The Gross-Pitaevskii Equation and the Hydrodynamic Expansion of a BEC

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Summary: The Gross-Pitaevskii Equation

The Gross-Pitaevskii equation is a mean field equation, which describes the zero-temperature properties of the non-uniform BEC.

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \Big(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g|\Psi(\mathbf{r},t)|^2\Big)\Psi(\mathbf{r},t)$$

$$\Big(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g|\Psi(\mathbf{r},t)|^2\Big)\Psi(\mathbf{r},t) = \mu\Psi(\mathbf{r},t)$$

conditions:

- $n|a|^3 \ll 1$
- $N \gg 1$
- depletion is negligible.
- One can only investigate phenomena taking place over distances much larger than the scattering length.

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properties:

- nonlinear
- condensate wave function has to be calculated self-consistently

The Effect of Interactions

Rescale stationary GPE

$$[-\nabla'^2 + r'^2 + 8\pi \frac{Na}{a_{ho}} \phi'^2(\mathbf{r'})]\phi'(\mathbf{r'})$$

$$= 2\mu'\phi'(\mathbf{r'})$$

$a_{ho} = \sqrt{\frac{\hbar}{m\omega_{ho}}}$, $\phi'(\mathbf{r'})$ normalized

$\longrightarrow$ only one parameter Na/a_{ho}.

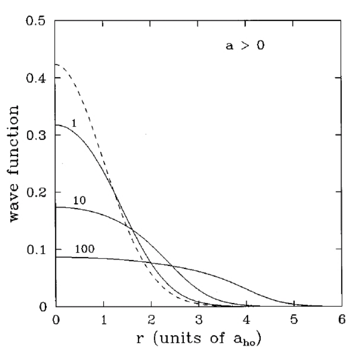


Figure: Condensate wave function at $T = 0$ obtained by numerical solution of the stationary GPE

Thomas-Fermi Approximation

Thomas-Fermi approximation: neglect the kinetic energy term in the GPE.

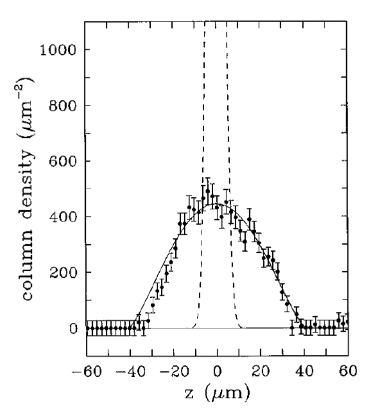
This approximation is good for $\frac{Na}{a_{ho}} \gg 1$.

$$n_{TF}(\mathbf{r}) = \begin{cases} \frac{1}{g}(\mu_{TF} - V_{ext}(\mathbf{r})) & : \quad \mu_{TF} > V_{ext} \\ 0 & : \quad \text{elsewhere} \end{cases}$$

For a harmonic potential:

- density distribution is an inverted parabola
- radius R of the BEC: $R_i^2 = \frac{2\mu}{m\omega_{ho,i}^2}$.

Ground State of the Condensate



Density distribution of 80000 Na-atoms by using near-resonant absorption imaging (Hau 1998)

solid line: numerical solution of the GPE

dashed line: prediction for a non-interacting gas

Hydrodynamic Equations

Define: $\Psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}e^{i\phi(\mathbf{r},t)}$

$$\frac{\partial}{\partial t}|\Psi|^2 = \Psi^*\frac{\partial}{\partial t}\Psi + (\frac{\partial}{\partial t}\Psi^*)\Psi = -\nabla\left[\frac{\hbar}{2mi}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)\right]$$

Continuity Equation $\quad \frac{\partial}{\partial t}n + \nabla(n\mathbf{v}) = 0$

BEC velocity field $\mathbf{v} = \frac{\hbar}{2mi|\Psi|^2}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)$

$\mathbf{v} = \frac{\hbar}{m}\nabla\phi \longrightarrow$ The flow of the BEC is rotationless, as long as ϕ contains no singularity, as e.g. in a vortex.

