1. Ideal quantum gases

The quantum statistical description of ideal, i.e., non-interacting gases is treated in detail by any standard textbook on Statistical Mechanics [e.g. K. Huang] as well as by the basic theory course. We will therefore restrict ourselves here to a summary of the main derivations and results.

We start with general elements of the quantum theory of many-body physics and discuss Bose and Fermi gases in the framework of the grand canonical ensemble, thereby giving expressions and discuss the physics as far as possible in parallel for the two different statistics.

We will, in particular, focus on the extension to trapped gases.

1.1 Fock space formulation

We aim at a grandcanonical formulation of the thermodynamics of ideal nonrelativistic Bose and Fermi gases.

Given be

- particles with mass m and dispersion $\epsilon(p) = p^2/2m$ in a volume V.
- their mean number $\bar{N} = \langle \hat{N} \rangle$ and mean energy $\bar{E} = \langle \hat{H} \rangle$ (1.0)

where $\hat{N}$ and $\hat{H}$ are the total number and Hamilton operators (see below), respectively. The grandcanonical formulation will provide the density operator $\hat{\varrho}$ w.r.t. which the means $\langle \cdot \rangle = Tr(\hat{\varrho}\, \cdot)$ are defined.

We now assume a cubic volume $V = L^3$ as well as periodic boundary conditions, such that our qm. basis will be labelled by the wave numbers $k_{\nu_j} = 2\pi \nu_j L^{-1}, j = 1,2,3.$

Without further discussion we introduce the Fock states

$$|\{n_i, \vec{k}_i\}\rangle = \prod_i (n_i!)^{-\frac{1}{2}} (\hat{a}_i^+)^{n_i} |0\rangle \tag{1.1}$$

where the index i counts through all modes labelled before by the triples (v_1, v_2, v_3), and n_i is a natural number giving the # of particles with wave number $\vec{k}_i$. The state (1.1) is created from the vacuum state $|0\rangle$ (with no particles in any of the momentum modes) by n_i-fold action of the creation operators $\hat{a}_i^+$, throughout all modes i. These and the annihilation ops. $\hat{a}_i$ are defined through

$$\begin{aligned} \hat{a}_i^+ \, |\{n_j, \vec{k}_j\}\rangle &= \sqrt{\pm n_j + 1}\, |\{n_j, \vec{k}_j\}_{j\setminus i}; (n_i+1, \vec{k}_i)\rangle, \\ \hat{a}_i \, |\{n_j, \vec{k}_j\}\rangle &= \sqrt{n_i}\, |\{n_j, \vec{k}_j\}_{j\setminus i}; (n_i-1, \vec{k}_i)\rangle, \\ \Rightarrow \hat{a}_i \, |0\rangle &= 0. \end{aligned} \tag{1.2}$$

Here we need to distinguish between Bosons ('+') and Fermions ('-') in the prefactor on the r.h.s. of the first relation, since there is at most 1 fermion in each mode.

Moreover the operators obey

$\boxed{B}$	$\boxed{F}$	
$[\hat{a}_i, \hat{a}_j^+] = \delta_{ij}$	$[\hat{a}_i, \hat{a}_j^+]_+ = \delta_{ij}$	(1.3)
commutation	anticommutation	

relations, while all other $\overset{\boxed{F}}{\text{(anti)}}\overset{\boxed{B}}{\text{commutators}}$ vanish. 'i' includes spin quantum no's. for $\boxed{F}$.
In terms of the operators $\hat{a}$, $\hat{a}^+$ one defines the

- particle number op.: $\hat{N}_i = \hat{a}_i^+ \hat{a}_i$
- total particle # op.: $\hat{N} = \sum_i \hat{N}_i$ (1.4)
- Hamilton op.: $\hat{H} = \sum_i \epsilon(\vec{k}_i) \hat{N}_i$ (1.5)

Note that, for fermions one has

$$\hat{a}_i^+ |0, \vec{k}_i\rangle = |1, \vec{k}_i\rangle \qquad (*)$$

$$\left.\begin{aligned} \hat{a}_i^+ |1, \vec{k}_i\rangle &\overset{(*)}{=} \hat{a}_i^+ \hat{a}_i^+ |0, \vec{k}_i\rangle \\ &\overset{(1.3.F)}{=} -\hat{a}_i^+ \hat{a}_i^+ |0, \vec{k}_i\rangle \end{aligned}\right\} \Rightarrow = 0$$

in accordance with Pauli's principle.
Furthermore, from (1.3), (1.4):

$$\begin{aligned} \hat{N}_i \hat{a}_i^+ |n_i\rangle &= \hat{a}_i^+ \hat{a}_i \hat{a}_i^+ |n_i\rangle = \hat{a}_i^+ (1 \pm \hat{a}_i^+ \hat{a}_i) |n_i\rangle \\ &= (1 \pm n_i) \hat{a}_i^+ \hat{a}_i |n_i\rangle \end{aligned}$$

$\Rightarrow \hat{a}_i^+ |n_i\rangle \propto |1 \pm n_i\rangle$. Since $\langle n_i | \hat{a}_i \hat{a}_i^+ | n_i\rangle \overset{!}{=} 1$
$= (1 \pm n_i) \langle n_i | n_i\rangle = (1 \pm n_i)$ one obtains (1.2), 1st relation.

1.2 Partition sum, no. distributions

The grandcanonical partition sum allows to find thermodynamic quantities by minimisation under the constraints (1.0) with the aid of the Lagrange multipliers

$$\beta = (k_B T)^{-1} \quad \text{inverse temperature,}$$
$$\mu \quad \text{chemical potential.} \tag{1.6}$$

The partition sum reads

$$Z_{gc}(\beta,\mu) = \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right], \tag{1.7}$$

the grandcanonical density matrix

$$\hat{\varrho}_{gc}(\beta,\mu) = Z_{gc}^{-1}\, e^{-\beta(\hat{H}-\mu\hat{N})}. \tag{1.8}$$

From these one finds, e.g.,

$$\bar{N} = \mathrm{Tr}\left[\hat{\varrho}_{gc}\hat{N}\right] = \beta^{-1}\frac{\partial}{\partial\mu}\ln Z_{gc}\Big|_{\beta=\mathrm{const.}}$$

$$\bar{E} = -\frac{\partial}{\partial\beta}\ln Z_{gc}\Big|_{\beta\mu=\mathrm{const.}} \tag{1.8a}$$

We use the Fock basis (1.7) to calculate the traces:

$\boxed{B}$

$$Z_{gc} = \prod_i \left(\sum_{n_i=0}^{\infty} \left[e^{-\beta(\epsilon_i - \mu)} \right]^{n_i} \right)$$

$$= \prod_i \frac{1}{1 - e^{-\beta(\epsilon_i - \mu)}}$$

if all $\epsilon_i > \mu \Leftrightarrow$ if $\epsilon_0 > \mu$

$\boxed{F}$

$$Z_{gc} = \prod_i \left(1 + e^{-\beta(\epsilon_i - \mu)} \right) \tag{1.9}$$

When using Z_{gc} as a generating function (cf. (1.8a)), we need $\ln Z_{gc}$. Thus we define

$$Z_{gc} = e^{-\beta \Omega_{gc}} \tag{1.10}$$

and find for the respective g.c. potentials:

$$\Omega_{gc} = \beta^{-1} \sum_i \ln\left(1 - e^{-\beta(\epsilon_i - \mu)}\right)$$

$$\Omega_{gc} = -\beta^{-1} \sum_i \ln\left(1 + e^{-\beta(\epsilon_i - \mu)}\right) \tag{1.11}$$

Defining

$$\bar{N} = \sum_i \bar{N}_i \qquad \bar{E} = \sum_i \epsilon_i \bar{N}_i \tag{1.12}$$

we find the distributions

$$\bar{N}_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$$

BOSE - EINSTEIN

$$\bar{N}_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1}$$

FERMI - DIRAC (1.13)

We introduce the fugacity

$$z = e^{\beta\mu}, \tag{1.14}$$

such that

$$\overline{N}_i = \frac{1}{z^{-1}e^{\beta\epsilon_i}-1} \qquad (0 \le z \le 1)$$

$$\overline{N}_i = \frac{1}{z^{-1}e^{\beta\epsilon_i}+1} \qquad (0 \le z \le \infty) \tag{1.13a}$$

Analytic expressions can be given in a clearer way if one assumes the many-body modes densely spaced in ϵ and simplifies the sums over i by approximating them as continuous integrals. For this we use the Bose and Fermi functions defined as

$$g_\alpha(z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{x^{\alpha-1}dx}{z^{-1}e^{x}-1} = \sum_{\nu=1}^\infty \frac{z^\nu}{\nu^\alpha}$$

$$f_\alpha(z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{x^{\alpha-1}dx}{z^{-1}e^{x}+1} = -g_\alpha(-z) \tag{1.15}$$

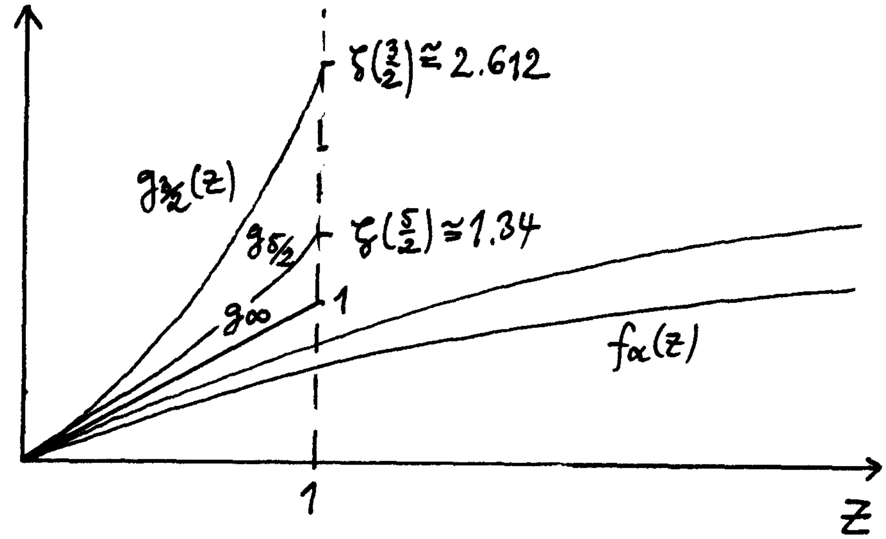
with

$$g_\alpha(z)\begin{cases} \cong z, & |z| \ll 1 \\ = \zeta(\alpha), & z=1 \\ = \dfrac{d\,g_{\alpha+1}(z)}{d\ln z} \end{cases}$$

$$f_\alpha(z)\begin{cases} \cong z, & |z| \ll 1 & (1.16) \\ = (1-2^{1-\alpha})\zeta(\alpha), & z=1 & (1.17) \\ = \dfrac{d f_{\alpha+1}(z)}{d\ln z} & & (1.18a) \end{cases}$$

