4 Not using Feynman rules

In these LHC lecture notes we always assume some field theory background which allows us to compute transition amplitudes on the basis of Feynman rules and phase space integrals. In other words, the corresponding lectures are meant to be heard after Quantum Field Theory I and II. Nevertheless, it can be useful to briefly repeat the steps which we have to go through to compute a transition amplitude from an action or from first principles. In this short section we will sketch this calculation and indicate where in the calculation Feynman ruled come in and significantly simplify our lives. In that sense this section is not actually needed to understand the other parts of these notes, but it might come in handy at times.

When we compute transition amplitudes for collider like LEP or LHC, we usually combine building blocks defined by Feynman rules in a way which does not make it obvious that we are dealing with a quantum field theory. For example, in Section 2.1 we compute the transition amplitude for the process $e^+e^- \to \gamma^* \to q\bar{q}$ through a photon, all starting from these Feynman rules. In this section, we will give a brief sketch of what we have to do to describe the quantum fields involved and to compute this transition amplitude without using Feynman rules.

From theoretical mechanics we remember that there are several ways to describe a physical system and calculate the time evolution of the states. One object to start from is the action as a function of one degree of freedom or field ϕ

$$\boxed{S = \int d^4x \, \mathscr{L}(\phi, \partial_\mu \phi)} \, . \tag{4.1}$$

With x we denote the four-vector including the time component $(x_0, \vec{x})$. The action has to be invariant under a variation $\delta S = 0$. We can translate this condition into the Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathscr{L}}{\partial \phi} \, , \tag{4.2}$$

Using the notation $\partial_\mu = \partial / \partial x^\mu$. A convenient second parameter in addition to x is the conjugate momentum $\pi(x) = \partial \mathscr{L} / \partial \dot{\phi}$. With this coordinate we define the third object which we can use to describe the dynamics of a system, the Hamiltonian density

$$\int d^3x \, \mathcal{H}(x) = \int d^3x \, \left(\pi \dot{\phi} - \mathscr{L} \right) \, . \tag{4.3}$$

While for example in quantum mechanics this Hamiltonian or energy functional is the basis of most calculations, in field theory we usually start from the Lagrangian.

We already know that for our scattering process we need to compute a transition amplitude between two kinds of matter particles, namely incoming electrons and outgoing quarks, interacting via their electric charges. The interaction is classically described by the electromagnetic Lagrangian based on the abelian $U(1)$ field theory,

$$\mathscr{L} \supset -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, , \tag{4.4}$$

in terms of a photon vector field A_μ. The Maxwell equations

$$0 = \partial^\mu F_{\mu\nu} = \partial^\mu \partial_\mu A_\nu - \partial^\mu \partial_\nu A_\mu = \Box A_\nu \tag{4.5}$$

are the equations of motion for this photon field. In the last step we assume the Lorentz gauge condition $\partial_\mu A^\mu = 0$ and find the d'Alembert equation for the vector potential A_μ.

To omit the vector index of the photon field we instead use a real scalar field ϕ to illustrate bosonic fields, their quantization, their equation of motions, and how they enter a calculation. Including a mass for this real scalar field we

can write down its equation of motion which is the same for a spin-zero scalar boson as for the spin-one vector boson of Eq.(4.5)

$$\left(\Box + m^2\right)\, \phi(x) = 0\,. \tag{4.6}$$

This Klein–Gordon equation corresponds to the d'Alembert equation for the electromagnetic vector potential in Lorentz gauge. This equation of motion corresponds to a contribution to the Lagrangian of the form

$$\mathscr{L} \supset \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2\,, \tag{4.7}$$

which we can easily confirm using the Euler-Lagrange equation Eq.(4.2).

Under a Lorentz transformation of the d'Alembert operator and of the scalar field,

$$\begin{aligned} x^\mu &\to x'^\mu = \Lambda^\mu{}_\nu\, x^\nu \\ \phi(x) &\to \phi'(x) = \phi(\Lambda^{-1}x)\,, \end{aligned} \tag{4.8}$$

the Klein–Gordon equation keeps its form in the new coordinates. It is a standard wave equation which we can solve using plane waves, and which modulo prefactors gives us

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2k_0}\, \left(e^{ik\cdot x}a^*(\vec{k}) + e^{-ik\cdot x}a(\vec{k})\right)\,. \tag{4.9}$$

Complex conjugates a and a^* are required for a real field ϕ. The value of k_0 is given by the dispersion relation, which means that if ϕ fulfills the Klein–Gordon equation it is $k_0^2 = \vec{k}^2 + m^2$. The Fourier transform a therefore explicitly only depends on $\vec{k}$. Up to this point the field ϕ is a real function of the space coordinates, *i.e.* it is not quantized.

Because in the remaining lecture we will only use quantized operator valued fields we will not change notation at this point. Switching from a field function to a field operator ϕ leaves the Klein–Gordon equation Eq.(4.6) the same, except that now it constrains an operator $\phi(x)$ which for example we cannot simply commute with other operators. ϕ used to be a real field, so now it will be a hermitian operator $\phi^\dagger = \phi$. In the plane wave solution Eq.(4.9) the coefficients a and (now) $a^\dagger$ are promoted to operators as well.
The Hilbert space in which our system resides includes a vacuum state $|0\rangle$ which has to be normalized, $\langle 0|0\rangle = 1$, and which has zero energy and momentum. We can show that the on–shell state $|k\rangle \equiv a^\dagger(\vec{k})|0\rangle$ is an eigenvector of the energy–momentum operator with eigenvalues k^μ. We can interpret $a^\dagger$ as a creation operator for a state with four-momentum k^μ. Successive application of $a^\dagger(\vec{k}_j)$ on the vacuum gives a basis of states with a varying number of states, defining the Fock space. Acting with the energy–momentum operator on $a(\vec{k})|0\rangle$ gives us a negative energy eigenvalue, which means that we have to set $a(\vec{k})|0\rangle = 0$; the operator $a(\vec{k})$ is an annihilation operator.
For operators it is crucial that we define their commutation relations. The basic property which we use to fix the commutators of a and $a^\dagger$ is causality, namely that field configurations outside each other's light cone cannot affect each other and therefore their field operators commute

$$\begin{aligned} & [\phi(x), \phi(x')] = 0 \qquad \text{for} \quad (x-x')^2 < 0 \quad \text{or} \quad (t-t')^2 < |\vec{x}-\vec{x}'|^2 \\ \text{or} \qquad & [\phi(\vec{x}), \phi(\vec{x}')] = 0 \qquad \text{for} \quad t = t',\ |\vec{x}-\vec{x}'| > 0\,. \end{aligned} \tag{4.10}$$

