$\mathcal{PT}$ quantum mechanics and some of its underlying mathematics

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In 1998, 1999 it was shown by Bender and collaborators that there are certain classes of Hamiltonians which at a first glance seem not selfadjoint in Hilbert spaces, but which nevertheless are having real spectra. Examples are Hamiltonians of the type $H = p^2 + x^2(ix)^\mu$. For parameters $\mu \in [0, 1]$ these Hamiltonians have positive real eigenvalues with square integrable eigenfunctions defined over the real line. It was found that the reality of the eigenvalues was connected with an underlying $\mathcal{PT}$ symmetry of the Hamiltonians and their eigenfunctions, i.e. the systems are in a sector of unbroken $\mathcal{PT}-$symmetry. There exist other sectors like $\mu \in (-1, 0)$ where this $\mathcal{PT}-$symmetry is spontaneously broken: although the Hamiltonian remains $\mathcal{PT}-$symmetric, part of its eigenfunctions loose $\mathcal{PT}-$symmetry and the corresponding eigenvalues are coming in complex conjugate pairs. A $\mathcal{PT}$ phase transition occurs at $\mu = -0$.

It turns out that the $\mathcal{PT}-$symmetry of the Hamiltonian H induces a natural indefinite metric structure in Hilbert space and that H, instead of being selfadjoint in a usual Hilbert space (with positive definite metric), is selfadjoint in a generalized Hilbert space with an indefinite metric — a so called Krein space. Similar to time-like, space-like and light-like vectors in Minkowski space a Krein space has elements of positive and negative type as well as neutral (isotropic) elements. Moreover in analogy to passing via Wick-rotation from Minkowski space to Euclidian space, in the sector of exact $\mathcal{PT}-$symmetry there exists an operator which allows to pass from a Krein space description of the system to a description in a Hilbert space with a highly nontrivial metric operator. At the $\mathcal{PT}$ phase transition point this operator becomes singular and the corresponding mapping breaks down.

In the talk, on an introductory level, some of the basic structures of $\mathcal{PT}-$symmetric quantum mechanics and their relation to corresponding Krein-space setups are sketched. For gaining some rough intuition, the facts are illustrated by simple matrix models. The richness of the systems is demonstrated on the simple example of a $\mathcal{PT}-$symmetric two-mode Bose-Hubbard model and the geometry of a $\mathcal{PT}$ symmetric brachistochrone setup.