University of Konstanz
Department of Physics

Preparatory Course in Mathematics

for Students in Physics, Chemistry and Biology

Kompaktkurs:

Einführung in die Rechenmethoden der Experimentalphysik

by

Dieter Ebner

e-mail: Dieter.Ebner@uni-konstanz.de
Tel: +49/7531/88-3785, Fax: -3195

typeset in LaTeX by
Elizabeth Smith and Maria Rosner
Figures in CorelDraw by Elizabeth Smith
Corrections due to Jeanette Bühler

Publisher:
Fachbereich Physik,
Universität Konstanz
D-78457 Konstanz
Tel: +49/7531/88-2413, Fax: -3888
e-mail: fachbereich.physik@uni-konstanz.de
downloadable from http://kaluza.physik.uni-konstanz.de/k

5. Edition, 220 copies, August 10, 2002

Contents

0 Preface: How to use this manuscript. How it originated and what its didactic principles are. xiv

1 The trigonometric functions and radian measure of angles 1

Q1: Circumference of a circle . . . 1

Q2: The irrational number π . . . 1

Q3: Radian measure . . . 2

Q4: Mathematically positive sense of rotation . . . 4

Q5: Cartesian coordinates and its quadrants . . . 8

Q6: Sine and cosine as projections . . . 8

Q7: Sine and cosine in a right triangle . . . 10

Q8: Graph, zeroes, domain, range, period . . . 11

Q9: Inverse function . . . 12

Q10: Cosine . . . 14

Ex11: Rope around the earth . . . 14

Ex12: Transforming radians into degrees . . . 15

Ex13: Folding wire into a sector . . . 16

Ex14: Folding to a cylinder . . . 17

Ex15: Application of trigonometric functions in a triangle . . . 18

Ex16: Addition of rotations . . . 19

Ex17: Sign of rotations. Angles greater than 2π . . . 19

Ex18: Graphical construction of trigonometric functions . . . 20

Ex19: ☻ Photon flux density on the earth . . . 21

2 Harmonic oscillator, tangent, Pythagoras 23

Q1: Harmonic oscillator . . . 23

Q2: Tangent . . . 25

Q3: Cotangent . . . 26

Q4: Pythagoras . . . 27

Qx5: Special values of sine and cosine . . . 27

Qx6: Parametric representation of a circle . . . 28

Qx7: Zeros of sine with phase shift . . . 30

Ex8: Slope of a street . . . 30

Ex9: Trigonometric functions used for calculations of triangles . . . 31

Ex10: Approximate calculation of π . . . 33

Ex11: Any trigonometric function expressed by any other one . . . 34

Ex12: ☻☻ Inverse function . . . 35

Ex13: Phase shifts of two harmonic oscillators . . . 36

3 Formulae for trigonometric functions. Absolute value **39**

Q1: Fundamental formulae for sin and cos . . . 39

Q2: Absolute value . . . 40

Q3: Zeros and poles of tan . . . 41

Qx4: Tangens and cotangens are odd functions . . . 41

Qx5: Period of tangens and cotangens is π . . . 41

Qx6: Tangens and cotangens for complementary angles . . . 41

Qx7: Addition theorem for trigonometric functions . . . 42

Ex8: Graphical significance of absolute value . . . 43

Ex9: Simplification of trigonometric functions . . . 44

Ex10: ☻ Multiple values of the arcus function . . . 47

Ex11: ☻ Ellipses . . . 49

Ex12: ☻ Addition theorem for tangens . . . 52

Ex13: ☻ Addition of sines expressed as a product 52

Ex14: Frequency ν, angular frequency ω and period T 53

Ex15: ☻ Superposition of waves 55

4 Powers, roots and exponential functions **60**

Q1: Powers 60

Q2: Square roots 60

Q3: General roots 61

Q4: General powers 61

Q5: Calculation rules for powers 62

Q6: ☻ Operator priority 63

Q7: The (natural) exponential function $y = e^x$ 65

Q8: General exponential function 66

Ex9: General powers on a calculator 67

Ex10: Simplification of general powers 67

Ex11: Space diagonal in a cube 68

Ex12: Mathematical properties of the square root function 69

Ex13: Calculation rules for powers 70

Ex14: Power series to calculate function values (trivial case) 71

Ex15: Power series to calculate function values (numeric example) 71

Ex16: ☻☻ Reflecting the graph of a function 71

Ex17: Evaluating a symbolic infinite sum 72

Ex18: Power series used to prove an inequality 72

Ex19: Permutations 73

Ex20: Indexed quantities arranged as matrices 75

5 Approximations **77**

Q1: Approximation of numbers . . . 77

Q2: Approximation of functions . . . 78

Q3: For sufficiently small x the linear approximation is always valid. . . 79

Q4: Approximations can save calculation time . . . 79

Q5: Several small quantities . . . 80

Ex6: Power series for some important cases . . . 80

Ex7: Fast calculations using approximations . . . 81

Ex8: Linear approximation in a simple case . . . 81

Ex9: Linear approximation of transcendental functions . . . 82

Ex10: ☻ Different meanings of 'very small' . . . 83

Ex11: Area of a ring in linear approximation of width . . . 85

Ex12: Propagation of error . . . 86