Similar commutators exist between the field ϕ and its time derivative $\dot{\phi}$. We can insert the operator–valued form of Eq.(4.9) into the equal-time commutators ($t = t', \vec{x} \neq \vec{x}'$) of ϕ and $\dot{\phi}$, where the latter allows us to vary the relative factor between a and $a^\dagger$ and derive several independent relations. One of them reads

$$\begin{aligned} 0 &= \left[\phi(\vec{x}), \dot{\phi}(\vec{x}')\right] \\ &= \int \frac{d^3k}{(2\pi)^3 2k_0}\, \frac{d^3k'}{(2\pi)^3 2k'_0}\, e^{-i(\vec{k}\vec{x}+\vec{k}'\vec{x}')}(ik'_0)\left[\left(e^{ik_0t}a^\dagger(\vec{k}) + e^{-ik_0t}a(\vec{k})\right), \left(e^{ik'_0t}a^\dagger(\vec{k}') - e^{-ik'_0t}a(-\vec{k}')\right)\right]\,. \end{aligned} \tag{4.11}$$

In this Fourier transform for free k_0 and k_0' the integral vanishes only if the integrand, and therefore all commutators under the integral vanish, at least as long as $\vec{x} \neq \vec{x}'$. For $\vec{x} = \vec{x}'$ the condition $[\phi(\vec{x},t), \dot{\phi}(\vec{x}',t)] = i\delta^3(\vec{x}-\vec{x}')$ fixes the third commutation relations for bosonic creation and annihilation operators

$$\begin{aligned}\left[a^\dagger(\vec{k}), a^\dagger(\vec{k}')\right] &= 0 \\ \left[a(\vec{k}), a(\vec{k}')\right] &= 0 \\ \left[a(\vec{k}), a^\dagger(\vec{k}')\right] &= (2\pi)^3 2k_0\ \delta^3(\vec{k}-\vec{k}')\ . \end{aligned} \tag{4.12}$$

Using these commutators we can analyze specific configurations values of the kind $\langle 0|a(\vec{k})a^\dagger(\vec{k})|0\rangle$, which do not vanish. It turns out that the integral defining the expectation value for the energy k^0 in such a basis diverges. This problem we can solve *ad hoc* by postulating that whenever sandwiched between vacuum states we only evaluate combinations where all annihilation operators a are moved to the right and all creation operators $a^\dagger$ to the left

$$\begin{aligned} :a^\dagger a: &= a^\dagger a \\ :a^\dagger a^\dagger: &= a^\dagger a^\dagger \\ :aa: &= aa \\ :aa^\dagger: &= a^\dagger a\ . \end{aligned} \tag{4.13}$$

We can interpret this normal–ordering as ordering the operators in a sensible way and neglecting the corresponding commutators sandwiched between vacuum states, for example $\langle 0|[a,a^\dagger]|0\rangle \equiv 0$. If we want a Hamilton operator to only give positive but finite energy states we need to include a time–ordering into its definition. The Wick theorem links normal–ordering and time–ordering in a way which makes them identical as long as we only compute tree level leading order processes.

Looking at the scattering process we want to evaluate we need to describe is four external fermions, their coupling to a photon, and the propagation of this boson from the e^+e^- annihilation to the point where is splits into a quark and antiquark pair. Let us start with this propagator. Such a propagator is defined as a time–ordered product of two field operators sandwiched between vacuum states. For scalar fields it reads

$$\Delta(x-x') \equiv i\ \langle 0|T\left(\phi(x)\phi(x')\right)|0\rangle\ . \tag{4.14}$$

The time–ordered product of two operators is defined as

$$T\left(A(x)B(x')\right) = \theta(x_0-x_0')A(x)B(x') + \theta(x_0'-x_0)B(x')A(x)\ . \tag{4.15}$$

We again use the operator version of Eq.(4.9) to evaluate this combination for free fields

$$\begin{aligned} \langle 0|\phi(x)\phi(x')|0\rangle\Big|_{x_0>x_0'} &= \int \frac{d^3k}{(2\pi)^3 2k_0}\ \frac{d^3k'}{(2\pi)^3 2k_0'}\ \langle 0|\ \left(e^{ikx}a^\dagger(\vec{k}) + e^{-ikx}a(\vec{k})\right)\ \left(e^{ik'x'}a^\dagger(\vec{k}') + e^{-ik'x'}a(\vec{k}')\right)\ |0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3 2k_0}\ \frac{d^3k'}{(2\pi)^3 2k_0'}\ e^{-i(kx-k'x')}\langle 0|a(\vec{k})a^\dagger(\vec{k}')|0\rangle \qquad \text{with vacuum } \langle 0|a^\dagger = 0 = a|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3 2k_0}\ \frac{d^3k'}{(2\pi)^3 2k_0'}\ e^{-i(kx-k'x')}\langle 0|[a(\vec{k})a^\dagger(\vec{k}')]|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3 2k_0}\ e^{-ik(x-x')}\langle 0|0\rangle \qquad\qquad \text{with } \left[a,a^\dagger\right] = (2\pi)^3 2k_0\ \delta^3(\vec{k}-\vec{k}') \\ &= \int \frac{d^3k}{(2\pi)^3}\ e^{i\vec{k}(\vec{x}-\vec{x}')}\ \frac{e^{-ik_0(x_0-x_0')}}{2k_0}\ . \end{aligned} \tag{4.16}$$

In the integral k_0 is given by its the on–shell value, so for $\vec{k} = \vec{k}'$ we also have $k_0 = k_0'$. Under the assumption $x_0 > x_0'$ the last ratio under the integral can be viewed as the result of a contour integration over the k_0 component, to

allow us to write the propagator as a four-dimensional integral over d^4k. We discuss this integral in detail in Section 2.1.2, where we find in Eq.(2.17)

$$\frac{i}{2\pi}\int dk_0 \frac{e^{-ik_0(x_0-x_0')}}{k^2-m^2+i\varepsilon} = \theta(x_0-x_0')\frac{e^{-ik_0^{(\text{os})}(x_0-x_0')}}{2k_0^{(\text{os})}} + \theta(x_0'-x_0)\frac{e^{ik_0^{(\text{os})}(x_0-x_0')}}{2k_0^{(\text{os})}} \; . \tag{4.17}$$

As in Eq.(4.16) $k_0^{(\text{os})} = \sqrt{\vec{k}^2+m^2}$ is the on–shell value of the energy component of the four-momentum. Using this result and slightly abusing our notation by now writing k_0 for the zero component of the four-dimensional k integration we can write

$$\begin{aligned}
\langle 0|\phi(x)\phi(x')|0\rangle\Big|_{x_0>x_0'} &= \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}(\vec{x}-\vec{x}')}\, \frac{e^{-ik_0(x_0-x_0')}}{2k_0}\, \theta(x_0-x_0') \\
&= i\int \frac{d^4k}{(2\pi)^4}\, e^{i\vec{k}(\vec{x}-\vec{x}')}\, \frac{e^{-ik_0(x_0-x_0')}}{k^2-m^2+i\varepsilon} \\
&= i\int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-x')}\, \frac{1}{k^2-m^2+i\varepsilon} \; .
\end{aligned} \tag{4.18}$$

