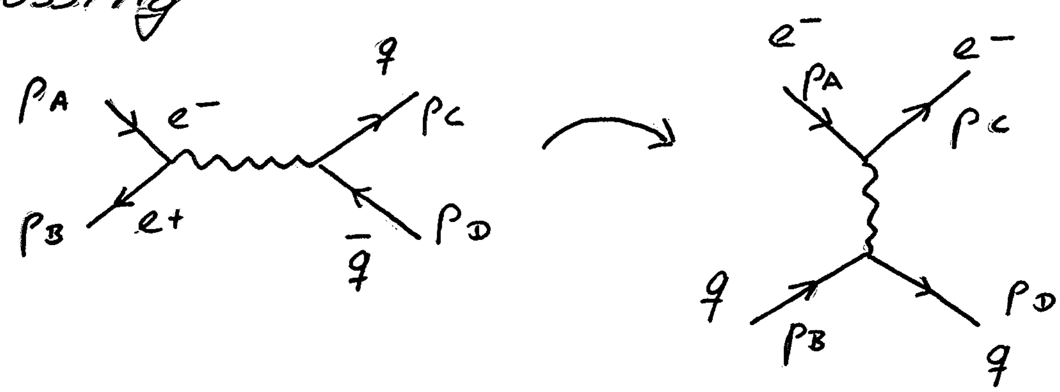
$m_1, m_2 \approx 0$

$$(k_1 - p_2)^2 = k_1^2 - 2k_1 p_2 + p_2^2 = -2k_1 p_2$$

$$|\mathcal{M}|^2 = 32 e^4 Q_e^2 Q_q^2 N_c \frac{1}{(k_1+k_2)^4}\left[(k_1p_1)(k_2p_2) + (k_1p_2)(k_2p_1)\right]$$

$$= 8 e^4 Q_e^2 Q_q^2 N_c \, \frac{t^2+u^2}{s^2}$$

" Crossing "



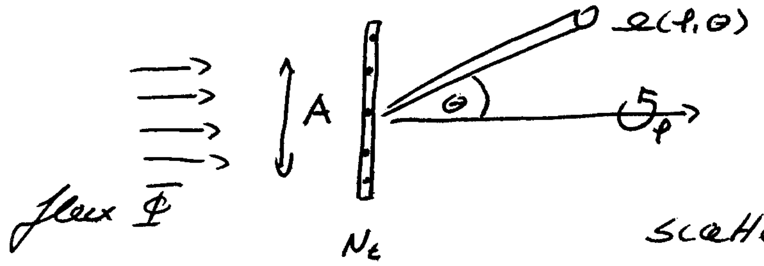
compare incoming particles in both diagrams and outgoing particles in both diagrams

$$\left.\begin{aligned} p_A &\longrightarrow p_A \\ p_B &\longrightarrow -p_C^{\,*)} \\ p_C &\longrightarrow p_D \\ p_D &\longrightarrow -p_B^{\,*)} \end{aligned}\right\} \quad \begin{aligned} s^2 &\to t^2 \\ t^2 &\to u^2 \\ u^2 &\to s^2 \end{aligned}$$

$$|\mathcal{M}|_t^2 = 8 e^4 Q_e^2 Q_q^2 N_c \, \frac{u^2+s^2}{t^2}$$

*) change of sign by transition from particle to antiparticle

Experimental cross-section



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area A

scattering rate: $\frac{dN_s}{dt} = \dot{N}_s$

incoming flux: $\Phi = \frac{\dot{N}_i}{A} = \frac{dN_i}{dt}\,\frac{dx}{A\,dx}$

$$= \frac{dN_i}{A\,dx}\,\frac{dx}{dt} = n_i v_i$$

Φ_i : incoming particle flux $[\frac{1}{s\,m^2}]$

$\dot{N}_i$: rate of incoming particles on surface A $[\frac{1}{s}]$

n_i : particle density in the beam $[\frac{1}{m^3}]$

v_i : velocity of incoming particles $[\frac{m}{s}]$

N_t : number of target particles

$\dot{N}_s$: rate of scattered particles

Typical units of cross-section : barn $= 10^{-28}\,m^2$

$= 10^{-24}\,cm^2$

rate of scattered particles

$$\dot{N}_s = \mathcal{L}\,\sigma$$

(instantaneous) luminosity $\frac{fb^{-1}}{s}$

$$\mathcal{L}_{int} = \int \mathcal{L}\,dt$$

integrated luminosity

integrated luminosity at LHC in 2016 :

$$\sim 40\,fb^{-1} = (10^{-15}\,barn)^{-1}$$

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Scattering operator S and transition amplitude