Ex13: ☻☻ Properties of the exponential function proved approximately . . . 87

Ex14: Pseudoprobability in the decimal expansion of π . . . 88

Ex15: A power series defined by the digits of π . . . 89

6 Logarithms **91**

Q1: Examples for logarithms . . . 91

Q2: General logarithms . . . 91

Q3: Calculation rules for logarithms . . . 93

Q4: Decay laws . . . 94

Q5: ☻ Transforming logarithms to a different base . . . 95

Ex6: Logarithms calculated with a calculator . . . 95

Ex7: Simplification of logarithms (numeric arguments) . . . 95

Ex8: Simplification of logarithms (algebraic arguments) . . . 96

Ex9: Equations involving logarithms . . . 96

7 Number systems, dimensional quantities **98**

Q1: Number Systems . . . 98

Q2: dimensioned quantities in physics . . . 99

T3: ☻ Systems of units in physics . . . 100

Ex4: ☻ Large dimensioned quantities with a calculator . . . 101

Ex5: Constant velocity . . . 103

Ex6: ☻☻ Logarithmic scaling . . . 107

Ex7: Periodic decimal as a quotient . . . 113

8 Infinite sequences and infinite series **115**

T1: Motivation for infinite sequences . . . 115

Q2: Different notations for limits . . . 116

Q3: Definition of a limit . . . 116

Ex4: Simple examples of limits . . . 117

Q5: Convergence and divergence . . . 119

Ex6: Insignificant changes in sequences . . . 119

Ex7: Infinite sums as infinite sequences of its partial sums . . . 120

Q8: Composite infinite sequences . . . 121

Ex9: Limits of infinite sequences . . . 121

Ex10: Limits of infinite sums . . . 123

Ex11: Limits of composite sequences and series . . . 125

9 Continuity and limits of functions **127**

Q1: Continuous functions . . . 127

Q2: Limit of a function . . . 130

Q3: Continuity expressed by limits of functions . . . 131

Ex4: Limits of series built from continuous functions . . . 131

Ex5: Removable singularities 133

Ex6: Limits of functions 133

10 Differential and differentiation **136**

Q1: Tangent, derivative, differential 136

Ex2: ☻ A second example 139

Q3: Derivatives of elementary functions 140

Q4: Derivatives of composite functions 141

Ex5: Derivative for a very simple case 142

Ex6: Differential quotient for a linear function 143

Ex7: Derivative of the exponential function 145

Ex8: The product rule 147

Ex9: The quotient rule 147

T10: Different notations for functions and their derivatives 147

Ex11: The chain rule, 1. example 149

Ex12: The chain rule, 2. example 149

Ex13: Velocity as the derivative with respect to time t. 150

Ex14: ☻☻ Velocity of a damped harmonic oscillator 151

Ex15: Derivative of x^x 151

11 Applications of differential calculus **153**

Q1: Minimax problems 153

Ex2: Shape of maximum volume with given surface 154

Ex3: Gold necessary for a gold ball 155

Ex4: ☻☻ The differential as the equation for the tangent 156

Ex5: Average as the best guess for a measured quantity 158

Ex6: Error propagation 159

12 Higher derivatives, Taylor's formula **162**

Q1: Higher derivatives . . . 162

Q2: Taylor's formula . . . 162

Q3: Distinguishing local minima from local maxima . . . 164

Ex4: Taylor's formula to construct power series . . . 165

Ex5: Taylor's formula in linear approximations . . . 166

Ex6: ☻☻ Qualitative analysis of the Gaussian bell-shaped curve . . . 166

Ex7: Extrapolation with Taylor's formula . . . 168

13 Integrals **171**

Q1: The integral as an area . . . 171

Q2: Indefinite integrals . . . 173

Q3: Integration as the inverse of differentiation . . . 174

Qx4: Linear combination of integrals . . . 178

Q5: Additivity of the integral in the integration range . . . 179

Ex6: Antiderivatives . . . 180

Ex7: Definite integrals . . . 182

Ex8: Indefinite integrals . . . 184

Ex9: Integration as the inverse of differentiation . . . 187

Ex10: Area under the sine curve . . . 188

Ex11: Area of a triangle calculated by an integral . . . 190

Ex12: ☻☻ Average of $\sin^2$ and $\cos^2$ is $\frac{1}{2}$ . . . 191

Ex13: Derivative of an integral with respect to its lower boundary . . . 193

14 Application of integrals to geometry **195**

Ex1: An amulet out of gold . . . 195

Ex2: Area of a circle . . . 197

Ex3: Area of a circle calculated in polar coordinates 199

Ex4: Volume of a cone 200

Ex5: Surface of a sphere 201

Ex6: Volume of a sphere 204

15 Substitution method and partial integration 206

Q1: Substitution method 206

Q2: ☻ Partial integration 208

Ex3: The substitution method 208

Ex4: Calculation of arc lengths (rectifications) 210

Ex5: ☻ Any quantity is the integral of its differentials 211

16 ☻ Improper integrals 213

Q1: Improper integrals 213