Qualitatively these functions behave as follows:

Fig. 1.1

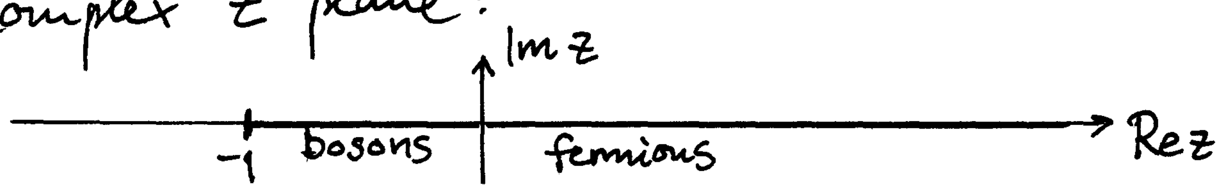


From (1.15) : $\lim_{\alpha \to \infty} f_\alpha(z) = z$

For large z one finds the following expansion for f_α:

$$f_\alpha(z) = \frac{(\ln z)^\alpha}{\Gamma(\alpha+1)} \left[1 + \sum_{l=1}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-2l)} \frac{2(1-2^{1-2l})\zeta(2l)}{(\ln z)^{2l}} + \mathcal{O}(1/z) \right], \quad z \to \infty \tag{1.18}$$

Note that (1.15) allows the interpretation of Bosons (fermions) living on the interval [-1, 0] (on the positive real axis) of the complex z plane:



Using (1.12), (1.13) and (1.15), we find, keeping $z<1$ for bosons:

$$\bar{N}(V,T,\mu) = \frac{V}{\lambda_T^3}\, g_{\frac{3}{2}}(z) \qquad \Bigg| \qquad \bar{N}(V,T,\mu) = g_s \frac{V}{\lambda_T^3} f_{\frac{3}{2}}(z) \tag{1.19}$$

where

$$\lambda_T := \left(\frac{2\pi\hbar^2}{m k_B T}\right)^{\frac{1}{2}} \tag{1.20}$$

is the thermal de Broglie wavelength obtained from $k_B T = \pi^{-1}\, \hbar^2 (2\pi/\lambda_T)^2/2m$, and g_s takes into account a spin degeneracy ($g_s = 2$ for spin 1/2)

1.3 classical limit

In the limit $z \to 0$ one has, using (1.16),

$$\frac{\bar{N}}{V}\lambda_T^3 =: \bar{\rho}\,\lambda_T^3 \cong z \qquad \Bigg| \qquad \bar{\rho}\,\lambda_T^3 \cong z \tag{1.21}$$

Hence, $z \to 0$ corresponds to small densities $\bar{\rho}$ and/or high temperatures, such that $\bar{\rho}\lambda_T^3 \ll 1$. In this limit one recovers, from (1.13), the MAXWELL – BOLTZMANN distribution:

$$\bar{N}_i \cong z e^{-\beta\epsilon_i} \qquad \Bigg| \qquad \bar{N}_i \cong z e^{-\beta\epsilon_i} \tag{1.22}$$

1.4 Trapped gases

Before carrying on with the thermodynamics of ideal degenerate quantum gases, let us widen our horizon and consider trapped gases.

The trap shows up in the density of states, i.e., the # of states in the energy shell $[\epsilon, \epsilon + d\epsilon[$, divided by $d\epsilon$:

Let's learn through examples:

(a) uniform gas in 3-dim volume V, periodic b.c.:

$$G(\epsilon) := \frac{V \cdot \text{momentum vol.} (\leq p)}{\text{phase space vol. per state}} = \frac{V \cdot \frac{4}{3}\pi p^3}{(2\pi\hbar)^3} \qquad (1.23)$$

$$\Rightarrow g(\epsilon) := \frac{dG}{d\epsilon} = \frac{V m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \epsilon^{1/2} \qquad (1.24)$$

for dispersion $p = \sqrt{2m\epsilon}$

(b) same as (a) but in d dimensions:

$$g(\epsilon) \propto \epsilon^{\frac{d}{2}-1} \qquad (1.25)$$

(c) harmonic oscillator $V = \frac{1}{2} m \sum_{i=1}^{3} \omega_i^2 r_i^2$:

$$G(\epsilon) = (\hbar\bar{\omega})^{-3} \int_0^{\epsilon} d\epsilon_1 \int_0^{\epsilon-\epsilon_1} d\epsilon_2 \int_0^{\epsilon-\epsilon_1-\epsilon_2} d\epsilon_3 = \frac{\epsilon^3}{6(\hbar\bar{\omega})^3} \quad (1.26)$$

with $\bar{\omega} = \left(\prod_i \omega_i\right)^{1/3}$. $\Rightarrow g(\epsilon) = \epsilon^2/2(\hbar\bar{\omega})^3$

(d) h.o. in d dim.^{s} :

$$g(\epsilon) = \epsilon^{d-1} \left[(d-1)! \prod_i \hbar \omega_i \right]^{-1} \quad (1.27)$$

Hence, we will consider a general d.o.s.

$$g(\epsilon) = C_\alpha \, \epsilon^{\alpha-1}. \quad (1.28)$$

Our above examples imply $\alpha = \frac{d}{2}$ for a quadratic and $\alpha = d$ for a linear dispersion in d dimensions.

We calculate generalised expressions for $\bar{N}$ (see (1.19)):

$$\bar{N} = \int_0^\infty \frac{d\epsilon \, g(\epsilon)}{z^{-1} e^{\beta\epsilon} \mp 1}$$

$$= C_\alpha \Gamma(\alpha) (k_B T)^\alpha \begin{cases} g_\alpha(z) \\ f_\alpha(z). \end{cases} \quad (1.29)$$

Bagnato et al. [PRA 35, 4354 (1987)] have shown that for a trapping potential of the form

$$V_{trap} = \sum_{i=1}^{3} \epsilon_i \, |r_i|^{p_i} \tag{1.30}$$

one finds

$$\alpha = \frac{3}{2} + \sum_{i=1}^{3} \frac{1}{p_i} \tag{1.31}$$

$$= 3\left(\tfrac{1}{2} + \tfrac{1}{p}\right) \quad \text{for spherical case.}$$

Cases of practical interest include:

Tab. 1.1

Trap	d	α	$\Gamma(\alpha)$	$\zeta(\alpha)$
HO	1	1	1	∞
L^3	3	$3/2$	$\frac{\sqrt{\pi}}{2} = 0.886$	2.612
HO	2	2	1	$1.645 = \pi^2/6$
r^3	3	$5/2$	$\frac{3}{4}\sqrt{\pi} = 1.329$	1.341
HO	3	3	2	1.202
$r^{3/2}$	3	$7/2$	$\frac{15}{8}\sqrt{\pi} = 3.323$	1.127
$r^{6/5}$	3	4	6	$1.082 = \pi^4/90$
$\lvert r \rvert$	3	$9/2$	$\frac{105}{16}\sqrt{\pi} = 11.632$	1.055

and:

- L^2-box, 2dim: $\alpha = d/2 = 1 \Rightarrow \zeta(\alpha) = \infty$
- harmonic confinement in 3rd direction: $p_3 = 2,\; p_1 = p_2 \to \infty \Rightarrow \alpha = 2,\; \zeta(\alpha) = 1.6$.
- '1D tube': $p_1 = p_2 = 2,\; p_3 \to \infty \Rightarrow \alpha = 5/2$.

1.5 Degenerate quantum gases

We turn back to the thermodynamics and consider the degenerate limit $z \gg 0$, where the gases show their nonclassical behaviour:

(1.19) and Fig 1.1 show that the maximum phase space density ~~(can be)~~ is finite, if $\alpha > 1$, e.g., for $\alpha = 3/2$:

$$\tilde{\omega} := \bar{\rho}\lambda_T^3 \leq \zeta\left(\tfrac{3}{2}\right) = 2.612\ldots$$

In the $\bar{\rho}$-T-plane:

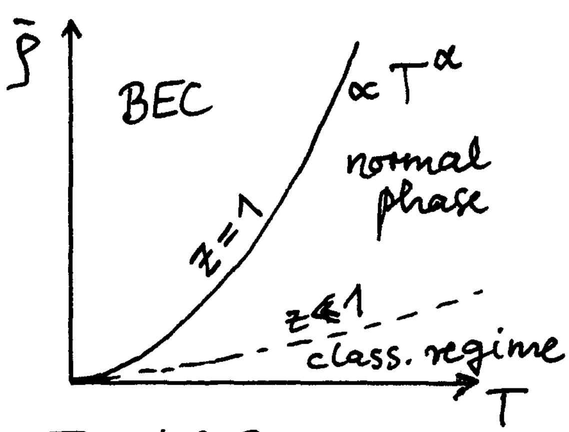


Fig 1.2-B

For fixed T: max. (crit.) density

$$\bar{\rho}_c = \frac{\zeta(3/2)}{\lambda_T^3} \qquad (1.32)$$

For fixed $\bar{\rho}$: crit. temp.

$$T_c = \frac{2\pi\hbar^2}{m k_B}\left(\frac{\bar{\rho}}{\zeta(3/2)}\right)^{\frac{2}{3}} \qquad (1.33)$$

For fermions, no geometric series had to be resummed such that there is no such restriction:

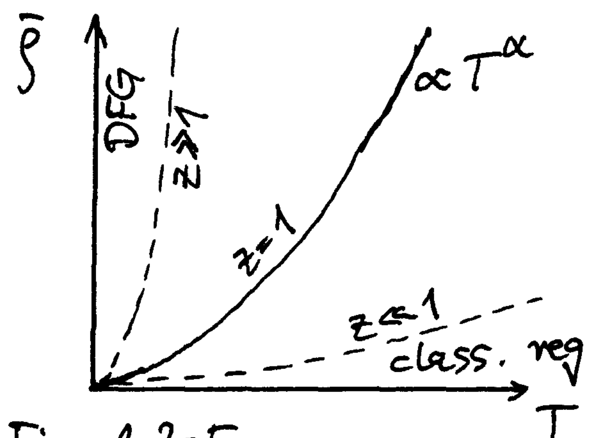


Fig. 1.2-F

The FD distribution approaches a sharp Fermi sea for $z \to \infty$.