Similarly, we can show the same relation for $x_0 < x_0'$, picking up the other theta function and returning for the combination of the two

$$\Delta(x-x') = -\int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-x')}\, \frac{1}{k^2-m^2+i\varepsilon} \; . \tag{4.19}$$

This Feynman propagator is a Green function for the Klein–Gordon equation Eq.(4.6), which we can explicitly confirm to read

$$\begin{aligned}
\left(\Box+m^2\right)\Delta(x-x') &= -\int \frac{d^4k}{(2\pi)^4}\, \left(\Box+m^2\right) e^{-ik\cdot(x-x')}\, \frac{1}{k^2-m^2} \\
&= \int \frac{d^4k}{(2\pi)^4}\, \left((ik)^2+m^2\right) e^{-ik\cdot(x-x')}\, \frac{(-1)}{k^2-m^2} \\
&= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} = \delta^4(x-x') \; .
\end{aligned} \tag{4.20}$$

This concludes our discussion of the bosonic propagator. Using a scalar field instead of a vector field we have shown how the field can be expressed in terms of creation and annihilation operators and what the commutation rules for the scalar fields as well as for the creation and annihilation operators are. A major problem arises when we sandwich these operators between vacuum states, which means that such insertions have to be normal–ordered. In addition, the time–ordered correlation function of two scalar fields is the Feynman propagator, defining an inverse of the Klein–Gordon equation over the entire position space.
All these properties we will later use for the photon field. The photon A^μ is a vector field, where each of the components obey the Klein–Gordon equation. The commutation relations and the photon propagator will not change, they will simply be dressed up by factors $g_{\mu\nu}$ where appropriate. For the propagator this generalization is strictly speaking gauge dependent, $g^{\mu\nu}$ corresponds to Feynman gauge. Nevertheless, from Section 2.1.2 we know that additional terms from other gauge choices do not contribute to our scattering amplitude.

The second object we need to describe for our scattering process are the external fermion fields. Most generally, scalars and gauge bosons are not the only particle we find in Nature. Matter particles or fermions, like electrons or quarks have a different equation of motion and a different contribution to the Lagrangian. No matter how it looks, the equation of motion for fermion fields ψ has to be invariant under a Lorentz transformation

$$\psi(x) \to \psi'(x) = \Lambda_{1/2}\, \psi(\Lambda^{-1}x) \; , \tag{4.21}$$

where $\Lambda_{1/2}$ is a different representation of the Lorentz transformations. It is called spinor representation and we can define it using the four Dirac matrices γ^μ with their anti–commutator Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1}\,. \tag{4.22}$$

The unit matrix has the same size as the γ matrices. That we usually write them as 4×4 matrices has nothing to do with the number of — also four — γ matrices. The explicit form of the γ_μ matrices is not relevant because it never appears in actual calculations. All we need is a few trace relations arising from their commutators. A representation of the Lorentz algebra in terms of the Dirac matrices is

$$S^{\mu\nu} = \frac{i}{4}\,[\gamma^\mu,\gamma^\nu] \qquad\qquad \text{implying} \quad \Lambda_{1/2} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\,. \tag{4.23}$$

Using the transformation property $\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu$ we can postulate an equation of motion for the fermions, the Dirac equation

$$(i\gamma^\mu\partial_\mu - m\mathbb{1})\;\psi(x) \equiv (i\partial\!\!\!/ - m\mathbb{1})\;\psi(x) = 0\,. \tag{4.24}$$

The unit matrix in the mass term is a four-by-four matrix, just like the Dirac matrices. Of course, we need to check that this equation is invariant under Lorentz transformations, keeping in mind that $\Lambda_{1/2}$ commutes with everything except for the Dirac matrices

$$\begin{aligned}
(i\gamma^\mu\partial_\mu - m\mathbb{1})\;\psi(x) &\to \left(i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\mathbb{1}\right)\;\Lambda_{1/2}\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\Lambda_{1/2}^{-1}\left(i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\mathbb{1}\right)\;\Lambda_{1/2}\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(i\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\mathbb{1}\right)\;\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(i\Lambda^\mu{}_\rho\gamma^\rho(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\mathbb{1}\right)\;\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(ig^\nu{}_\rho\gamma^\rho\partial_\nu - m\mathbb{1}\right)\;\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(i\gamma^\nu\partial_\nu - m\mathbb{1}\right)\;\psi(\Lambda^{-1}x) = 0\,.
\end{aligned} \tag{4.25}$$

An interesting side remark is that we can multiply the Dirac equation with $(-i\gamma^\mu\partial_\mu - m\mathbb{1})$ and obtain the Klein–Gordon equation $(\partial^2 + m^2)\psi = 0$, which will be useful when we construct a fermion propagator.
An additional problem is that for example to define a mass term in the Lagrangian we need to form Lorentz scalars or invariants out of the fermion fields ψ. Naively, $(\psi^\dagger\psi)$ would work if the Lorentz transformations in $(\psi^\dagger\Lambda_{1/2}^\dagger\Lambda_{1/2}\psi)$ cancelled. Unfortunately $\Lambda_{1/2}$ is not a unitary transformation, which means we have to go beyond $(\psi^\dagger\psi)$. One can show that the Dirac adjoint

$$\overline{\psi} = \psi^\dagger\gamma^0 \qquad\qquad \text{with} \quad \overline{\psi} \to \overline{\psi}\Lambda_{1/2}^{-1} \qquad \text{and} \quad \overline{\psi}\psi \to \overline{\psi}\psi \tag{4.26}$$

has the correct transformation property. This allows us to write down the Lagrangian for a fermion field

$$\mathscr{L} \supset \overline{\psi}(i\partial\!\!\!/ - m\mathbb{1})\psi\,. \tag{4.27}$$

Just like for bosons we can show that this term produces the Dirac equation of motion. Because we will later need the fermion–photon interaction in the form of a Hamiltonian or Lagrangian we introduce the convenient form of the covariant derivative

$$\mathscr{L} \supset \overline{\psi}\,(i\not{D} - m\mathbb{1})\,\psi \equiv \overline{\psi}\,(i(\partial\!\!\!/ + ieQ\not{A}) - m\mathbb{1})\,\psi = \overline{\psi}\,(i(\partial\!\!\!/ - m\mathbb{1})\,\psi + eQA_\mu\,\overline{\psi}\gamma^\mu\psi \tag{4.28}$$

The last term describes the coupling of a vector photon field A_μ to a vector-like expression $\overline{\psi}\gamma^\mu\psi$ which we call a vector current.