Slide 4

Recap:

$$\lim_{t\to\infty} |t\rangle = S|i\rangle$$

Measurement selects specific state $|f\rangle$

$\langle f|t\rangle = \langle f|S|i\rangle = S_{fi} \equiv$ transition amplitude

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta(p_f - p_i) M_{fi}$$

Probability density

$$P_{fi} = |S_{fi}|^2 = (2\pi)^8 [\delta(p_f - p_i)]^2 |M_{fi}|^2$$

To compute transition probability has to consider number of possible states for each of the outgoing particles $\Rightarrow$ phase space

Look again at $\quad A + B \rightarrow C + D$
$\qquad\qquad\qquad\quad |i\rangle \qquad\quad |f\rangle$

LI phase space: $\quad dN_f = \dfrac{d^3p_C}{2E_C(2\pi)^3} \dfrac{d^3p_D}{2E_D(2\pi)^3}$

Interlude: LI normalisation and phase space

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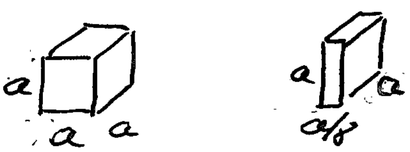
Normalisation: one particle/volume $(V = a^3)$

$\int d^3x\, \psi^+\psi = 1/V \qquad$ (often V defined as 1, will cancel later in any case)



$\Rightarrow V = \beta c$

boosted system

$V = a^3 \qquad N' = \frac{a^3}{\gamma} \Leftarrow$ not LI normalisation

Fix this by choice of normalisation

$$\int d^3x\ \psi^\dagger \psi = 2E/V$$

not LI form

$$|V_{fi}|^2 = |\langle f|V|i\rangle|^2 \Rightarrow 2E_A 2E_B 2E_C 2E_D |V_{fi}|^2 = |M_{fi}|^2$$

$$dN_f = \frac{d^3p}{(2\pi)^3} \Rightarrow dN_f = \frac{dp}{(2\pi)^3\, 2E_C\, 2E_D}$$

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_A 2E_B}\int |M_{fi}|^2\, \delta(p_A+p_B-p_C-p_D)\, \frac{d^3p_C}{2E_C(2\pi)^3}\frac{d^3p_D}{2E_D(2\pi)^3}$$

Γ_{fi}: transition rate (definition see later)

$\frac{(2\pi)^4}{2E_A 2E_B}$: not LI

$|M_{fi}|^2$: LI

$\frac{d^3p_C}{2E_C(2\pi)^3}\frac{d^3p_D}{2E_D(2\pi)^3}$: LI

as expected rate becomes smaller due to time dilatation

Amplitude, cross section and phase space

$$\sigma = \frac{\dot{N}_S}{\Phi N_t} = \frac{\dot{N}_S}{V}\frac{1}{\Phi/V\, N_t} = \underbrace{\frac{\text{transition probability}}{V\, T}}_{\text{transition rate per volume}}\ \frac{1}{\Phi/V\, N_t}$$

$$\omega_{fi} = \Gamma_{fi}/V$$

$$\omega_{fi} = \frac{|S_{fi}|^2\, dN_f}{T V} = \frac{(2\pi)^8 [\delta^4(p_f-p_i)]^2\, |M_{fi}|^2\, dN_f}{V T}$$

Fermi's trick in 1D

$$[2\pi\,\delta(x-x')]^2 = \underbrace{\int_{-\infty}^{+\infty} dt\, e^{-(x-x')t}}_{2\pi\delta(x-x')}\ 2\pi\,\delta(x-x')$$

$$= \underbrace{\int_{-\infty}^{+\infty} dt}_{=T}\ 2\pi\,\delta(x-x')$$

since relevant contribution only for $e^{i\Delta x t} = 1$ e.g. $\Delta x = 0$

argument:
no experiment runs forever, just choose T large enough

$$\omega_{fi} = (2\pi)^4 \, \delta^4(p_f - p_i) \, |M_{fi}|^2 \, dN_f$$

<u>Next step: compute incident flux</u> $\Phi/V \cdot N_t$

Lab $\quad 1 \longrightarrow \cdot 2$

CMS $\quad 1 \longrightarrow \longleftarrow 2 \qquad \vec{p}_1 = -\vec{p}_2 = \vec{p}_i^{\,*}$