We compare the different distributions:

Next we consider the dependence of the chemical potential μ on temperature T for fixed $\bar{\rho}$: Before showing this graphically, we discuss the

- classical limit: From (1.21):

$$\mu = \beta^{-1} \ln z \cong k_B T \ln \bar{\rho} T^{-\alpha} \sim -T \ln T \qquad (1.32)$$

- degenerate Fermi gas: From (1.18):

$$f_\alpha(z) \sim (\ln z)^\alpha \left(1 + (\ln z)^{-2} + \mathcal{O}(|\ln z|^{-4})\right)$$

$$\overset{(1.19)}{\Rightarrow} \quad \bar{\rho} T^{-\alpha} \sim \left(\frac{\mu}{T}\right)^\alpha \left[1 + \left(\frac{k_B T}{\mu}\right)^2 + \dots\right]$$

$$\Rightarrow \quad \mu \sim \bar{\rho}^{-\alpha} \left[1 - \left(\frac{T}{T_F}\right)^2 + \dots\right] \qquad (1.33)$$

with the Fermi temperature defined through

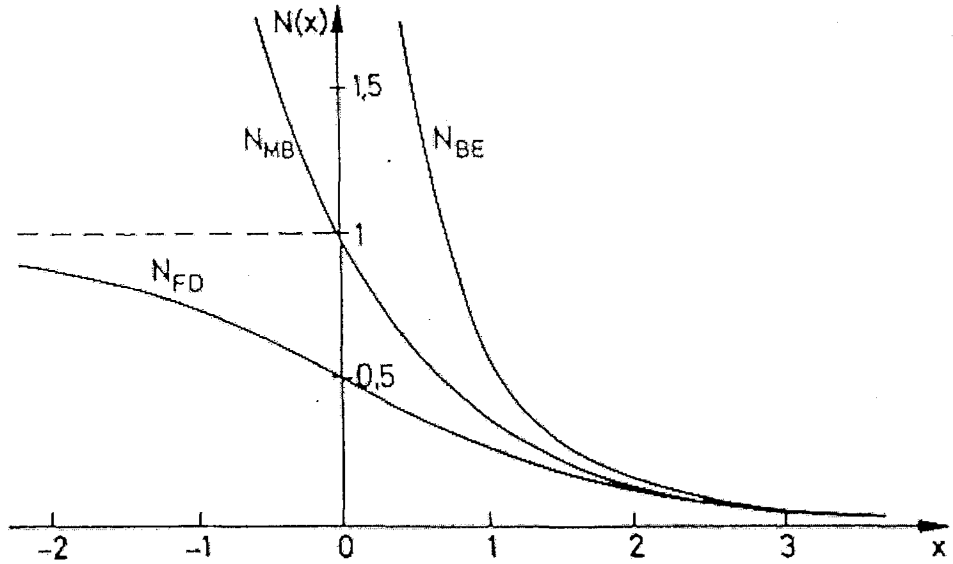
$$k_B T_F = \epsilon_F \qquad (1.34)$$

in terms of the Fermi energy ϵ_F:

$$\epsilon_F = \mu(T=0) = \frac{\hbar^2}{2m} \left(3\pi^2 \bar{\rho}\right)^{\frac{2}{3}} \qquad (1.35)$$

which is the energy where at $T=0$ the FD distribution is cut off.

To see this we consider the number distributions:



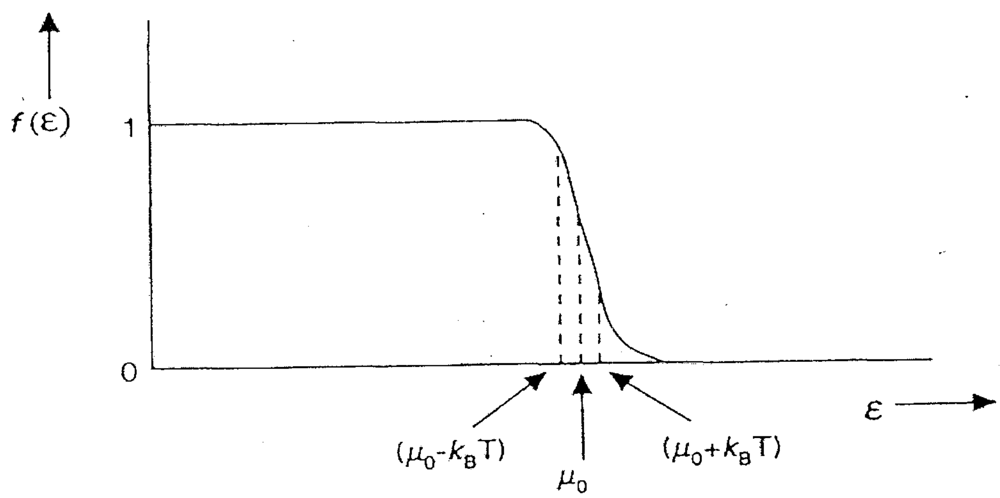


Abb. 7.2: Fermi-Funktion für eine Temperatur T mit $T_F \gg T > 0$.

$$\sigma = e^{\frac{\mu}{kT}}, \quad x := \frac{\epsilon - \mu}{kT},$$

$$N_{iBE} = \frac{1}{\sigma^{-1} e^{\epsilon_i/kT} - 1} \longrightarrow N_{BE}(x) = \frac{1}{e^x - 1}$$

$$N_{iMB} = \frac{1}{\sigma^{-1} e^{\epsilon_i/kT}} \longrightarrow N_{MB}(x) = \frac{1}{e^x}$$

$$N_{iFD} = \frac{1}{\sigma^{-1} e^{\epsilon_i/kT} + 1} \longrightarrow N_{FD}(x) = \frac{1}{e^x + 1}$$

We now turn to the μ - T - diagram:

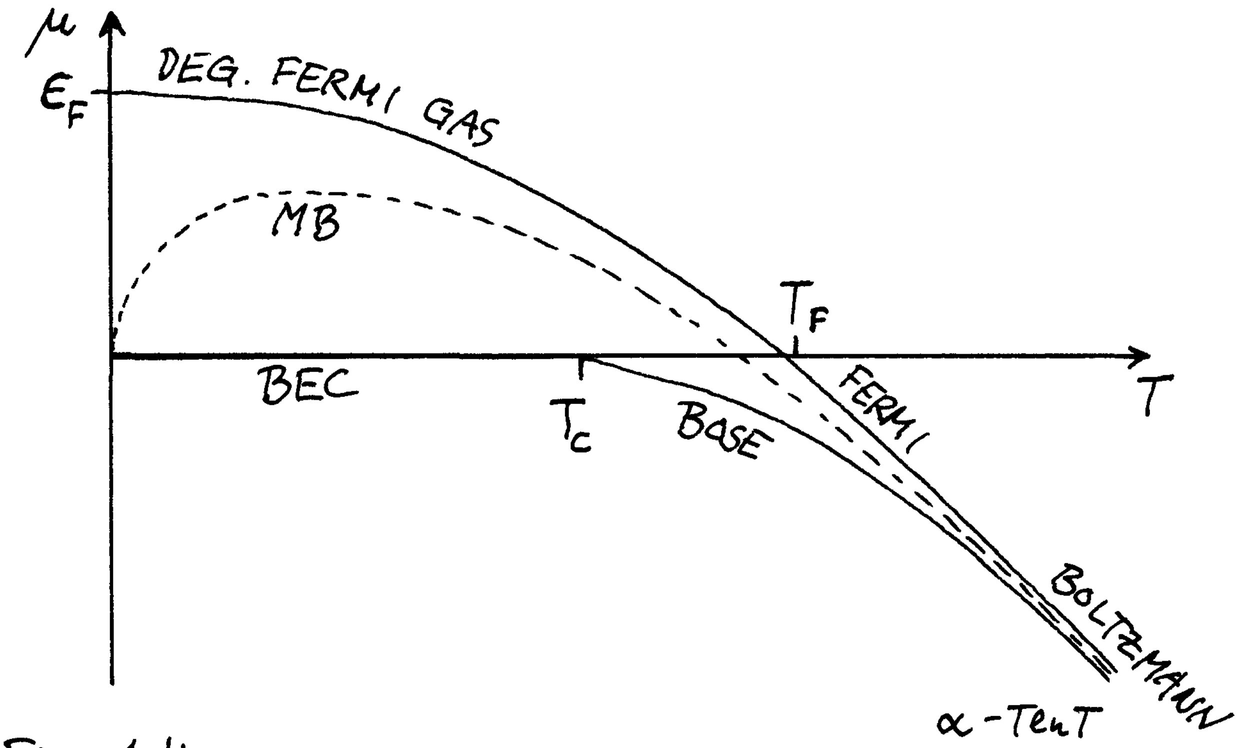


Fig 1.4

The critical temperature for BEC is found from (1.33) to be related to T_F (1.34) by

$$\frac{T_F}{T_c} = \frac{\pi^{\frac{1}{3}}}{4^{2/3}} \left[3 \zeta\left(\tfrac{3}{2}\right) \right]^{\frac{2}{3}} \cong 1.4 \qquad (1.36)$$

$$= \left(\Gamma(\alpha+1)\, \zeta(\alpha) \right)^{\frac{1}{\alpha}} . \qquad (1.36a)$$

The fermionic μ vanishes where $z = 1$, i.e. for the temperature T with

$$\frac{T}{T_F} = \left[\Gamma(\alpha+1)\, f_\alpha(1) \right]^{-\frac{1}{\alpha}} \cong 0.989 . \qquad (1.37)$$

for $\alpha = \frac{3}{2}$

Bose - Einstein condensation

From (1.13a-B) it follows that the ground-state occupation number is

$$\bar{N}_0 = \frac{z}{1-z} \tag{1.38}$$

which diverges for $z \to 1$. In turn,

$$z = \frac{\bar{N}_0}{1+\bar{N}_0} \tag{1.39}$$

We consider the thermodynamic limit

$$V \to \infty \quad \text{and} \quad \bar{\rho}_0 = \frac{\bar{N}_0}{V} < \infty . \tag{1.40}$$