Everything written above we could as well apply to classical fields. Just like in the bosonic case we need to define the Dirac field operator in terms of plane wave coefficients

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\text{spin}} \left(e^{ikx} v_s(k) b^\dagger(\vec{k}) + e^{-ikx} u_s(k) a(\vec{k}) \right)$$
$$\overline{\psi}(x) = \int \frac{d^3k}{(2\pi)^2 2k_0} \sum_{\text{spin}} \left(e^{ikx} \bar{u}(k) a^\dagger(\vec{k}) + e^{-ikx} \bar{v}(k) b(\vec{k}) \right) , \tag{4.29}$$

where the fermion spin can be $\pm 1/2$. In the absence of any other constraints we have four generating operators, $a, a^\dagger, b, b^\dagger$. Acting on the vacuum a and b are annihilation operators and $a^\dagger, b^\dagger$ are creation operators, a for particles and b for anti–particles. These operators only depend on the momentum three-vector because the fourth component follows from the dispersion relation of the on–shell particles. The way we introduce the spinors u and v the same would hold for them, but there are instances where we use them also for off–shell states and we have to take into account their dependence on the complete momentum four-vector. Again following causality we postulate the anti–commutation relations, for example at equal time $t = t'$

$$\begin{aligned} \{\psi(x), \psi(x')\} &= 0 = \{\overline{\psi}(x), \overline{\psi}(x')\} \\ \{\psi(x), \overline{\psi}(x')\} &= \gamma^0 \delta^3(\vec{x} - \vec{x}') \, . \end{aligned} \tag{4.30}$$

Trying the same thing with commutators simply does not work, as Michael Peskin nicely shows in his book. These anti–commutators we can link to anti–commutators for the creation and annihilation operators in momentum space

$$\begin{aligned} \left\{ a_r(\vec{k}), a_s^\dagger(\vec{k}') \right\} &= \delta_{rs} (2\pi)^3 \, 2k^0 \, \delta^3(\vec{k} - \vec{k}') \\ \left\{ b_r(\vec{k}), b_s^\dagger(\vec{k}') \right\} &= \delta_{rs} (2\pi)^3 \, 2k^0 \, \delta^3(\vec{k} - \vec{k}') \\ \left\{ a_r(\vec{k}), b_s(\vec{k}') \right\} &= 0 \qquad\qquad \text{for all other } a^{(\dagger)} \text{ and } b^{(\dagger)}, \end{aligned} \tag{4.31}$$

provided we know the spin sums for the spinors u and v and their Dirac adjoints

$$\begin{aligned} \sum_{\text{spin}} u_s(k) \bar{u}_s(k) &= \not{k} + m \mathbb{1} \\ \sum_{\text{spin}} v_s(k) \bar{v}_s(k) &= \not{k} - m \mathbb{1} \, . \end{aligned} \tag{4.32}$$

Strictly speaking, $\not{k}$ is a (4×4) matrix, so in the mass term we need to include a unit matrix which is often omitted. Most of the time this is not a problem, unless we for example compute traces of chains of Dirac matrices and need to remember that $\text{Tr}\, \mathbb{1} \neq 1$. To produce such a matrix u and v are four-dimensional objects.
These anti–commutator relation have the fundamental consequence that for two fermion states generated from the vacuum we have to keep track of the ordering

$$|e^-(k,r) e^-(k',r')\rangle = a_r^\dagger(k) a_{r'}^\dagger(k') |0\rangle = -a_{r'}^\dagger(k') a_r^\dagger(k) |0\rangle = -|e^-(k',r') e^-(k,r)\rangle \, . \tag{4.33}$$

This factor (-1) needs to be taken into account when we apply normal–ordering to fermions. For $k = k'$ and $r = r'$ this leads to Pauli's exclusion principle: two identical fermion states cannot co-exist exactly in the same point.
Again, this is all we need to say about fermions to compute our electron–positron scattering process. We know the Dirac equation and the corresponding contribution to the Lagrangian, including the definition of the Dirac adjoint to construct Lorentz scalars. The quantized fermion field obeys anti–commutation relations, as do its creation and annihilation operators. To link them we need to know the form of the spin sums over the spinors u and v.

To illustrate how we can compute a transition amplitude without using Feynman rules we use our usual scattering process

$$\boxed{e^-(k_1, s_1) + e^+(k_2, s_2) \to q(p_1, s_3) + \bar{q}(p_2, s_4)} \, , \tag{4.34}$$

where k_j, p_j and s_j are the four-momenta and spin orientations of the external fermions. In the future, or more specifically asymptotically for $t \to +\infty$, the initial state $\lim_{t\to-\infty}|t\rangle \equiv |i\rangle$ will have evolved into the final state $\lim_{t\to\infty}|t\rangle = \mathcal{S}|i\rangle$ via a yet unknown linear operator $\mathcal{S}$. To describe this scattering into a final state $\langle f|$ we need to compute the transition amplitude

$$S \equiv \langle f|t\to\infty\rangle = \langle f|\mathcal{S}|i\rangle = \langle q_3\bar{q}_4|\mathcal{S}|e_1^+e_2^-\rangle = \langle 0|a_3b_4\,\mathcal{S}\,a_1^\dagger b_2^\dagger|0\rangle\ . \tag{4.35}$$

We use one index to indicate the momenta and spins of the external particles.
The transition matrix element $\mathcal{S}$ we compute from the time evolution of the initial state $i\partial_t|t\rangle = \mathcal{H}(t)|t\rangle$ in the interaction picture with a time dependent Hamilton operator. The evolution equation then reads

$$\begin{aligned}
|t\rangle &= |i\rangle - i\int_{-\infty}^{t} dt'\,\mathcal{H}(t')|t'\rangle \\
&= |i\rangle - i\int_{-\infty}^{t} dt'\,\mathcal{H}(t')\left[|i\rangle - i\int_{-\infty}^{t'} dt''\,\mathcal{H}(t'')|t''\rangle\right] \\
&= |i\rangle - i\int_{-\infty}^{t} dt'\,\mathcal{H}(t')\left[|i\rangle - i\int_{-\infty}^{t'} dt''\,\mathcal{H}(t'')\left[|i\rangle - i\int_{-\infty}^{t''} dt'''\,\mathcal{H}(t''')|t'''\rangle\right]\right]\ ,
\end{aligned} \tag{4.36}$$

just inserting the same evolution twice. The problem with this form is that it still involves $|t'''\rangle$ at the very end. What we instead want is something that is only proportional to $|i\rangle$. We can achieve this by looking at the integration boundaries: the integration range becomes smaller in each step of primed variables. In the limit of infinitely many insertions the remaining integrals should be over less and less time, starting at $t \to -\infty$. Neglecting higher powers of the Hamilton operator $\mathcal{H}$ or, as we will see later, neglecting powers of a coupling mediating the interaction between the states involves we can rewrite this form as