17 ☻ Partial derivatives and total differential. Implicit functions 216

Q1: Partial derivatives 216

Q2: Taylor's formula in 2 variables 217

Q3: Implicit functions 220

Q4: Implicit differentiation 224

Ex5: Error propagation of multiple error sources 225

Ex6: Container with maximum volume 226

Ex7: Complete differential as the tangential plane 229

18 ☻ Multiple Integrals 235

T1: Double integral as an integral of an integral 235

Ex2: Area of a triangle calculated as a double integral 236

Ex3: Center of mass of a half-moon 238

Ex4: The cardioid . . . 242

19 Differential equations **250**

Q1: What are differential equations? . . . 250

Q2: Separation of variables . . . 255

Ex3: Growth equation solved again by separation of variables . . . 257

Ex4: Further examples for separation of variables . . . 259

Ex5: The oscillation equation . . . 263

Ex6: Constant velocity . . . 264

Ex7: Constant acceleration . . . 267

20 ☻ Binomial theorem **270**

Q1: Binomial theorem . . . 270

21 Introduction of vectors **272**

Q1: Introduction of vectors . . . 272

Q2: Addition of vectors . . . 277

T3: Computer graphics . . . 279

Ex4: $\vec{a}+\vec{b}$, $\lambda\vec{a}$ and $\hat{a}$ . . . 281

Ex5: Vector addition by parallelogram construction . . . 282

Ex6: Equation of a sphere . . . 284

Ex7: Construction of a regular tetrahedron . . . 285

Ex8: Bisectors intersect at a single point . . . 292

22 Vectors in physics. Linear combinations **295**

Q1: Forces as vectors . . . 295

Q2: Vectors depending on a scalar variable . . . 296

Q3: Velocity as a vector . . . 297

Q4: Vector fields 297

Q5: Vector spaces 298

Q6: ☻ Vector space versus geometrical space 298

Q7: Linear combinations, linear dependence 299

Q10: Components of vectors in some directions 301

Ex8: Linear combinations 302

23 Scalar product **303**

Q1: Scalar product (= dot product) 303

Ex2: Angles in an equilateral triangle 307

Ex3: Shortest distance from a straight line 308

Ex4: Shortest distance from a plane 309

Ex5: Invariance of the scalar product under rotations 311

24 Vector product **315**

Q1: Vector product 315

Ex2: Products of coordinate unit vectors 319

Ex3: Area of a parallelogram expressed by a determinant 321

Ex4: Linear (in)dependence expressed by vector product 323

Ex5: ☻ The vector product as a pseudo-vector; axial and polar vectors . . 324

25 Wedge product. Multiple vector products. **331**

Q1: Wedge product 331

Q2: Multiple vector products 332

Ex3: Other formulas for multiple vector products 332

Ex4: Purely vectorial treatment of a regular tetrahedron 333

Ex5: Volume of a cuboid calculated by wedge product and determinants . 339

26 Leibniz's product rule for vectors **343**

Q1: Leibniz's product rule for vectors 343

Ex2: Proof of Leibniz's product rule for vectors 343

Ex3: Velocity and acceleration of circular motion 344

Ex4: Mathematical pendulum 349

Ex5: Conservation of angular momentum 356

27 ☻ Complex numbers **358**

Q1: Complex numbers 358

Ex2: Hieronimo Cardano's problem from the year 1545 358

Q3: Addition and multiplication in components 359

T4: Fundamental theorem of algebra 360

Ex5: Example: square root of i 360

Q6: Real models of $\mathbb{C}$ 361

Ex7: Proof of the multiplication law 364

Q8: Complex conjugation 365

Ex9: Reflexion at the imaginary axis is not an automorphism 367

Q10: $\Re$ and $\Im$ expressed by $*$ 368

Q11: $|z|$ expressed by $*$ 368

Q12: No $<$ relation in $\mathbb{C}$ 369

Q13: Quotients decomposed in real- and imaginary parts 369

Ex14: Scalar product expressed by complex multiplication and $*$ 370

28 ☻ Complex functions **371**

Q1: Complex functions 371

Ex2: $w = z^2$ decomposed as two real valued functions 371

Q3: Euler's formula 372

Ex4: Proof of Euler's formula 372

Q5: Phase: complex number on the unit circle 373

Ex6: Plane rotation 374

T7: Properties of complex functions 375

Ex8: Functional properties derived from power series 375

Q9: Trigonometric functions as exponentials 376

Ex10: Parity of sine and cosine derived from Euler's formula 376

Q11: e-function has an imaginary period 377

Ex12: $*$ of an exponential 377

Ex13: Moivre's formula 377

Ex14: Addition theorem for trigonometric functions derived via $\mathbb{C}$ 378

0 Preface: How to use this manuscript. How it originated and what its didactic principles are.