Note that the other possibilities

→ $\bar{\rho}_0 \to 0 \quad \Rightarrow \quad z < 1$

→ $\bar{\rho}_0 \to \infty$

either lead back to the non-degenerate gas or are unphysical. Assuming the validity of (1.13a-B) in the above limit implies, with (1.39)

$$\bar{N}_i = \left(e^{\beta\epsilon_i} - 1 + \frac{1}{\bar{\rho}_0 V} e^{\beta\epsilon_i}\right)^{-1}$$
$$\to \left(e^{\beta\epsilon_i} - 1\right)^{-1} \tag{1.41}$$

Hence, $\bar{\rho}_i = \bar{N}_i / V \to 0$ for all $i \neq 0$.
This pitfall can be overcome by singling out the zero-mode:

$$\bar{\rho} = \bar{\rho}_0 + \frac{1}{\lambda_T^3} g_{3/2}(z) \tag{1.42}$$

From Fig. 1.2-B and (1.32) one finds, if $\bar{\rho} > \bar{\rho}_c$: $\bar{\rho} = \bar{\rho}_0 + \bar{\rho}_c$, if $\bar{\rho} < \bar{\rho}_c$: $\bar{\rho}_0 = 0$.
For general α we have

$$\bar{N} = \bar{N}_0 + C_\alpha \Gamma_\alpha (k_B T)^\alpha g_\alpha(z). \tag{1.42a}$$

From the discussion in Sect. 1.4 and Fig. 1.2-B, and Eqs. (1.32), (1.33) it is clear that there will be no such Bose-Einstein condensation if $\alpha \leq 1$, since $g_\alpha(z)$ diverges before $z = 1$ is reached.
This can be seen more easily already, e.g., from the integral of the BE-distribution in d dimensions: For $z = 1$ one has

$$\int d^d p \frac{1}{e^{\beta p^2/2m} - 1} \propto \int \frac{p^{d-1} dp}{e^{\beta p^2/2m} - 1} \overset{\infty}{\approx} \int_0^{\Delta p} \frac{p^{d-1} dp}{p^2} + \text{finite} \tag{1.43}$$

Hence the integral diverges logarithmically for $d=2$ and polynomially for $d=1$. Therefore, lowering the temp. at fixed $\bar{\rho}$ or increasing $\bar{\rho}$ at fixed T can always be compensated by lowering $|\mu|$ in order to still satisfy (1.12), without reaching $\mu=0$ at any non-zero T or finite $\bar{\rho}$.
In contrast, for $\alpha>1$ there is a finite maximum $\bar{\rho}_c$ or $\bar{N}_c$.

1.6 Thermodynamic quantities

In order to easily compute th.dyn. bulk quantities it is now convenient to also express the grandcanonical potentials in terms of g_α and f_α:
From (1.8a), (1.18a), and (1.29) we find

$$\Omega_{gc} = -C_\alpha \Gamma(\alpha)(k_B T)^{\alpha+1} \times \begin{cases} g_{\alpha+1}(z) & \boxed{B} \\ g_s f_{\alpha+1}(z) & \boxed{F} \end{cases} \qquad (1.44)$$

$$= -(k_B T)\frac{V}{\lambda_T^3} \times \begin{cases} g_{5/2}(z) \\ f_{5/2}(z)\, g_s \end{cases}, \alpha = \frac{3}{2} \qquad (1.44a)$$

For BEC: $\Omega_{gc} \to \Omega_{gc} - k_B T \ln(1-z)$.

1. Ideal quantum gases

1.1 The ideal Bose-gas

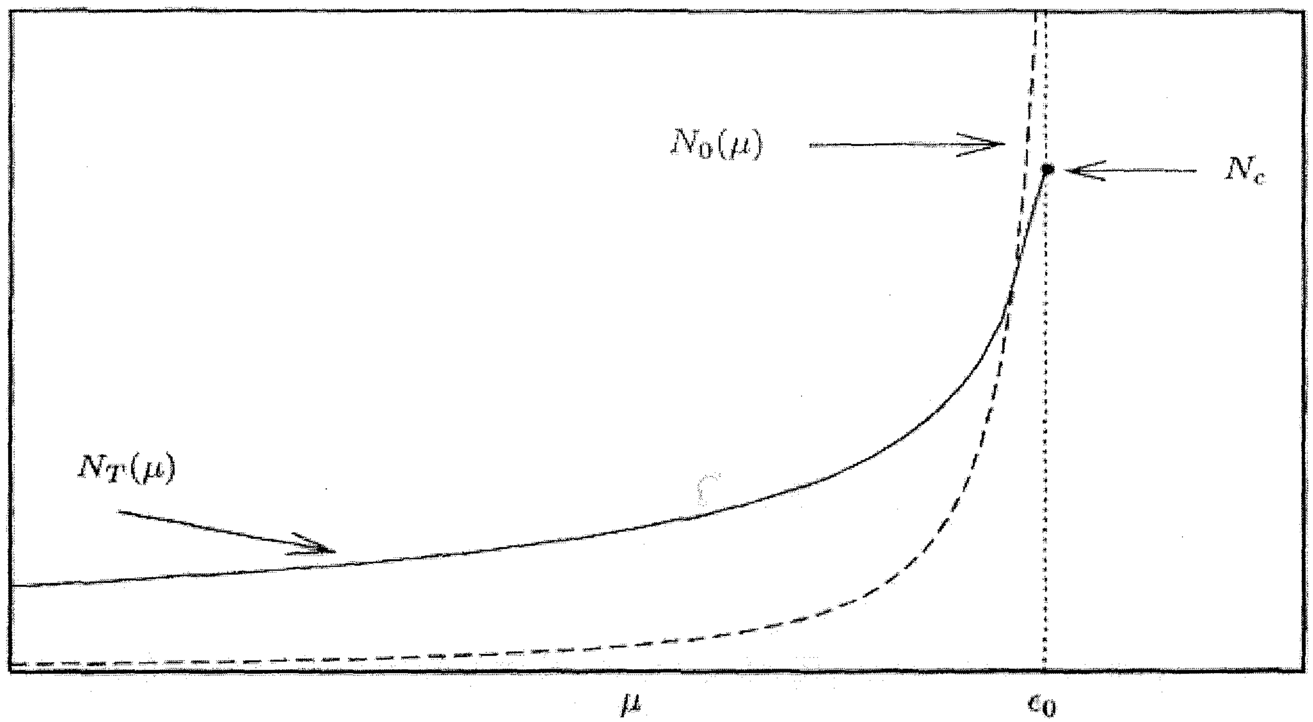


FIG. 3.1. Ideal gas model. The number of particles out of the condensate (N_T, solid line) and in the condensate (N_0, dashed line) as a function of the chemical potential, for a fixed value of T. The actual value of μ is fixed by the normalization condition $N = N_0 + N_T$. If $N > N_c$ the solution is given by $\mu \sim \epsilon_0$ and the system exhibits Bose–Einstein condensation ($N_0/N \neq 0$) in the thermodynamic limit

Fig. 1.1

Condensate fraction

Fore bosons we find the condensate fraction below T_c:

$$\frac{\bar{N}_0}{\bar{N}} = 1 - \left(\frac{T}{T_c}\right)^{\alpha} \qquad (1.45)$$

Mean energy

From (1.8a); (1.44a):

$$\bar{E} = \frac{\partial}{\partial\beta}\left(\beta\,\Omega_{gc}\right)_{z=\text{const.}}$$

$$= C_\alpha\,\Gamma(\alpha+1)\,(k_B T)^{\alpha+1}\begin{cases} g_{\alpha+1}(z) \\ f_{\alpha+1}(z)\end{cases} \qquad (1.46)$$

$$= \frac{3}{2}\,k_B T\,\frac{V}{\lambda_T^3}\begin{cases} g_{5/2}(z) \\ f_{5/2}(z)\end{cases},\quad \alpha=\frac{3}{2} \qquad (1,46a)$$

In the class. limit we obtain as expected (cf. (1.21), (1.29)):

$$\bar{E} = \alpha\,\bar{N}\,k_B T \qquad (1.47)$$

which reveals the connection between α and the # of d.o.f. per particle.