$$\begin{aligned}
|t\to\infty\rangle &= |i\rangle + (-i)\int_{-\infty}^{\infty} dt'\,\mathcal{H}(t')\,|i\rangle + (-i)^2\int_{-\infty}^{\infty} dt'\mathcal{H}(t')\int_{-\infty}^{t'} dt''\,\mathcal{H}(t'')\,|i\rangle + \mathcal{O}(\mathcal{H}^3) \\
&= |i\rangle + (-i)\int_{-\infty}^{\infty} dt'\,\mathcal{H}(t')\,|i\rangle + (-i)^2\int_{-\infty}^{\infty} dt'dt''\,\theta(t'-t'')\,\mathcal{H}(t')\mathcal{H}(t'')\,|i\rangle + \mathcal{O}(\mathcal{H}^3) \\
&= |i\rangle + (-i)\int_{-\infty}^{\infty} dt'\,\mathcal{H}(t')\,|i\rangle + \frac{(-i)^2}{2}\int_{-\infty}^{\infty} dt'dt''\,T(\mathcal{H}(t')\mathcal{H}(t''))\,|i\rangle + \mathcal{O}(\mathcal{H}^3) \\
&\stackrel{!}{=} \mathcal{S}\,|i\rangle\ ,
\end{aligned} \tag{4.37}$$

where the time–ordered product only contributes a factor two for two identical and hence commuting operators. The last line of Eq.(4.37) with the time–ordered Hamilton operators and the corresponding factor $1/(2!)$ is important because it means that we can sum S to an exponential series

$$\mathcal{S} = T\left(e^{-i\int dt\,\mathcal{H}(t)}\right)\ , \tag{4.38}$$

and ensure that it generates a unitary transformation. For our computation we will be fine with the quadratic term which we explicitly list.

The form of the interaction Hamiltonian for two fermionic currents each involving a different particle species j with charge Q_j follows from the covariant derivative Eq.(4.28)

$$\mathcal{H}_{\text{int}}(t) = -\int d^3x\,\mathscr{L}_{\text{int}}(x) \supset \sum_j -eQ_j\int d^3x\,A_\mu\ :\overline{\psi}_j\gamma^\mu\psi_j:\ , \tag{4.39}$$

in terms of the four-vector x including its first entry $t = x_0$, the fermion current $(\overline{\psi}\gamma_\mu\psi)$, and the photon vector field A_μ. The current is normal–ordered, which means that annihilation operators a, b are moved to the right and creation operators $a^\dagger, b^\dagger$ are moved to the left. For fermions an exchange of fields includes a minus sign, while for bosons the two operators are simply exchanged.

To connect four creation and annihilation operators arising from the external states we need four such operators from $\mathcal{S}$, which means the first term which will contribute to the scattering process is the quadratic term in $\mathcal{H}$. The two Hamiltonians contribute one electron and one quark current each. It is not hard to check that the two possible assignments give the same result, so we only follow one of them and include an additional factor two in the formula for

$$\begin{aligned}
S &= 2 \times \frac{(-i)^2}{2} \int dt' dt'' \langle 0 | a_3 b_4 \, T(\mathcal{H}(t') \mathcal{H}(t'')) \, a_1^\dagger b_2^\dagger | 0 \rangle \\
&= -e^2 Q_e Q_q \int d^4x' d^4x'' \langle 0 | a_3 b_4 \, T \left(: \overline{\psi}_q(x') \gamma_\mu \psi_q(x') : \, A^\mu(x') : \overline{\psi}_e(x'') \gamma_\nu \psi_e(x'') : \, A^\nu(x'') \right) \, a_1^\dagger b_2^\dagger | 0 \rangle \\
&= -e^2 Q_e Q_q \int d^4x' d^4x'' \langle 0 | T \left(A^\mu(x') A^\nu(x'') \right) | 0 \rangle \\
&\qquad\qquad \langle 0 | a_3 b_4 T \left(: \overline{\psi}_q(x') \gamma_\mu \psi_q(x') : : \overline{\psi}_e(x'') \gamma_\nu \psi_e(x'') : \right) \, b_1^\dagger a_2^\dagger | 0 \rangle \; .
\end{aligned} \tag{4.40}$$

The first of the time–ordered products is a gauge boson propagator in Feynman gauge

$$\langle 0 | T \left(A^\mu(x') A^\nu(x'') \right) | 0 \rangle = -i \int \frac{d^4q}{(2\pi)^4} \, e^{-iq(x'-x'')} \frac{g^{\mu\nu}}{q^2} \equiv g^{\mu\nu} \Delta(x'-x'') \; . \tag{4.41}$$

We still have to evaluate the second time–ordered product by properly combining the creation and annihilation operators with the fermion fields. For example, we can write

$$\begin{aligned}
\langle 0 | \psi(x) a^\dagger(\vec{p}) | 0 \rangle &= \int \frac{d^3k}{(2\pi)^3 2E} \sum_{\text{spins}} \langle 0 | \left(e^{ikx} b^\dagger(\vec{k}) v(k) + e^{-ikx} a(\vec{k}) u(k) \right) \, a^\dagger(\vec{p}) | 0 \rangle \\
&= \int \frac{d^3k}{(2\pi)^3 2E} \sum_{\text{spins}} e^{-ikx} u(k) \, \langle 0 | a(\vec{k}) a^\dagger(\vec{p}) | 0 \rangle \\
&= \int \frac{d^3k}{(2\pi)^3 2E} \sum_{\text{spins}} e^{-ikx} u(k) \, \langle 0 | [a(\vec{k}) a^\dagger(\vec{p})] | 0 \rangle \\
&= \int \frac{d^3k}{(2\pi)^3 2E} \sum_{\text{spins}} e^{-ikx} u(k) \, (2\pi)^3 2E \, \delta^3(\vec{k} - \vec{p}) \, \langle 0 | 0 \rangle \; \sim \; e^{-ipx} \, u(p) \; .
\end{aligned} \tag{4.42}$$

In the last step we remove the spin sum to later add it to the transition amplitude. The normal–ordering of the fermion currents in this case is never really needed after properly defining the interaction Hamiltonian. Correspondingly, we find the other non-zero normal–ordered combinations

$$\begin{aligned}
\langle 0 | b(\vec{p}) \psi(x) | 0 \rangle &= e^{ipx} \, v(p) \\
\langle 0 | a(\vec{p}) \overline{\psi}(x) | 0 \rangle &= e^{ipx} \, \bar{u}(p) \\
\langle 0 | \overline{\psi}(x) b^\dagger(\vec{p}) | 0 \rangle &= e^{-ipx} \, \bar{v}(p) \; .
\end{aligned} \tag{4.43}$$