At the university of Konstanz, as in most German universities, an intensive course ('**Kompaktkurs**') in mathematics is offered to physics, chemistry and biology students. It is optional and in Konstanz it is held the week before the first term. The intention of this preparatory course is to **refresh school mathematics**, to compensate for deficits in the mathematical education in special types of high schools and to deliver some additional important topics necessary immediately for the beginning lectures in physics, even before the regular mathematical lectures, hampered by their necessity to give a systematic treatment, are able to provide them.

Unfortunately, these aims are unachievable, at least in one week, and only a fraction of the present manuscript can be worked through in our 'Kompaktkurs'. But it is hoped that some students will use the manuscript in subsequent weeks to acquire some more topics by self-education.

The author of this manuscript is of opinion that in mathematics memorizing also plays an important role. Nothing goes into our **long-time memory** without **repetition** and without **motivation**. Since the 'Kompaktkurs' is also intended for students not particulary interested in mathematics itself, their motivation may be very low, which they should compensate by even more repetitions.

Therefore, the manuscript is organized as a **Question/Answer game**. In the first reading the **Q**-units bring a minimum of the material in compact form, which should simply be **learnt by heart**. You should work through these Q-units several times at appropriate intervals, corresponding to your memory abilities, until the material sits in your long-time memory (or until you have passed your examinations).

A real understanding of the Q-units will hopefully be achieved in the following exercises (**Ex**-units). An attempt was made to choose exercises that were as simple as possible, while still exemplifying a certain point as clearly as possible.

To enhance motivation, an attempt was made to try and find exercises which have real meaning or appeal to the students, so they are confident that they learn for themselves and not for their teachers. However, realistic and important examples have a tendency to be very long and complicated and will therefore distract from the essential point. A student who is very motivated will be better off with so called **nonsense examples**, which as such, have no real application but will present a certain point with utmost clarity and not at the expense of unneccessary complication by inessentials.

Most examples are taken from geometry, elementary physics or every day life.

In our brain we have several different memory systems. One type of memory system is called '**procedural memory**', which, for example, controls our ability to ride a bicycle. Corresponding mathematical abilities are contained in the **Qx**-units of this manuscript. These very simple exercises should be done repeatedly along with the Q-units.

Besides genetic factors, trainig, motivation and the number of repetitions, our memory also depends on the amount of related material we already know. Those who know the results of all football games of all preceeding years will have an easier time learning the results of the current year, as would be the case for myself. This means that a systematic treatment is the death of all education.

We can only **learn piecewise**. When one piece is assimilated, it is only at a later time that we are able to digest the next bite. Because of that reason, the student should also go through a subject several times, preferably using different books. In the first run one simply hears all the terminology and thus creates '**empty boxes**' in the brain with labels with these terms. In later runs the boxes are filled successively and interrelations between them are established.

Therefore, **systematicity** was not of high priority in the present manuscript. Instead we would like to lead the student through several valleys of the mathematical landscape as fast as possible.

We even offer the reader two speeds. Some material is marked by ☻ and those who want the slower speed can omit it in the first run. (☻☻ in a category, e.g. in c) ☻☻ means to omit all successive units of the same category, i.e. d) e) f) ... of that exercise.)

A further important condition for long-time memory, perhaps belonging to the category of motivation, is **genuine self activity**, in contrast to a boring systematic two-hour lecture given as a monologue as is still practised far too often at universities. The Italian physician **Maria Montessori** (1870-1952) was the first to have recognized this: her disabled children made more progress than a corresponding normal primary school class which was educated conventionally. She simply gave all kinds of material to her children and enhanced their motivation to do something with it.