Condensate fraction of the homogeneous ideal gas

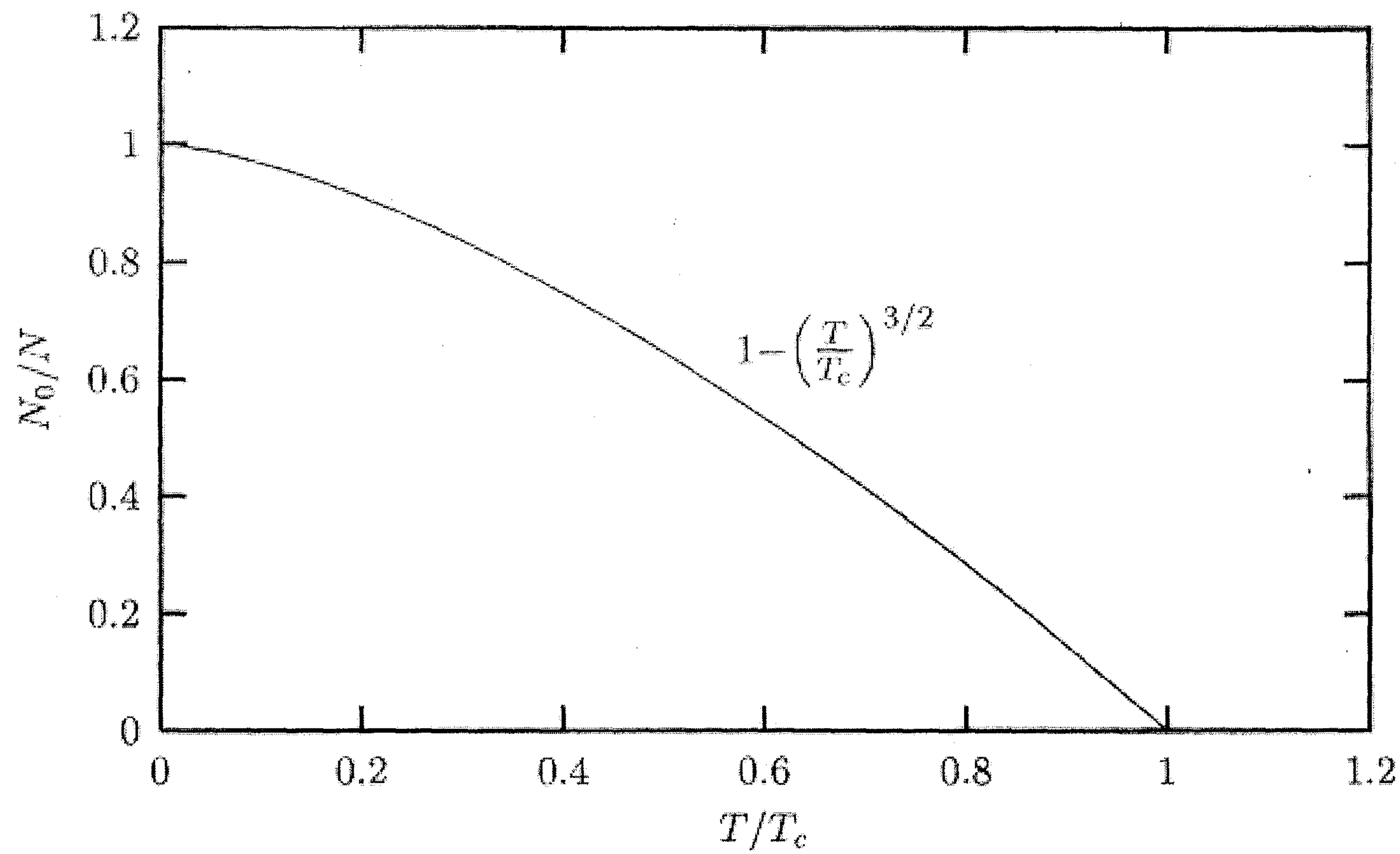


FIG. 3.2. Condensate fraction N_0/N versus temperature for a uniform ideal Bose gas. The condensate fraction is different from zero below T_c where the system exhibits Bose–Einstein condensation

Fig. 1.2

Condensate fraction of the ideal/real Bose gas in a harmonic trap

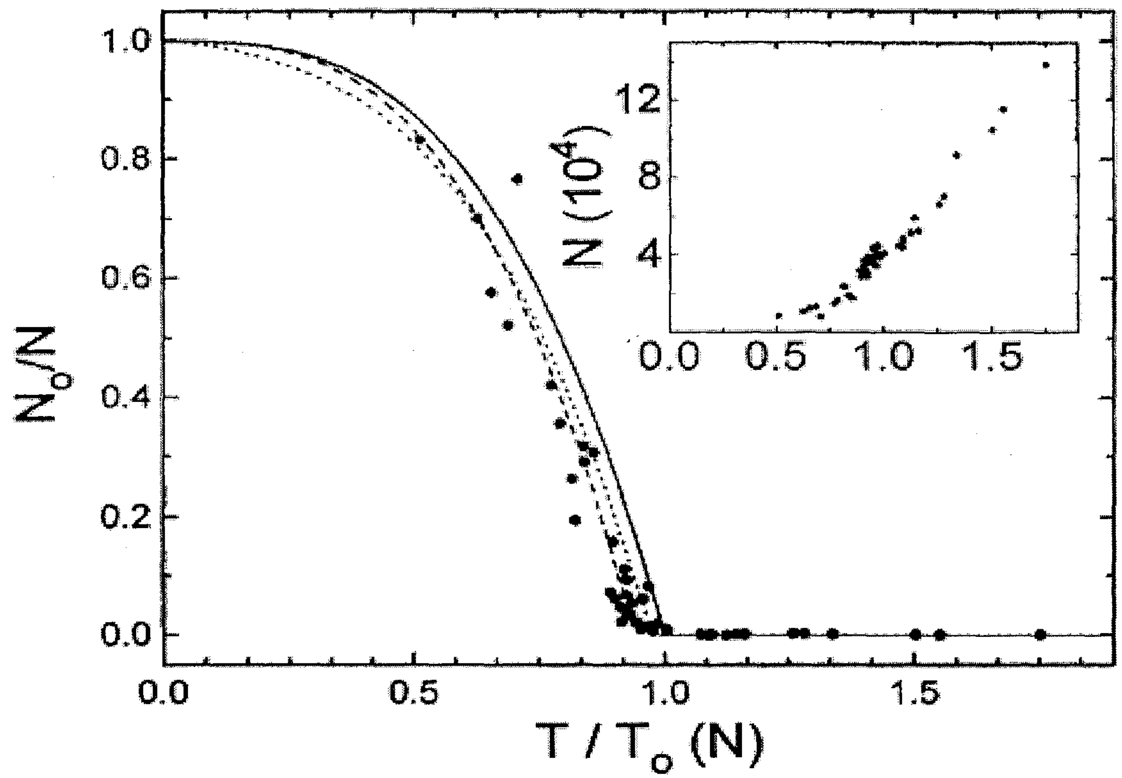


FIG. 1. Total number N (inset) and ground-state fraction N_o/N as a function of scaled temperature T/T_o. The scale temperature $T_o(N)$ is the predicted critical temperature, in the thermodynamic (infinite N) limit, for an ideal gas in a harmonic potential. The solid (dotted) line shows the infinite (finite) N theory curves. At the transition, the cloud consists of 40 000 atoms at 280 nK. The dashed line is a least-squares fit to the form $N_o/N = 1 - (T/T_c)^3$ which gives $T_c = 0.94(5)T_o$. Each point represents the average of three separate images.

Fig. 1.3 [J.R. Ensher et al., PRL 77, 4984 (1996)]

In the degenerate regime, $z \gg 0$, we have

$$\boxed{B} : \quad \bar{E} \cong C_\alpha \Gamma_{\alpha+1} \zeta(\alpha+1) \, (k_B T)^{\alpha+1} \qquad (1.48)$$

and from (1.33):

$$\boxed{F} \qquad \bar{E} \cong C_\alpha \Gamma(\alpha+1) \, \mu^{\alpha+1} \left[1 + \left(\frac{k_B T}{\epsilon_F} \right)^2 + \dots \right]$$

$$= \alpha \Gamma(\alpha+1) \, \bar{N} k_B T_F \left[1 + \left(\frac{\cancel{k_B} T}{T_F} \right)^2 + \dots \right] \qquad (1.48\text{-}F)$$