All other combinations of $a, a^\dagger, b, b^\dagger$ with ψ and $\overline{\psi}$ vanish when we sandwich them between vacua. Before we contract the four creation and annihilation operators we need to keep in mind that electromagnetic currents only link one set of particle, they do not convert quarks into electrons. This limits the number of permutations we need to take into account. We find one unique non–vanishing combination of external states and current creators and annihilators, namely matching $a_3 \overline{\psi}$ and $b_4 \overline{\psi}$ for the quarks and $\overline{\psi} b_1^\dagger$ as well as $\psi a_1^\dagger$ for the electrons.

$$\langle 0 | a_3 b_4 : \overline{\psi}_q(x') \gamma_\mu \psi_q(x') : : \overline{\psi}_e(x'') \gamma_\nu \psi_e(x'') : \, a_1^\dagger b_2^\dagger | 0 \rangle = e^{i(p_1+p_2)x'} \bar{u}_3 \gamma_\mu v_4 \, e^{-i(k_1+k_2)x''} \bar{v}_2 \gamma_\nu u_1 \; . \tag{4.44}$$

Inserting the different contributions into Eq.(4.40) we find

$$\begin{aligned}
S &= -e^2 Q_e Q_q \int d^4x' d^4x'' \int \frac{d^4q}{(2\pi)^4}\, e^{iq(x'-x'')} \frac{-ig^{\mu\nu}}{q^2}\, e^{i(p_1+p_2)x'} \bar{u}_3 \gamma_\mu v_4\, e^{-i(k_1+k_2)x''} \bar{v}_2 \gamma_\nu u_1 \\
&= ie^2 Q_e Q_q \int \frac{d^4q}{(2\pi)^4}\, \bar{u}_3 \gamma_\mu v_4 \frac{1}{q^2} \bar{v}_2 \gamma^\mu u_1 \int d^4x'\, e^{i(q+p_1+p_2)x'} \int d^4x''\, e^{-i(q+k_1+k_2)x''} \\
&= ie^2 Q_e Q_q\, (2\pi)^8 \int \frac{d^4q}{(2\pi)^4}\, \bar{u}_3 \gamma_\mu v_4 \frac{1}{q^2} \bar{v}_2 \gamma^\mu u_1\, \delta^4(q+k_1+k_2) \delta^4(-q-p_1-p_2) \\
&= i(2\pi)^4 \delta^4(k_1+k_2-p_1-p_2)\, e^2 Q_e Q_q\, \bar{u}_3 \gamma_\mu v_4 \frac{1}{(k_1+k_2)^2} \bar{v}_2 \gamma^\mu u_1 \,. \qquad (4.45)
\end{aligned}$$

Stripping off unwanted prefactors we can define the transition matrix element for quark–antiquark production in QED as

$$\mathcal{M} = e^2 Q_e Q_q\, (\bar{u}_3 \gamma_\mu v_4) \frac{1}{(k_1+k_2)^2} (\bar{v}_2 \gamma^\mu u_1) \,, \qquad (4.46)$$

with $(k_1+k_2)^2 = (p_1+p_2)^2$. This matrix element or transition amplitude we have to square to compute the transition probability. Part of the squaring is the sum over all spins which uses the spin sums Eq.(4.32) to get rid of the spinors and then some trace rules to get rid of all Dirac matrices. Neither for the spinors nor for the Dirac matrices we need to know their explicit form

$$\begin{aligned}
|\mathcal{M}|^2 =& \sum_{\text{spin, color}} e^4 Q_e^2 Q_q^2 \frac{1}{(k_1+k_2)^4}\, (\bar{v}_4 \gamma_\nu u_3)(\bar{u}_1 \gamma^\nu v_2)\, (\bar{u}_3 \gamma_\mu v_4)\, (\bar{v}_2 \gamma^\mu u_1) \\
=& e^4 Q_e^2 Q_q^2 N_c \frac{1}{(k_1+k_2)^4} \sum_{\text{spin}} (\bar{v}_4 \gamma_\nu u_3)(\bar{u}_1 \gamma^\nu v_2)\, (\bar{u}_3 \gamma_\mu v_4)\, (\bar{v}_2 \gamma^\mu u_1) \,. \qquad (4.47)
\end{aligned}$$

The color factor N_c is the number of color singlet states we can form out of a quark and an antiquark with opposite color charges. Because color only appears in the final state we sum over all possible color states or multiply by N_c. In the next step we can observe how the crucial structure of transition amplitudes with external fermions, namely traces of chains of Dirac matrices, magically form:

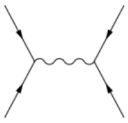
$$\begin{aligned}
|\mathcal{M}|^2 =& e^4 Q_e^2 Q_q^2 N_c \frac{1}{(k_1+k_2)^4} \sum_{\text{spin}} (\bar{v}_4)_i (\gamma_\nu)_{ij} (u_3)_j (\bar{u}_3)_k (\gamma_\mu)_{kl} (v_4)_l \cdots && \text{for one trace} \\
=& e^4 Q_e^2 Q_q^2 N_c \frac{1}{(k_1+k_2)^4} \left(\sum_{\text{spin}} (v_4)_l (\bar{v}_4)_i \right) \left(\sum_{\text{spin}} (u_3)_j (\bar{u}_3)_k \right) (\gamma_\nu)_{ij} (\gamma_\mu)_{kl} \cdots \\
=& e^4 Q_e^2 Q_q^2 N_c \frac{1}{(k_1+k_2)^4} (\not{p}_4)_{li} (\not{p}_3)_{jk} (\gamma_\nu)_{ij} (\gamma_\mu)_{kl} \cdots && \text{using Eq.(4.32)} \\
=& e^4 Q_e^2 Q_q^2 N_c \frac{1}{(k_1+k_2)^4} \operatorname{Tr}(\not{p}_4 \gamma_\nu \not{p}_3 \gamma_\mu) \operatorname{Tr}(\not{p}_1 \gamma^\nu \not{p}_2 \gamma^\mu) && \text{both traces again.} \qquad (4.48)
\end{aligned}$$

In the final step we need to use a know expression for the Dirac trace. More complicated and longer traces become very complicated very fast and we use FORM to evaluate them on the computer. We find

$$\begin{aligned}
|\mathcal{M}|^2 =& e^4 Q_e^2 Q_q^2 N_c \frac{1}{(k_1+k_2)^4} 4\,(p_{3\nu} p_{4\mu} + p_{3\mu} p_{4\nu} - g_{\mu\nu}(p_1 p_2))\; 4\,(k_1^\nu k_2^\mu + k_1^\mu k_2^\nu - g_{\mu\nu}(k_1 k_2)) \\
=& 16 e^4 Q_e^2 Q_q^2 N_c \frac{1}{(k_1+k_2)^4} \left[2(k_1 p_1)(k_2 p_2) + 2(k_1 p_2)(k_2 p_1) + 0 \times (p_1 p_2)(k_1 k_2) \right] && \text{with } g_{\mu\nu} g^{\mu\nu} = 4 \\
=& 32 e^4 Q_e^2 Q_q^2 N_c \frac{1}{(k_1+k_2)^4} \left[(k_1 p_1)(k_2 p_2) + (k_1 p_2)(k_2 p_1) \right] \,, \qquad (4.49)
\end{aligned}$$

This result for the matrix element and the matrix element squared is the same expression as we obtain from Feynman rules in Eq.(2.8).