As much in accordance with the Montessori pedagogics as is possible for a manuscript, we present all material in the form of **questions**. The Q-units are the material and the Ex-units should be your own activity. We have deferred to the Ex-units as much as was possible. This was achieved by a lot of Hints presented before the Results and the extensive Solutions, and by bisecting the exercises into a large number of small subunits a) b) c) ..., ensuring that the reader keeps on the right track.

Thus, important material is presented in the Ex-units and their results are summarized in bold boxes. They should be memorized together with the Q-units.

Working through all the exercises is much more demanding than simply reading a text. But it is rewarding and it is definitely recommended to the reader, although you will proceed much slower. If he, nevertheless, simply wants to read through the mansuscript, he can do so by immediately turning to the solutions. However, in the long term he will be less successful than his more active fellow student.

Some HINTS ask you to consult a **formulary**. Even more important than knowing many things by heart is knowing where they can be found in a book or formulary. Working with a formulary of your choice should be practiced extensively and is encouraged from the beginning.

Other HINTS about previous material contained in this manuscript is not given in the form of page or formula numbers, but instead verbally, e.g. 'Pythagoras', and you should consult the **index** to find the corresponding item in the manuscript. This practice enhances the probability that you will develop a corresponding box in your brain, since lexical long-time memory is intimately related to language abilities in the temporal lobe of our brain.

Considerable effort has been made to keep the Q-units as small and as effective as possible so that the student learns by heart only what is absolutely necessary or economic. Q-units are supplemented by Qx-units in order to make these as beneficial as possible to the student. It is common practise among students, especially those with difficulties, to manage mathematics by the memorization method, and they have discredited the learning of anything at all by heart in mathmematics We hope with our Q-units we have given a better choice of what to learn by heart, than that made by those students.

'A book is only as good as its index' is true for almost any scientific book. Therefore, I should devote much more energy to it in the future.

Interspersed you will find a lot of comments or remarks (REM). You should read them, but some of them you will understand only at a later time.

Some text is presented in **T**-units (theory-units), because it seemed impractical to cast them in the Q/A-scheme and/or because they do not contain material which should be learnt by heart or could possibly be remembered by a single reading.

In an attempt to be globally competitive it is now welcomed at our university to give lectures and manuscripts in the English language. For the convenience of the German readers, difficult English words are immediately translated into German ($\stackrel{\mathbf{G}}{=}$). These words are also included in the index, which could thus serve as a vocabulary.

Needless to say, we did not attempt to give **rigorous mathematics**. Instead we tried to present the subject as **intuitively** as possible, giving it in the form of **cooking-recipies** and explaining it with the help of **examples**. Thereby, unfortunately, the beautiful logical edifice of mathematics does not become apparent.

Only in some exercises do we cast a glance at the logical interrelation of mathematical truth. But this is more a type of **surfing in mathematics** than of learning what mathematical deduction and proving really means.

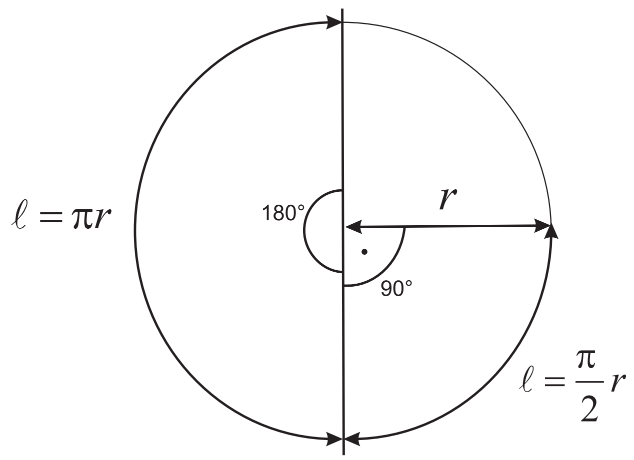
Therefore, this manuscript should not be your last book in mathematics, but it could be your first.

1 The trigonometric functions and radian measure of angles

1.**Q 1: Circumference of a circle**
Give the formula for the length of the **circumference**[$\overset{\mathbf{G}}{=}$ `Umfang`] c of a (full) circle of radius r, of a half circle and of the quarter of a circle.

(Solution:)

$$\boxed{c = 2\pi r} \quad \text{circumference of a circle} \tag{1}$$



`Fig`1.1. **1**: Circumference c of a full circle, and l = circumference of half (quarter) of a circle

REM 1: c = circumference = perimeter = length of the **periphery**
$d = 2r$=**diameter**[$\overset{\mathbf{G}}{=}$ `Durchmesser`]
(1) is also the definition of π.