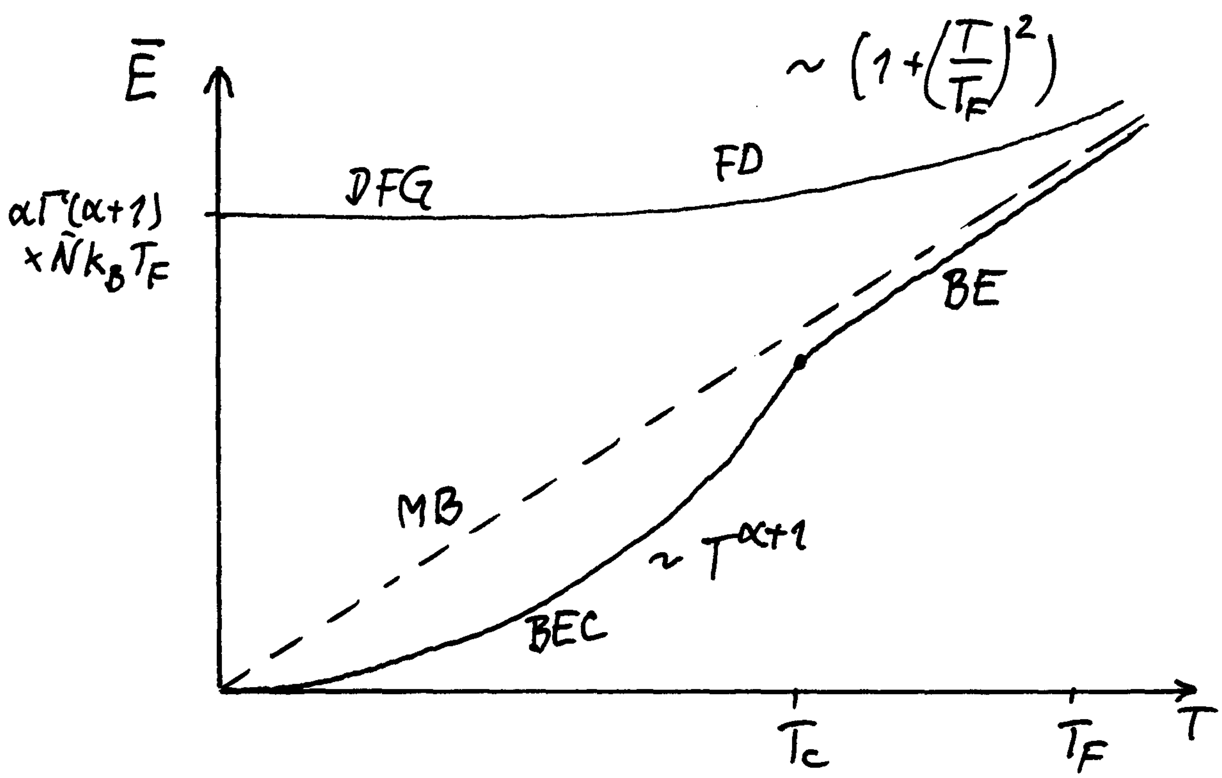


FIG 1.5

Mean energy per particle of ideal Fermi, Boltzmann and Bose gases

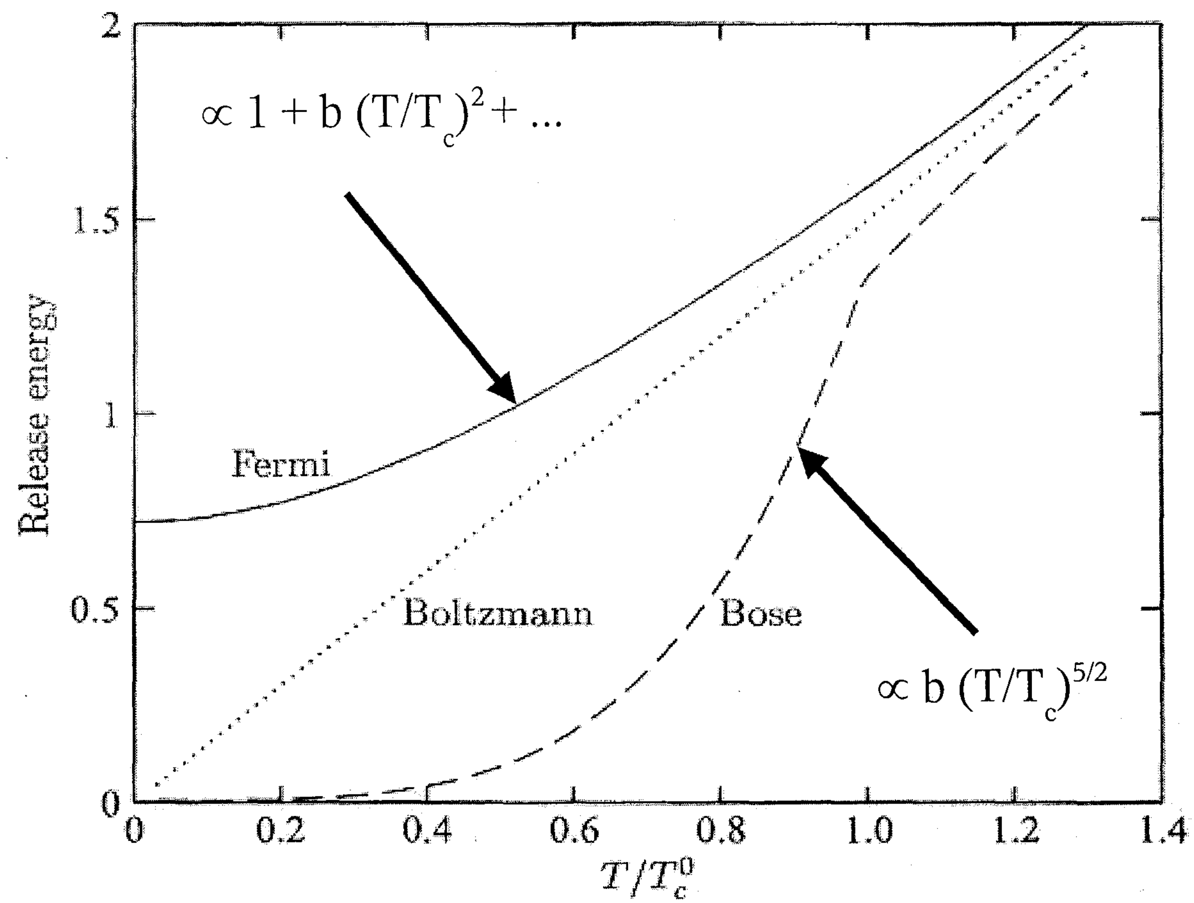


FIG. 18.2. Release energy as a function of temperature calculated for an ideal Fermi (solid line), classical (dotted line) and Bose (dashed line) gas. Here T_c^0 is the critical temperature for Bose–Einstein condensation

Fig. 1.8 [Pitaevskii, Stringari: BEC]

Energy per atom for a ^{87}Rb Bose gas

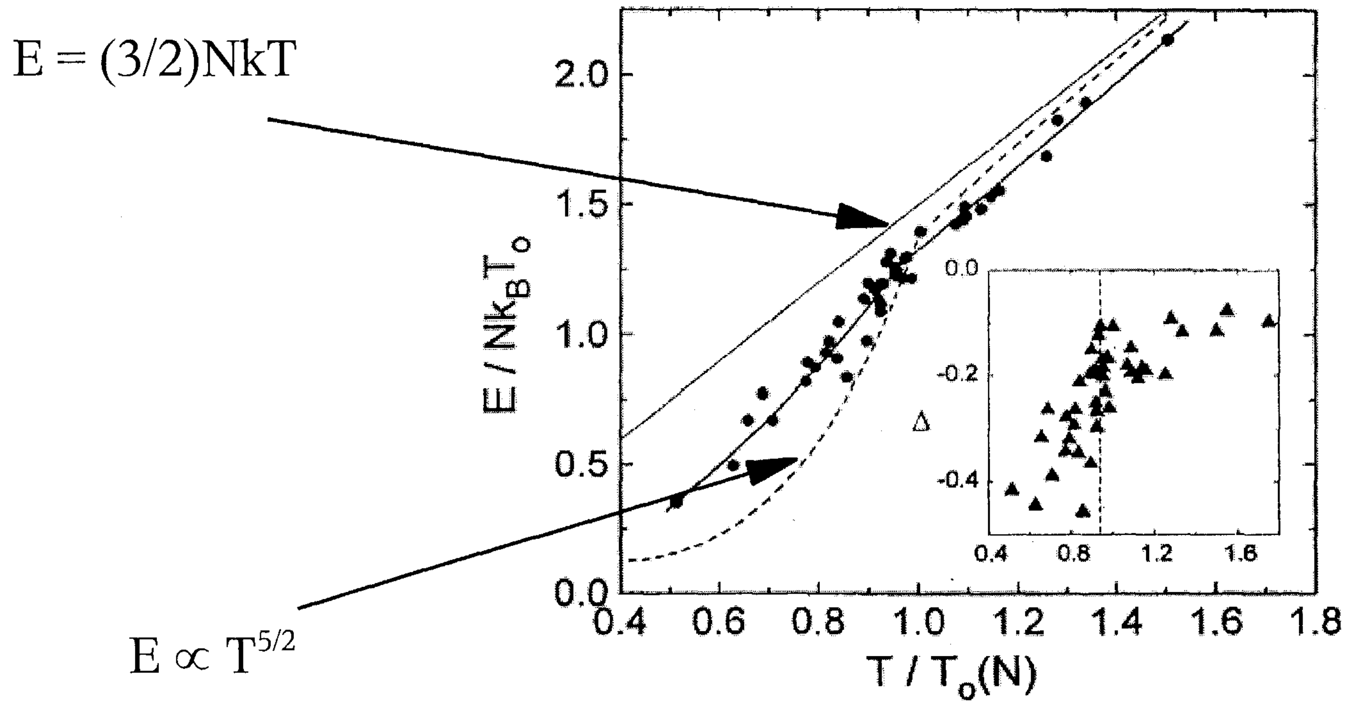


FIG. 2. The scaled energy per particle E/Nk_BT_o of the Bose gas is plotted vs scaled temperature T/T_o. The straight, solid line is the energy for a classical, ideal gas, and the dashed line is the predicted energy for a finite number of noninteracting bosons [22]. The solid, curved lines are separate polynomial fits to the data above and below the empirical transition temperature of $0.94T_o$. (inset) The difference Δ between the data and the classical energy emphasizes the change in slope of the measured energy-temperature curve near $0.94T_o$ (vertical dashed line).

Fig. 1.5 [J.R. Ensher et al., PRL 77, 4984 (1996)]

Mean pressure

The pressure is obtained as

$$\bar{p} = -\frac{\partial}{\partial V}\Omega_{gc} \qquad (1.49)$$

For the 3dim homogeneous gas, $\alpha = 3/2$ one obtains

$$\bar{p} = \frac{k_B T}{\lambda_T^3} \times \begin{cases} g_{5/2}(z) \\ g_s f_{5/2}(z) \end{cases} \qquad (1.50)$$

which yields an equation of state

$$\bar{p} V = \bar{N} k_B T \frac{g_{5/2}(z)}{g_{3/2}(z)} \qquad (1.51)$$

$$\rightarrow \begin{cases} \bar{N} k_B T, & z \to 0 \\ \bar{N} k_B T \frac{\zeta(5/2)}{\zeta(3/2)}, & z = 1 \end{cases} \qquad (1.51a)$$

Note that in the BEC phase, the pressure is constant in V

$$\bar{p}_{BEC} = k_B T \lambda_T^{-3} \zeta(5/2) \qquad (1.52)$$

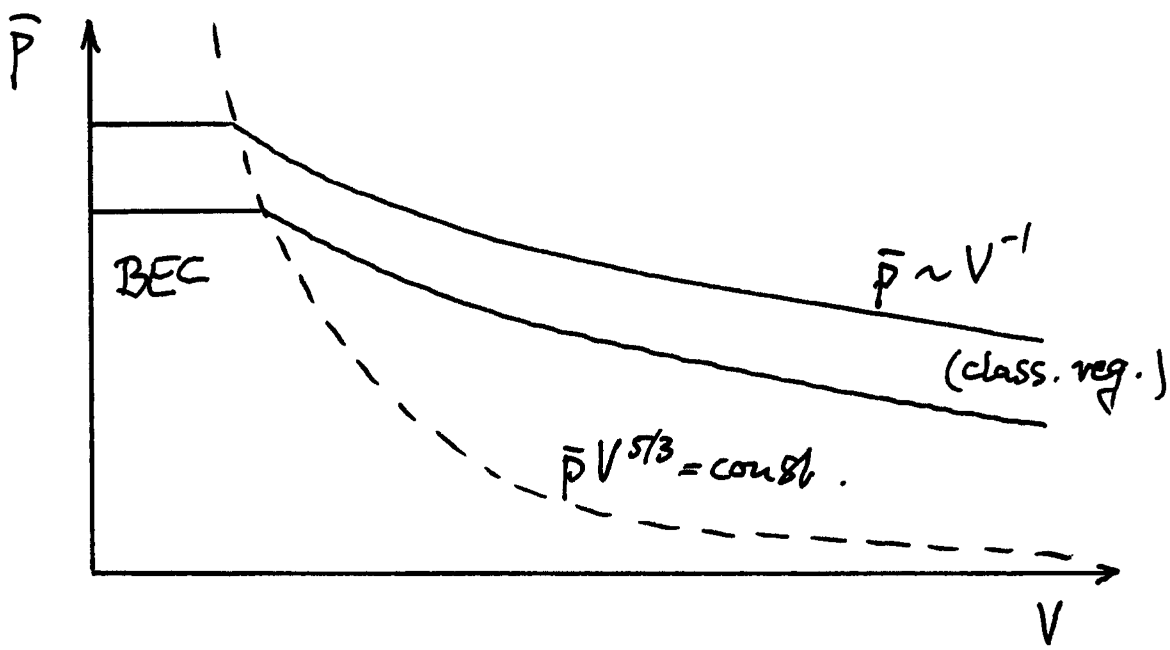
whereas, in the normal phase, from

$$V = \bar{N} \lambda_T^3 g_{3/2}(z)^{-1}$$

the pressure decreases with V.