Summary: Hydrodynamic Equations

$$\frac{\partial}{\partial t} n + \nabla(n\mathbf{v}) = 0$$

$$m\frac{\partial}{\partial t}\mathbf{v} + \nabla\left(V_{ext}(\mathbf{r}) + gn + \frac{mv^2}{2} - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\right) = 0$$

- The BEC behaves like a perfect fluid and can be described by the local density and local velocity only.
- The flow of the BEC is rotationless, as long as ϕ contains no singularity, as e.g. in a vortex.

Expansion of a BEC

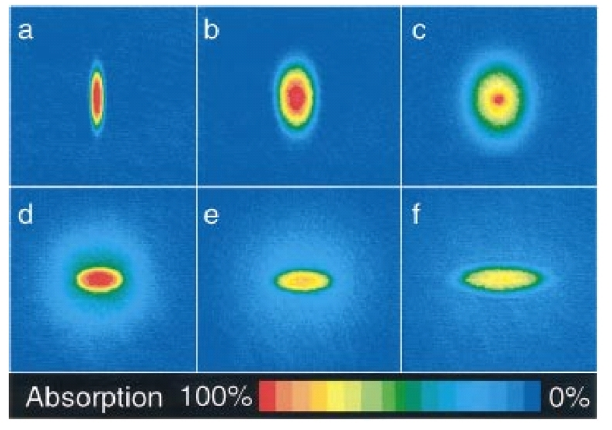


Figure: Time-of-flight images of expanding mixed clouds. Flight times for (a)(f) are 1, 5, 10, 20, 30, and 45 ms, respectively. (Ketterle et al 1996)

Expansion of a BEC: Theoretical Description

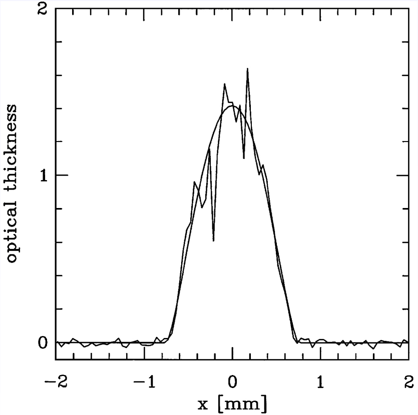


Figure: Spatial density of an expanding condensate (expansion time 40 ms)

The shape of the density distribution looks pretty much as in the ground state

$\longrightarrow$ describe evolution by a scaling argument

Expansion of a BEC: Theoretical Description

(Dalfovo(1996), Castin(1996))

- use the ansatz

$$n(\mathbf{r},t) = n_0(t)\left(1 - \frac{x^2}{R^2(0)b_x^2(t)} - \frac{y^2}{R^2(0)b_y^2(t)} - \frac{z^2}{R^2(0)b_z^2(t)}\right)$$

$$\mathbf{v} = \frac{1}{2}[\alpha_x(t)x^2 + \alpha_y y^2(t) + \alpha_z z^2(t)]$$

- the class of density distribution is a solution of the hydrodynamic equations and the ansatz corresponds to a scaling solution in which the parabolic shape is preserved.

- the radius of the BEC involves in time as

$$R_i(t) = R_i(0)b_i(t)$$

Expansion of a BEC: Theoretical Description

The hydrodynamic equations provides for the scaling factors

$$\ddot{b_i} + \omega_{ho,i}^2(t) b_i - \frac{\omega_{ho,i}^2(0)}{b_i b_x b_y b_z} = 0$$