REM 2: The divison of the **full angle** into 360° has Babylonean origin.
An attempt to make popular the division of the **right angle** into 100 so called new grades has failed.

1.**Q 2: The irrational number π**
Give the value of π **approximately**[$\overset{\mathbf{G}}{=}$ `angenähert`] as a decimal number.

(Solution:)

$$\boxed{\pi = 3.1415926\ldots} \tag{1}$$

REM: π is **irrational**, i.e. its decimal number never becomes periodic.
The word 'irrational' was coined because in former times people believed that such numbers cannot be understood rationally.

1.**Q 3: Radian measure**

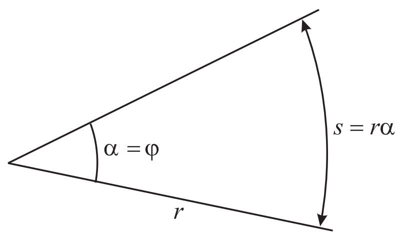


Fig1.3. **1**: s = arc's length, α = angle in radians, φ = angle in degrees

1.3. **a**) Give the length s of the **arc**[$\stackrel{\mathbf{G}}{=}$ Bogen] with radius r and **centriangle**[$\stackrel{\mathbf{G}}{=}$ Zentriwinkel] φ.

(Solution:)

$$\boxed{s = \underbrace{\frac{\pi}{180^\circ}\,\varphi}_{\alpha}\, r = r\alpha} \quad \text{s = length of arc, } \varphi \text{ in degrees} \tag{1}$$

$$\boxed{s = r\alpha} \quad \text{s = length of arc, } \alpha \text{ in radians} \tag{2}$$

REM 1: s is **proportional** to (also called **linear in**) both φ and r.

1.3. **b**) Say in words, what is the **radian measure**[$\stackrel{\mathbf{G}}{=}$ Bogenmaß] for angles.

(Solution:)

The division of the right angle into 90 degrees (90°) seemed unnatural in mathematics. Therefore the length of arc divided by r (or alternatively: the length of arc of the unit circle) was used to measure angles. We call it the radian measure for angles.

The important formula (1) thus gets simplified and takes the form (2).

1.3. **c**) Give in degrees φ and in **radians** α: right angle, full angle, half of the full angle.

(Solution:)

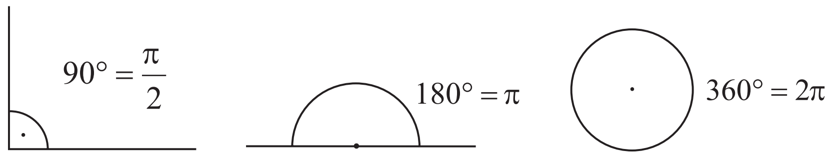


Fig1.3. **2**: Some important angles in degrees and in radians

REM 2: rad is an **abbreviation**[$\stackrel{\mathbf{G}}{=}$ `Abkürzung`] for radian.

REM 3: rad is simply 1 and it could be omitted, as was done in all cases where π is involved. When an angle is given as a decimal number, it is usual to give the unit radian to make clear that the number is an angle in radians, and not, e.g. the number of cows on a meadow.

1.3. **d**) For a general angle give the correspondence between its measure in radians (α) and in degrees (φ).

(Solution:)

$$\boxed{\alpha = \frac{\pi}{180°}\varphi} \quad \varphi \text{ in degrees, } \alpha \text{ in radians} \tag{3}$$

REM 1:
To devise (3): α and φ must be proportional. Test (3) for $\varphi = 180°$.

REM 2:

$$\boxed{1 \text{ rad} = 1 \text{ radian} = 1 = \frac{180°}{\pi} \approx 57.3°} \tag{4}$$

REM 3: As lengths can be expressed in different units, e.g. $l = 5$ cm $= 0.05$ m, angles can be expressed in different units. The radian unit ist 57.3 times larger than the degree unit. That the degree unit (1°) is written with a superscripted circle is an irrelevant typographical detail.

REM 4: (3) is consistent with $\alpha = \varphi$, as can be seen from (4).