Lab $\quad \Phi/V \, N_t = n_1 v_1 \cdot \frac{N_t}{V} = 2E_1 2E_2 \left(\frac{|\vec{p}_1|}{E_1} \right)$

CMS

$$\begin{aligned} \Phi/V \, N_t &= 2E_1 2E_2 \, |\vec{v}_1 - \vec{v}_2| \\ &= 2E_1 2E_2 \left| \frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right| \\ &= 4\left(|\vec{p}_1| E_2 + |\vec{p}_2| E_1\right) = 4\left(|\vec{p}_i|^* (E_1 + E_2)\right) \\ &= 4 |\vec{p}_i|^* \sqrt{s} \end{aligned}$$

with little math can show

$$4\left(|\vec{p}_1| E_2 + |\vec{p}_2| E_1\right) = 4\left[(p_1 p_2)^2 - m_1^2 m_2^2\right]^{1/2}$$

$\Rightarrow$ flux in CMS is LI

<u>Differential cross-section</u>

$$d\sigma = \frac{|M_{fi}|^2}{4\left[(p_1 p_2)^2 - m_1^2 m_2^2\right]^{1/2}} \underbrace{(2\pi)^4 \, \delta^4(p_f - p_i) \, \frac{d^3 p_c}{2E_c (2\pi)^3} \, \frac{d^3 p_D}{2E_D (2\pi)^3}}_{\substack{LIPS_2 \,\equiv \\ \text{Lorentz invariant 2-body} \\ \text{phase space}}}$$

LIPS for n particles

$$d\,LIPS_n(p_i, \underbrace{p_1, p_2, \ldots, p_n}_{\text{final state particles}}) = (2\pi)^4 \, \delta^4\left(p_i - (p_1 + \ldots + p_n)\right) \prod_{\substack{\text{final} \\ \text{state} \\ \text{particles}}} \frac{d p_f^3}{(2\pi)^3 2E_f}$$

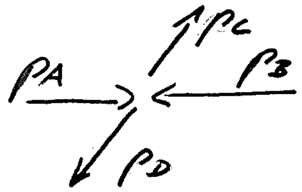
Differential cross-section

$$\frac{d\sigma}{d\Omega}$$

↑ not LI => need to define system
angle depends on reference frame

* indicate CMS

In CMS



$|\vec{p}_A| = |\vec{p}_B| = |\vec{p}_i^{\,*}|$
$|\vec{p}_C| = |\vec{p}_D| = |\vec{p}_f^{\,*}|$

$$d\Omega_C = d\phi_C \, d\cos\theta_C$$

$$\int dLIPS_2 = \frac{1}{4\pi^2}\int \delta^3(\vec{p}_C + \vec{p}_D)\,\delta(E_A + E_B - E_C - E_D)\,\frac{d^3p}{2E_C}\,\frac{d^3p}{2E_D}$$

$d^3p = d\Omega^* p^2 dp$

$$= \int d\Omega_C^* \frac{1}{16\pi^2}\int \delta(E_A + E_B - E_C - E_D)\,\frac{|p_C|^2\, d|\vec{p}_C|}{E_C E_D}$$

using $\omega = E_C + E_D = \sqrt{m_C^2 + p_C^2} + \sqrt{m_D^2 + p_C^2}$

$$\frac{d\omega}{dp_C} = p_C\left(\frac{1}{E_C} + \frac{1}{E_D}\right) \Rightarrow p_C\, dp_C = d\omega\, \frac{E_C E_D}{E_C + E_D}$$

$$\int dLIPS_2 = \int d\Omega_C^* \frac{1}{16\pi^2}\int \delta(E_A + E_B - \omega)\,\frac{|p_C|}{E_C + E_D}\, d\omega$$

$$= \int d\Omega_C^* \frac{1}{16\pi^2}\,\frac{|p_C|}{E_C + E_D} = \int d\Omega_C \frac{1}{16\pi^2}\,\frac{|p_C|}{\sqrt{s}}$$

Putting everything together:

$A + B \rightarrow C + D$

CMS: $$d\sigma = \frac{|M_{fi}|^2}{4|p_i^*|\sqrt{s}}\,\frac{1}{16\pi^2}\,\frac{1}{\sqrt{s}}\,|p_f^*|\, d\Omega_f^*$$

$$\boxed{\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2}\,\frac{|p_f^*|}{|p_i^*|}\,\frac{|M_{fi}|^2}{s}}$$

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Rapidity

$\frac{d\sigma}{d\Omega}$ is not LI

$\Rightarrow$ define rapidity $y = \frac{1}{2} \ln \left(\frac{E+p_z}{E-p_z} \right)$

$\left[\int d\phi = 4\pi \right.$

head on collisions

$\left. \text{are symmetric in } \phi \right]$

for highly relativistic particles

$\theta \sim 0 \quad p_z \sim E \quad \Rightarrow \quad y \to \infty$

$\theta \sim \pi \quad p_z \sim E \quad \Rightarrow \quad y \to -\infty$

$\theta \sim \pi/2 \quad p_z = 0 \quad \Rightarrow \quad y = 0$

$$y = \frac{1}{2} \ln \left(\frac{E+p_z}{E-p_z} \right) = \ln \sqrt{\frac{E+p_z}{E-p_z}}$$

$$= \ln \frac{E+p_z}{\sqrt{E^2-p_z^2}} = \tanh^{-1} \tanh \ln \frac{E+p_z}{\sqrt{E^2-p_z^2}}$$

use $\tanh \theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$

$$y = \tanh^{-1} \left[\frac{\frac{E+p_z}{\sqrt{E^2-p_z^2}} - \frac{\sqrt{E^2-p_z^2}}{E+p_z}}{\frac{E+p_z}{\sqrt{E^2-p_z^2}} + \frac{\sqrt{E^2-p_z^2}}{E+p_z}} \right]$$

$$= \ldots = \tanh^{-1} \frac{p_z}{E}$$

Rapidity transformation under Lorentz boost parallel to z-axis

$$y' = \frac{1}{2} \ln \frac{\gamma E - \beta\gamma p_z + \gamma p_z - \beta\gamma E}{\gamma E - \beta\gamma p_z - \gamma p_z + \beta\gamma E}$$

$$= \frac{1}{2} \ln \frac{\gamma(E+p_z) - \beta\gamma(E+p_z)}{\gamma(E-p_z) + \beta\gamma(E-p_z)}$$

$$= \frac{1}{2} \ln \frac{E+p_z}{E-p_z} \cdot \frac{\gamma-\beta\gamma}{\gamma+\beta\gamma}$$

$$= \frac{1}{2} \ln \frac{E+p_z}{E-p_z} + \ln\sqrt{\frac{1-\beta}{1+\beta}}$$

$$y' = y + \ln\sqrt{\frac{1-\beta}{1+\beta}}$$

$\Delta y = y_1 - y_2 = y_1' - y_2' = \Delta y' \quad \Rightarrow \Delta y$ is LI!

$\Rightarrow \frac{d\sigma}{dy}$ is LI!

Pseudorapidity:

For highly relativistic particles use approximation $m \ll p$

$$y = \frac{1}{2} \ln \left(\frac{E+p_z}{E-p_z}\right) = \frac{1}{2} \ln \frac{(p^2+m^2)^{1/2} + p_z}{(p^2+m^2)^{1/2} - p_z}$$

$$= \frac{1}{2} \ln \frac{\left(1+\frac{m^2}{p^2}\right)^{1/2} + \frac{p_z}{p}}{\left(1+\frac{m^2}{p^2}\right)^{1/2} - \frac{p_z}{p}}$$

$$\simeq \frac{1}{2} \ln \frac{1 + \frac{p_z}{p} + \frac{m^2}{2p^2} + \dots}{1 - \frac{p_z}{p} + \frac{m^2}{2p^2} + \dots}$$

$$1+\frac{p_z}{p} = 1+\cos\theta = 1+\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) = 2\cos^2\theta/2$$

$$1-\frac{p_z}{p} = 2\sin^2\theta/2$$

$$y \simeq \frac{1}{2}\ln\frac{\cos^2\theta/2}{\sin^2\theta/2} = -\ln\tan\theta/2$$

Define pseudorapidity $\eta = -\ln\tan\theta/2$
for highly relativistic particles $\eta = y$

In experiment η is faster to compute

slides