Feynman rules are calculational rules which we can extract from the Lagrangian. These building blocks representing external and internal particles we combine to construct $\mathcal{M}$. To compute the matrix element in Eq.(4.44) while skipping everything we did to get this formula, we start by drawing Feynman diagrams representing all ways we can link the given initial and final states through interaction vertices and internal propagators. For $q\bar{q}$ production in e^+e^- scattering described by QED there exists only one such diagram:



It consist of four external fermions, one internal photon, and two interaction vertices. From Eq.(4.44) we know how to describe external fermions in terms of spinors:

symbol	diagram
$u(p,s)$	incoming fermion (e^-, q) with momentum p and spin s
$\bar{v}(p,s)$	incoming anti–fermion $(e^+, \bar{q})$
$\bar{u}(p,s)$	outgoing fermion (e^-, q)
$v(p,s)$	outgoing anti–fermion $(e^+, \bar{q})$

Spin sums are the only way to get rid of spinors in the computation. Equation (4.32) shows that as long as we neglect fermion masses the two spinors u and v for particles and antiparticles are identical. To link external particles to each other and to internal propagators we need vertices. In Eq.(4.44) we see that two fermions and a gauge boson interact via a vector current proportional to γ^μ. As a convention, we add one factor i, so the vertex rule in QED becomes

$$ieQ_f\,\gamma^\mu \qquad\qquad (f-\bar{f}-\gamma). \tag{4.50}$$

This factor i we can consistently change for all three-point and four-point vertices in our theory. Finally, there is the intermediate photon which propagates between the γ^μ and the γ^ν vertices. The wave line in the Feynman diagram corresponds to

$$\frac{-ig^{\mu\nu}}{p^2+i\epsilon}\,. \tag{4.51}$$

Again, the factor $-i$ is conventional. For a bosonic propagator it does not matter in which direction the momentum flows. Blindly combining these Feynman rules gives us directly Eq.(4.44), so all we need to do is square the matrix element, insert the spin sums and compute the Dirac trace.
Whenever we compute such a matrix element starting from a Feynman diagram nothing tells us that the lines in the Feynman diagrams are not actual physical states propagating from the left to the right. Even including loop diagrams will still look completely reasonably from a semi–classical point of view. Feynman rules define an algorithm which hides all field theory input in the calculation of scattering amplitudes and are therefore perfectly suited to compute the differential and total cross sections on the computer.

The vector structure of the QED couplings, for example mediated by a covariant derivative Eq.(4.28) we did not actually motivate. It happens to work on the Lagrangian level and agrees with data, so it is correct. We can write a completely general interaction of two fermions with a boson in terms of basis elements

$$g\,\overline{\psi}M\psi = \sum_{\text{basis } j} g_j\,\overline{\psi}M_j\psi\,. \tag{4.52}$$

For a real (4×4) matrix M the necessary 16 basis elements can be organized such that they are easy to keep track of using Lorentz transformation properties. This eventually leads to the Fierz transformation used in Section 1.9.1. The

vector γ^μ from the QED interaction gives us four such basis elements, the unit matrix a fifth. Another six we already know as well, they are the generators of the spinor representation $[\gamma^\mu, \gamma^\nu]$. We can check that all of them are linearly independent. Five basis elements in a handy form are still missing.
To define them, we need to know that there exists another (4×4) matrix which is invariant under proper Lorentz transformations. We can write it in terms of the four Dirac matrices in two equivalent forms

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4 = \frac{i}{4!}\,\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \;, \tag{4.53}$$

using the totally anti–symmetric Levi–Civita tensor $\epsilon_{\mu\nu\rho\sigma}$. This form already shows a major technical complication in dealing with γ_5: in other than four space–time dimensions we do not know how to define the Levi–Civita tensor, which means that for example for regularization purposes we cannot analytically continue our calculation to $n = 4 - 2\epsilon$ dimensions. The main property of γ_5 is equivalent to that fact that it is another basis element of our (4×4) matrices, it commutes with the other four Dirac matrices $[\gamma_\mu, \gamma_5] = 0$. Combining this new object as $(\gamma^\mu\gamma_5)$ and $i\gamma_5$ gives us all 16 basis element for the interaction of two spinors with a third scalar, vector, or tensor field:

	degrees of freedom	basis elements M_j
scalar	1	$\mathbb{1}$
vector	4	γ^μ
pseudoscalar	1	$i\gamma_5$
axialvector	4	$\gamma^\mu\gamma_5$
tensor	6	$\frac{i}{2}[\gamma^\mu, \gamma^\nu]$

The field indices need to contract with the indices of the object $\overline{\psi}M\psi$. Again, the factors i are conventional. In the Standard Model as a fundamental theory, tensor interactions do not play a major role. The reason is the dimensionality of the Lagrangian. The mass dimension of a fermion field ψ or $\overline{\psi}$ is $m^{3/2}$ while the mass dimension of a scalar field, a photon field, or a derivative is m. For example from Eq.(4.28) we see that every term in the QED Lagrangian is of mass dimension four. This is required for a renormalizable fundamental field theory. Introducing a tensor coupling we have to contract two indices, μ and ν, and not with the metric tensor. The only other objects coming to mind have mass dimension m^2, which means that together with the fermion fields the term in the Lagrangian has mass dimension of at least m^5 and is therefore not allowed.

The second obvious question is: what does it mean to include a factor γ_5 in the interaction, *i.e.* what distinguishes a scalar from a pseudoscalar and a vector from an axialvector? We can give an easy answer by defining three transformations of our field in space and time. The first one is the parity transformation P which mirrors the three spatial coordinates $(t, \vec{x}) \to (t, -\vec{x})$. The second is charge conjugation C which converts particles into their anti–particles. Both of them leave the Dirac equation intact and can be represented by a unitary transformation. The third transformation is time reversal T which converts $(t, \vec{x}) \to (-t, \vec{x})$, also leaves the Dirac equation intact, but only has an anti–unitary representation. Every single one of them is violated in our Standard Model.
Instead of writing out the representation of these transformations in terms of Dirac matrices we characterize them using the basic interactions from Eq.(4.52). Parity symmetry does not allow any interaction including γ_5, which means it forbids pseudoscalars and axialvectors. Time reversal symmetry does not allow any complex couplings g_j. Because any field theory described by a Lagrangian not including some kind of external field is invariant under CPT, and we have never observed CPT violation, a combined CP invariance is essentially the same as T invariance.