REM 5: Corresponding to (3) we have

$$\boxed{L = \frac{1\text{m}}{100 \text{ cm}} l} \quad l \text{ in cm, } L \text{ in m} \tag{5}$$

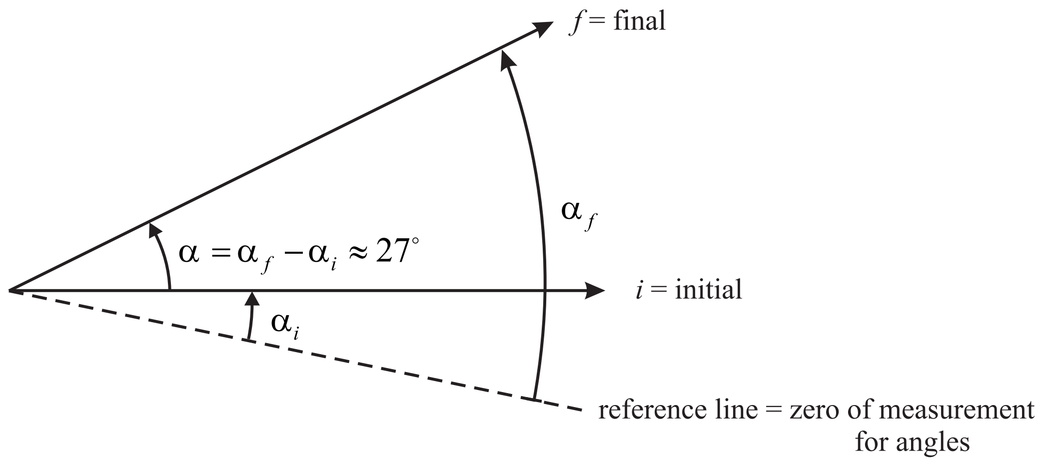
consistent with $L = l$

1.**Q 4: Mathematically positive sense of rotation**

1.4. **a**) What is a **mathematically positive** rotation?

(Solution:)

Rotation **counter-clockwise**[$\stackrel{G}{=}$ `gegen den Uhrzeiger`] is called mathematically positive.



`Fig`1.4. **1**: A rotation α from an initial direction (i) to a final direction (f) is called positive if it is counter-clockwise. An arbitrary direction is given by angles (e.g. α_f, α_i) which are viewed as rotations from a fixed reference line.

1.4. **b**) Why the word *mathematically*?

(Solution:)

In every day life one would rather be inclined to choose the clockwise rotation. For historical reasons, mathematicians have chosen the *counter*-clockwise direction as positive. Very often the adjective 'mathematically' is omitted.

1.4. **c**)☻☻ What are the restrictions (or assumptions) for that definition?

(Solution:)

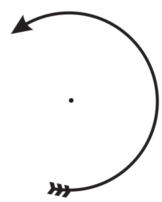


Fig1.4. **2**: A positive rotation seems to be negative when viewed from the opposite side of the paper.

In the above figure you see a positive rotation. Regard the sheet from the opposite side and you will observe the rotation is negative.

An ideal geometrical plane (approximated by a decent sheet of paper) is completely **smooth** [$\overset{\mathbf{G}}{=}$ `glatt`] (**homogeneous**) and has no **inherent** [$\overset{\mathbf{G}}{=}$ `innewohnend`] orientation. By writing down the above figure, we have **promoted** [$\overset{\mathbf{G}}{=}$ `befördert`] the plane to an **oriented plane** (also called: a plane with an orientation). You must look unto it from the correct side, so that its orientation is positive.

A real plane (e.g. a sheet of paper) has two sides: it makes a difference if you drop a **blob** [$\overset{\mathbf{G}}{=}$ `Tropfen`] of ink on the one side or on the other side. That is not the case for a mathematical plane, when you denote a point P on it. By sitting always on one side of the plane and observing it from there and writing only unto that side, we give an orientation to that plane.

Thus, 'mathematically positive' is meaningless for an arbitrary plane **immersed** [$\overset{\mathbf{G}}{=}$ `eingebettet`] in 3-dimensional space.

1.4. **d**) What is the **sign** [$\overset{\mathbf{G}}{=}$ `Vorzeichen`] of an angle?

(Solution:)

An angle as the space between two (equivalent) straight lines has no orientation or sign (even if the plane itself has an orientation).

However, there are two *conventions* for giving signs to the angles. Convention I is explained in fig. 3, convention II in fig. 4.

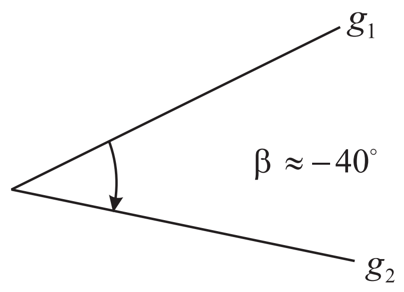


Fig1.4. **3**: In sign-convention I (limited to a plane situation and when viewing at the plane always from the same side, thus rarely used in physics) an angle with an **arc-arrow**[$\stackrel{\mathbf{G}}{=}$ `Winkelböglein`] in the clock-wise direction is counted negative.