Example: cigar shaped trap $\omega_x = \omega_y \gg \omega_z$

$$b_\perp(t) = \sqrt{1+\tau^2}$$

$$b_z(t) = 1 + \lambda^2[\tau \arctan \tau - \ln\sqrt{1+\tau^2}]$$

$\tau = \omega_\perp(0) \cdot t$, $\lambda = \omega_z(0)/\omega_\perp(0) \ll 1$

Expansion of a BEC: Experiment

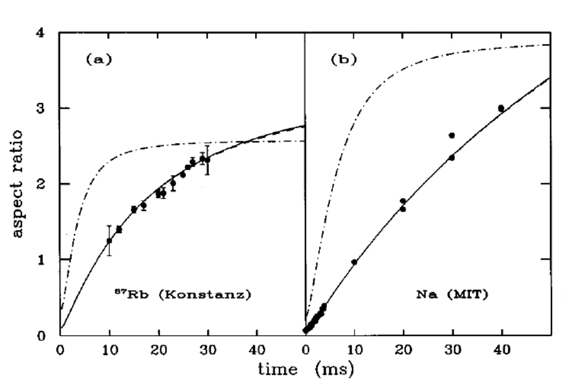


Figure: Aspect ratio, $R_\perp/Z$ of an expanding condensate as a function of time. The solid lines are obtained by solving the equation for the scaling parameter. The dot-dashed lines are the predictions for noninteracting atoms. (Stringari)

Summary

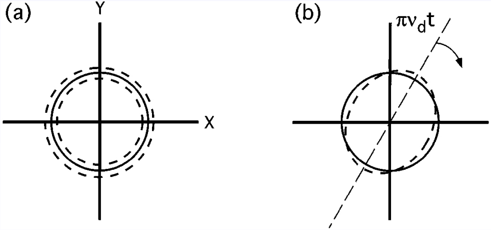
1. The GPE describes the features of atomic BEC's very well.
2. The expansion of an atomic BEC can not be described as a dispersion of a initially confined wave packet. Instead the experimental data agree nicely with the hydrodynamic picture.

Thank you for your attention

Collective Oscillations

Experimental Setup

- TOP trap with potential $V(\mathbf{r}) = \frac{m}{2}(\omega_r^2 r^2 + \omega_z^2 z^2)$.
- Perturb V by applying a sinusoidal current to the trap coils



- free evolution of the cloud in the unperturbed trap for a variable time length
- switch off of the confining potential and imaging of the cloud shape after 7 ms

Collective Oscillations

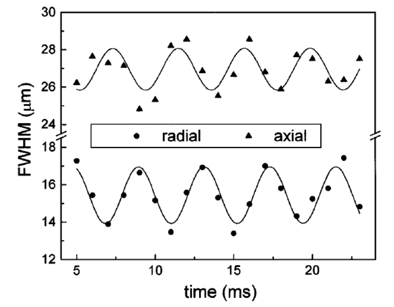


Figure: Oscillation of the shape of the expanding condensate as function of the expansion time ($m = 0$ mode) (Jin et al 1996)

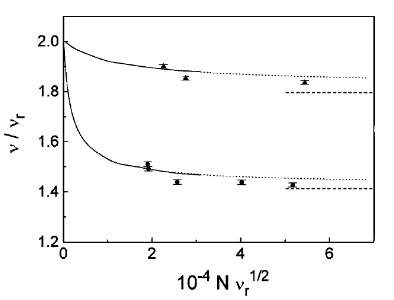


Figure: Frequency of the $m = 0$ (triangles) and $m = 2$ (circles) condensate modes as function of the interaction strength (Jin et al 1996)

Collective Oscillations

Theoretical description:

- Collective oscillations are described by the linearized solutions of the time-dependent GPE
- Ansatz: $\Psi(\mathbf{r}, t) = \exp\left(-i\mu t/\hbar\right)\Big(\Psi_g(\mathbf{r}) + u_i(\mathbf{r}, t)\exp\left(-i\omega t\right) + v_i^*(\mathbf{r}, t)\exp\left(i\omega t\right)\Big)$
- Inserting into the time-dependent GPE and linearizing in $u_i((\mathbf{r}, t))$ and $v_i^*((\mathbf{r}, t)) \longrightarrow$ linear coupled DGL in $u_i(\mathbf{r}, t)$ and $v_i^*(\mathbf{r}, t)$.
- Frequencies ω_i are obtained by diagonalization of the matrix