To again look at the same question we rotate the $\{\mathbb{1}, \gamma_5\}$ plane and define the two (4×4) matrix valued objects which we already use in Eq.(1.15),

$$\mathbb{P}_{R,L} = \frac{1}{2}\,(\mathbb{1} \pm \gamma_5) \;. \tag{4.54}$$

It is easy to show that the two are orthogonal

$$\mathbb{P}_L\mathbb{P}_R = \frac{1}{4}\,(\mathbb{1} - \gamma_5)\,(\mathbb{1} + \gamma_5) = \frac{1}{4}\,\left(\mathbb{1} - \gamma_5^2\right) = 0 \qquad \text{using} \quad \gamma_5^2 = \mathbb{1} \;, \tag{4.55}$$

and projectors

$$\mathbb{P}_{R,L}^2 = \frac{1}{4}\left(\mathbb{1} \pm 2\gamma_5 + \gamma_5^2\right) = \frac{1}{4}\left(2\mathbb{1} \pm 2\gamma_5\right) = \frac{1}{2}\left(\mathbb{1} \pm \gamma_5\right) = \mathbb{P}_{R,L} \,. \tag{4.56}$$

In QED these combinations do not play any role. Their effect on kinetic and mass terms we compute in Eqs.(1.16) and (1.17). Looking at interactions, we can for example define a combined vector–axialvector coupling as $\gamma^\mu \pm \gamma^\mu\gamma_5 = 2\gamma_\mu \mathbb{P}_{R,L}$. Sandwiching this coupling between fermion fields gives for example

$$\begin{aligned}
\overline{\psi}\,\gamma_\mu \mathbb{P}_L\,\psi &= \overline{\psi}\,\gamma_\mu \mathbb{P}_L^2\,\psi \\
&= \psi^\dagger \mathbb{P}_L\,\gamma_0\,\gamma_\mu\,\mathbb{P}_L \psi \qquad\qquad \text{with} \qquad \{\gamma_5,\gamma_\mu\} = 0 \\
&= \left(\mathbb{P}_L\psi\right)^\dagger \gamma_0\,\gamma_\mu\,\mathbb{P}_L\psi \qquad\qquad \text{with} \qquad \gamma_5^\dagger = \gamma_5 \\
&= \overline{\psi}_L\,\gamma_\mu\,\psi_L \qquad\qquad \text{with} \qquad \psi_{L,R} \equiv \mathbb{P}_{L,R}\,\psi \,.
\end{aligned} \tag{4.57}$$

If we call the eigenstates of $\mathbb{P}_{R,L}$ right handed and left handed fermions $\psi_{L,R}$ this chirality allows us to define a vector coupling between only left handed fermions by combining the vector and the axialvector couplings with a relative minus sign. The same is of course true for right handed couplings. In Section 1.1.3 we show that kinetic terms can also be defined independently for left and right handed fermions, while mass terms or scalar interactions mix the two chiralities

$$\begin{aligned}
\overline{\psi}\,\partial\!\!\!/\,\psi &= \overline{\psi}_R\,\partial\!\!\!/\,\psi_R + \overline{\psi}_L\,\partial\!\!\!/\,\psi_L \\
\overline{\psi}\,\mathbb{1}\,\psi &= \overline{\psi}_R\,\mathbb{1}\,\psi_L + \overline{\psi}_L\,\mathbb{1}\,\psi_R \,.
\end{aligned} \tag{4.58}$$

In other words, we can write for example QED in terms of independent left and right handed fields as long as we neglect all fermion masses. This defines the chiral limit where the Lagrangian is symmetric under $\psi_L \leftrightarrow \psi_R$. Introducing fermion masses breaks this chiral symmetry, or turning the argument around, to introduce fermion masses we need to combine a left handed and a right handed fermion fields and give them one common Dirac mass.

At this stage it is not obvious at all what chirality means in physics terms. However, we will see that in the Standard Model the left handed fermions play a special role: the massive W bosons only couple to them and not to their right handed counter parts. So chirality is a property of fermions known to one gauge interaction of the Standard Model as part of the corresponding charge. The Higgs mechanism breaks it and only leaves the QCD–like gauge symmetry intact.

There exists a property which is identical to chirality for massless fermions and has an easy physical interpretation: the helicity. It is defined as the projection of the particle spin onto its three-momentum direction

$$h = \vec{s}\cdot\frac{\vec{p}}{|\vec{p}|} = \left(\vec{s}+\vec{L}\right)\cdot\frac{\vec{p}}{|\vec{p}|} = \vec{J}\cdot\frac{\vec{p}}{|\vec{p}|} \qquad\qquad \text{with} \quad \vec{p}\perp\vec{L}\,, \tag{4.59}$$

or equivalently the projection of the combined orbital angular momentum and the spin on the momentum direction. From quantum mechanics we know that there exist discrete eigenvalues for the z component of the angular momentum operator, symmetric around zero. Applied to fermions this gives us two spin states with the eigenvalues of h being $\pm 1/2$. Unfortunately, there is no really nice way to show this identity. What we need to know is that the spin operator is in general given by

$$\vec{s} = \gamma_5\gamma^0\,\vec{\gamma}\,. \tag{4.60}$$

We can show this by writing it out in terms of Pauli matrices, but we will skip this here and instead just accept this general form. We then write the solution ψ to the massless Dirac equation after transforming it into momentum space

$\psi(\vec{x}) = u(\vec{p}) \exp(-ip \cdot x)$

$$
\begin{aligned}
\left(\gamma^0 p_0 - \vec{\gamma}\vec{p}\right)\ u(\vec{p}) &= 0 \\
\gamma_5 \gamma^0\ \gamma^0 p_0\ u(\vec{p}) &= \gamma_5 \gamma^0\ \vec{\gamma}\vec{p}\, u(\vec{p}) \\
\gamma_5 p_0\ u(\vec{p}) &= \vec{s}\cdot\vec{p}\, u(\vec{p}) \qquad\qquad \text{with } \left(\gamma^0\right)^2 = \mathbb{1} \\
\gamma_5\ u(\vec{p}) &= \frac{\vec{s}\cdot\vec{p}}{p_0}\ u(\vec{p}) \\
\gamma_5\ u(\vec{p}) &= \pm\frac{\vec{s}\cdot\vec{p}}{|\vec{p}|}\ u(\vec{p}) = \pm h\ u(\vec{p})\ . \qquad (4.61)
\end{aligned}
$$

In other words, the chirality operator γ_5 indeed gives us the helicity of the particle state, modulo a sign depending on the sign of the energy. For the helicity it is easy to argue why for massive particles this property is not Lorentz invariant and hence not a well defined property: massless particles propagate with the speed of light, which means we can never boost into their rest frame or pass them. For massive particles we can do that and this way switch the sign of $\vec{p}$ and the sign of h. Luckily, for almost all Standard Model fermions we can at the LHC neglect their masses.