The BEC phase boundary is given by

$$\bar{p}\, V^{5/3} = \frac{2\pi\hbar^2}{m} \frac{\zeta(5/2)}{[\zeta(3/2)]^{5/3}} \bar{N}^{5/3} \qquad (1.53)$$



The Bose gas w/o interactions is an infinitely compressible gas below T_c.

One obtains a 1st-order phase transition (which is not the case once interactions are taken into account.)

We finally plot the $\bar{p}$ - T diagram. The phase boundary to the BEC phase is given by

$$\bar{p}\, T^{5/2} = \text{const.} \qquad (1.53a)$$

Isothermic curves below and above the critical line for Bose-Einstein condensation

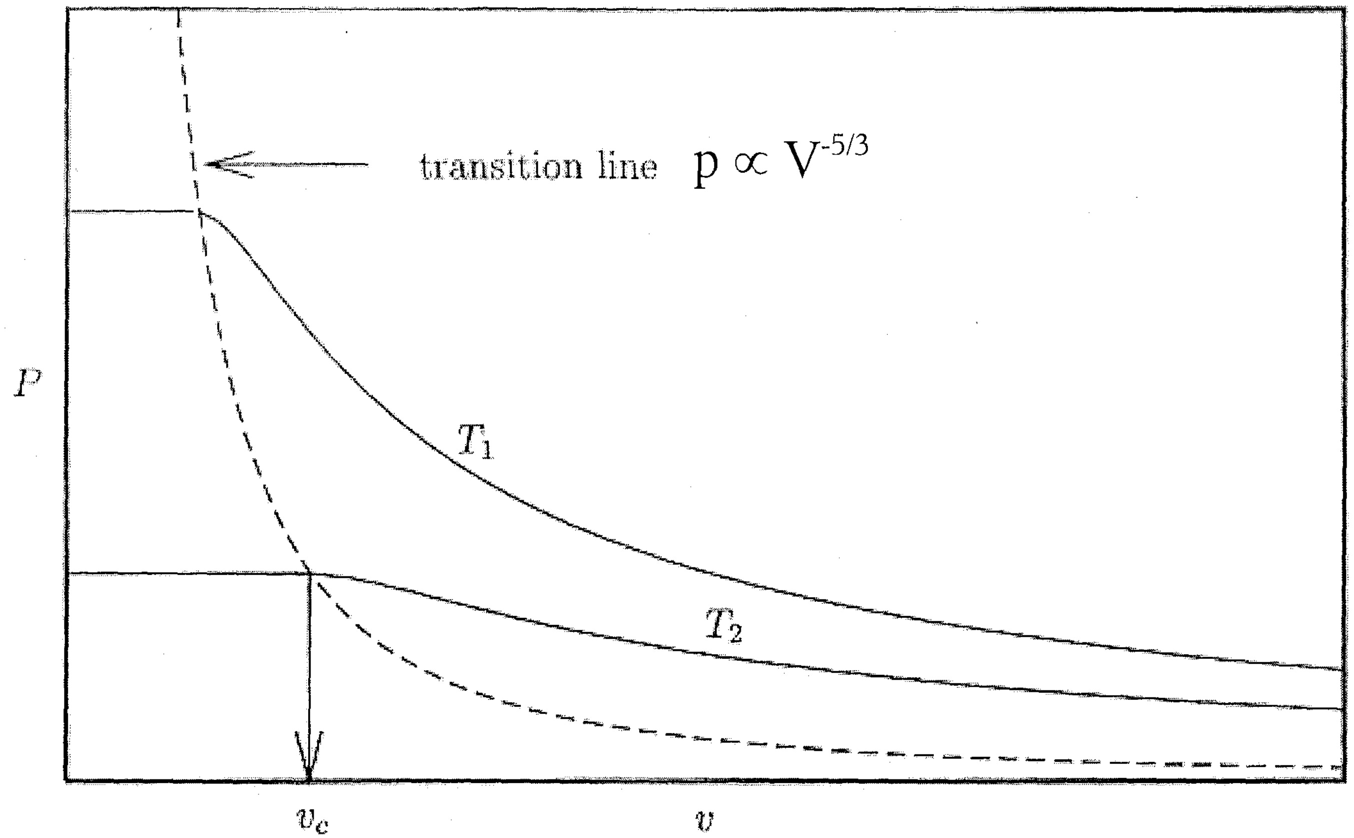
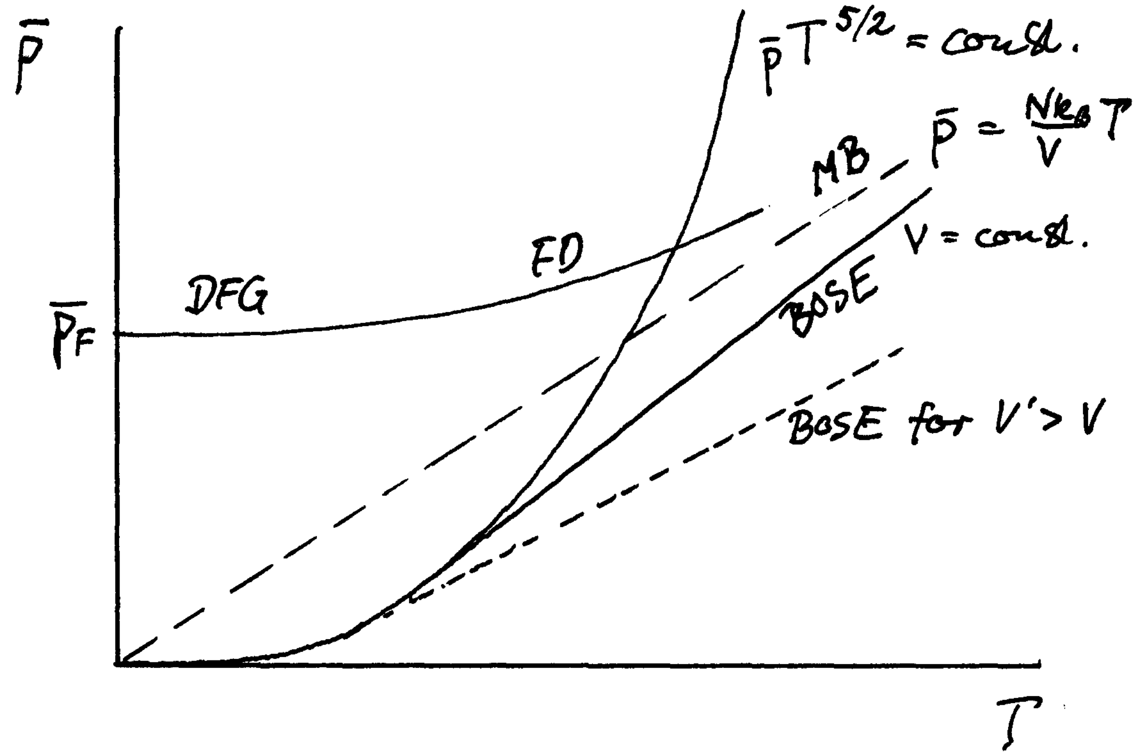


FIG. 3.4. Pressure of the ideal Bose gas as a function of $v = V/N$ for two different temperatures. Below v_c the pressure is constant. The transition line (dashed curve) is also shown (see text)

Fig. 1.7 [Pitaevskii, Stringari: BEC]

One obtains:



The Fermi gas near $T=0$ behaves as

$$\bar{p} = \frac{2}{5}(3\pi^2\bar{\rho})^{\frac{2}{3}}\bar{\rho}\frac{\hbar^2}{2m}\left\{1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2+\cdots\right\} \quad (1.53b)$$

Due to the Pauli repulsion there remains the Fermi pressure

$$\bar{p}_F = \frac{2}{5}\bar{\rho}\,\epsilon_F = \frac{2}{5}\frac{\hbar^2}{2m}(3\pi^2)^{\frac{2}{3}}\bar{\rho}^{\frac{5}{3}}. \quad (1.53c)$$

Entropy

Entropy is obtained as

$$\bar{S} = -\frac{\partial}{\partial T}\,\Omega_{gc} \tag{1.54}$$

For bosons:

$$\bar{S} = k_B C_\alpha \Gamma_\alpha (k_B T)^\alpha \left[(\alpha+1)\, g_{\alpha+1}(z) - g_\alpha(z) \ln z\right], \tag{1.55}$$

For fermions replace g_α by f_α. For $\alpha = \frac{3}{2}$:

$$\bar{S} = k_B \frac{V}{\lambda_T^3}\left[\frac{5}{2} g_{5/2}(z) - g_{3/2}(z) \ln z\right] \tag{1.55a}$$

The usual class. expressions are easily obtained. In the BEC phase one needs to add

$$\bar{S}_{BEC} = -k_B\left[\ln(1-z) - \frac{z}{1-z}\ln z\right] + \bar{S}(T > T_c). \tag{1.56}$$

$$\xrightarrow[\bar{\rho}_0 < \infty]{V\to\infty} k_B \ln(\rho_0 V) + \frac{5}{2} k_B \frac{V}{\lambda_T^3}\,\zeta(5/2) \tag{1.57}$$

$\Rightarrow$ $s = S/V \to 0$ for $T \to 0$ as required.

Fermi gases in the deg. limit obey ($\alpha = \frac{3}{2}$)

$$\bar{S} = \frac{\pi^2}{2} k_B \bar{N} \frac{k_B T}{\epsilon_F} \left\{ 1 + \mathcal{O}\left(\left(\frac{T}{T_F}\right)^2 \right) \right\} \qquad (1.58)$$

We finish by discussing the

<u>Specific heat at V=const.:</u>

$$C_V' = V^{-1} T \left(\frac{\partial S}{\partial T} \right)_{V, \bar{N}} \qquad (1.59)$$

In the BEC phase:

$$C_V' = \frac{15}{4} k_B \zeta(5/2) \lambda_T^{-3}$$
$$\xrightarrow{T=T_c} \frac{15}{4} \bar{\rho} k_B \frac{\zeta(5/2)}{\zeta(3/2)} . \qquad (1.60)$$

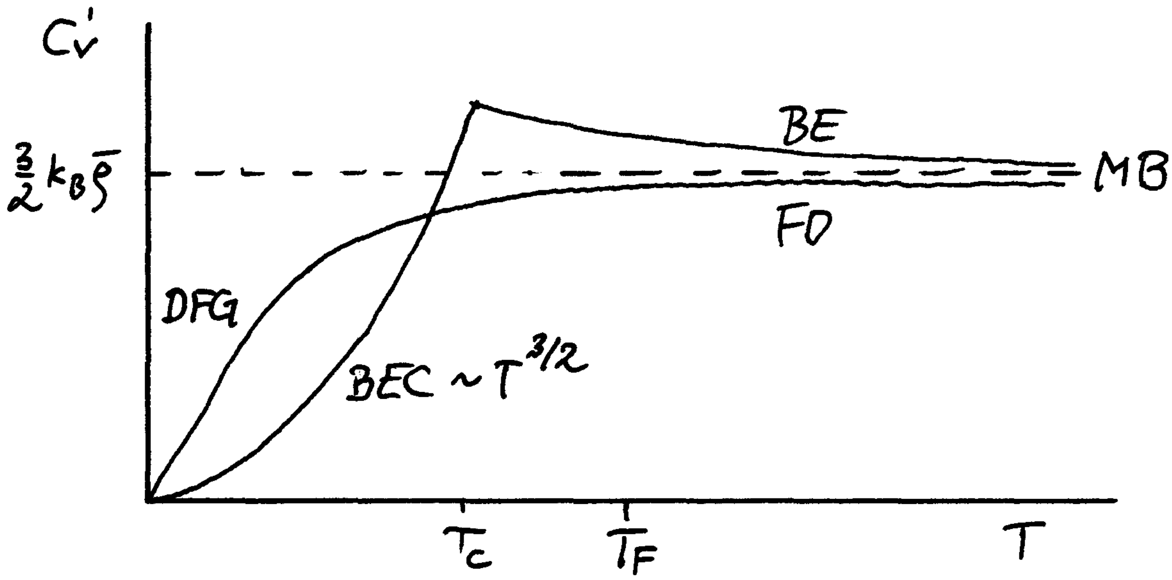
In the normal phase:

$$C_V' = k_B \bar{\rho} \left[\frac{15}{4} \frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4} \frac{g_{3/2}(z)}{g_{1/2}(z)} \right] \qquad (1.61)$$
$$\xrightarrow{T \to \infty \ (z \to 0)} \frac{3}{2} k_B \bar{\rho}$$

Fermions for $z \to \infty$:

$$C_V' = \frac{\pi^2}{2} k_B \bar{\rho} \frac{k_B T}{\epsilon_F} \left\{ 1 + \mathcal{O}\left(\left(\frac{T}{T_F} \right)^2 \right) \right\} \qquad (1.62)$$

This gives the following diagram:



For general α one has, for bosons:

(a) $z \to 0$: $C_V' \to \alpha k_B \bar{N}$ (1.63)

(b) $z = 1$: $C_V' = (\alpha+1) \dfrac{\zeta(\alpha+1)}{\zeta(\alpha)} \left(\dfrac{T}{T_c}\right)^{\alpha} \alpha k_B \bar{N}$ (1.64)

(c) $0<z<1$: $C_V' =$

$$C_V' = \alpha k_B \bar{N} \left[(\alpha+1) \frac{g_{\alpha+1}(z)}{g_\alpha(z)} - \alpha \frac{g_\alpha(z)}{g_{\alpha-1}(z)} \right] \qquad (1.65)$$

Thus, for $\alpha > 2$, there is a discontinuity at $z=1$, which disappears at $\alpha = 1$, where $g_{\alpha-1}(1) \to \infty$.

Specific heat of the ideal Bose gas for different densities of states

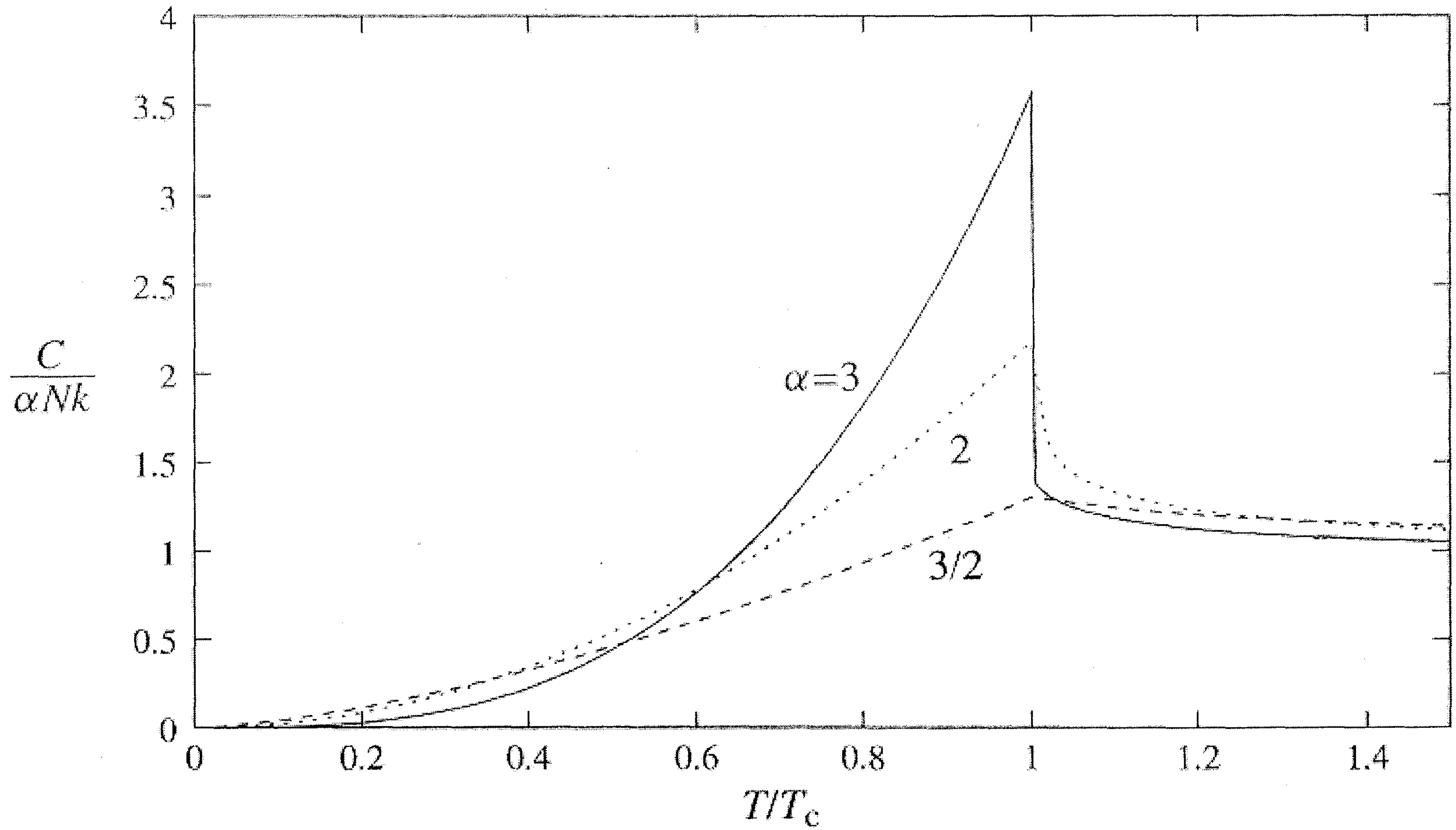


Fig. 2.3. The specific heat C, in units of αNk, as a function of the reduced temperature T/T_c for different values of α.

Fig. 1.4 (Density of states: $g(\epsilon) = C_\alpha \, \epsilon^\alpha$)

Specific heat of a ^{87}Rb-BEC in a 3-dimensional harmonic trap

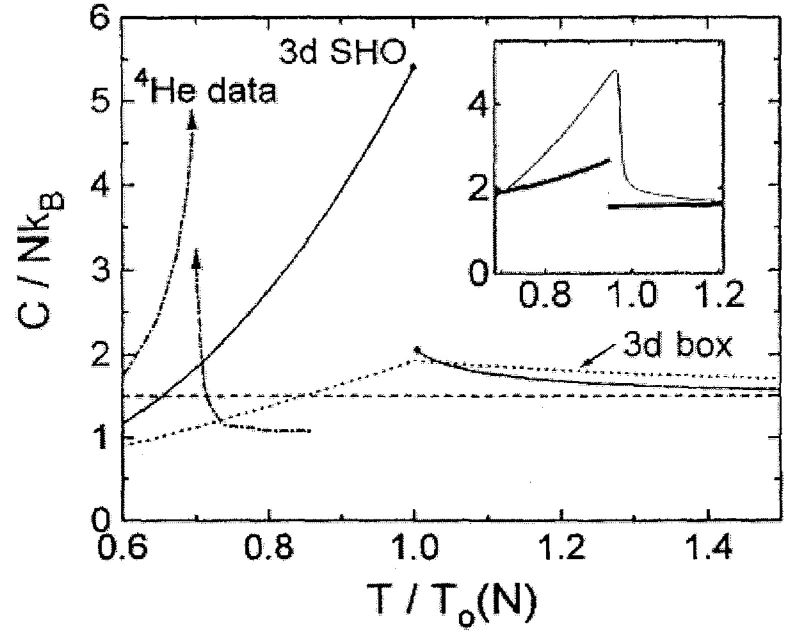


FIG. 3. Specific heat, at constant external potential, vs scaled temperature T/T_o is plotted for various theories and experiment: theoretical curves for bosons in a anisotropic 3D harmonic oscillator and a 3D square well potential, and the data curve for liquid ^{4}He [25]. The flat dashed line is the specific heat for a classical ideal gas. (inset) The derivative (bold line) of the polynomial fits to our energy data is compared to the predicted specific heat (fine line) for a finite number of ideal bosons in a harmonic potential.

Fig. 1.6 [J.R. Ensher et al., PRL 77, 4984 (1996)]