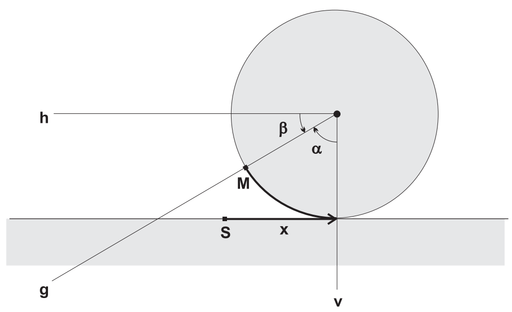


Fig1.4. **4**: In sign-convention II (standard in physics) one draws a **typical situation**, as in this figure for a wheel rolling on a horizontal plane.
In the *typical* situation all variables (α, β, x), by definition, are positive.
The arc-arrows are **optional**[$\stackrel{\mathbf{G}}{=}$ `freiwillig`] and do not have any significance.
The wheel has a mark M, permanently burnt in, defining a rotating straight line g. α and β are the angles of g relative to the vertical and horizontal directions, respectively.
S is the starting point, when we had S = M.

REM: When it is not possible to draw a typical situation where all variables (for angles or distances) are positive, e.g. γ is negative, then write $-\gamma$ into the angle, or introduce the **auxiliary variable**[$\stackrel{\mathbf{G}}{=}$ `Hilfvariable`] $\gamma' = -\gamma$ and write γ' into the angle.

1.4. **e**) The wheel of fig. 4 is rolling on a plane without **slipping**[$\stackrel{\mathbf{G}}{=}$ `rutschen`]. Therefore the **bold**[$\stackrel{\mathbf{G}}{=}$ `fett`] lines are equal.

Calculate:
i) the relation between x and α,
ii) the relation between the angles α and β,
iii) measure the value of α and β (including signs) for the case of fig. 4,
iv) calculate x (including sign) if the wheel has radius $R = 1.5$m.

(Solution:)

i)

$$x = R\alpha \tag{1}$$

since the bold lines are equal.
ii)

$$\alpha + \beta = \frac{\pi}{2} \tag{2}$$

as can be seen from the figure. Note that α, β, π, x and R are positive, since fig. 4 is a *typical* situation, where all variables are positive.
iii) $\alpha \approx 60°, \beta \approx 30°$.
iv)

$$x = R\alpha = 1.5\text{m} \cdot 60° = 1.5\text{m}\,\frac{60°}{180°}\pi = 1.57\text{m} \tag{3}$$

1.4. **f)**

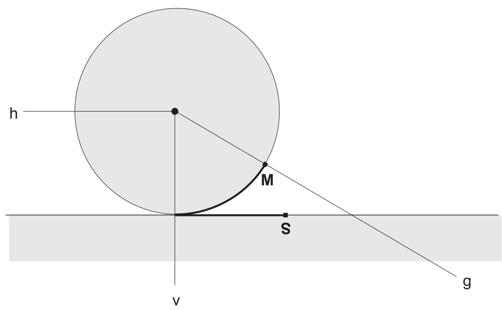


Fig1.4. **5**: **Particular** [$\stackrel{G}{=}$ `spezielle`] situation of the **typical** situation of fig. 4: The wheel has rolled by the same distance, but to the left of the starting point S = M. In this particular situation, α, β and x are negative, as can most easily be seen by inserting the analogous arc-arrows, which are opposite to those of fig. 4.

Fig. 5 is a *particular* situation of the same wheel as in the *typical* situation of fig.4. Answer the same questions as in the previous exercise e).

HINTS: draw into fig. 5 the analogous arc-arrows as in fig. 4, i.e. where the arc-arrow for α for α goes from v to g.

(Solution:)

i) ii) The same as (1)(2), since relations derived *generally* in a *typical* situation remain unchanges and have not to be rederived.
iii) $\alpha \approx -60°, \beta \approx -30°$, because the arc-arrows, introduced into fig.5, are now opposite to the corresponding arc-arrows of the typical situation of fig.4.
iv) $x = -1.57$m.

REM: In convention II, fig. 4, (arc-)arrows are optional, and in fact superfluous for the definition of the signs of the variables, always positive in the typical situation. However, intuitively, arc-arrows are chosen from a *fixed* element (S, h, v) to a *variable* element (foot point, g and again g). The arc-arrows are useful, since the signs in a special situation can then easily be decided: the sign is negative if the arrow in the special situation is opposite to the corresponding arrow in the typical situation.

1.**Q 5: Cartesian coordinates and its quadrants**

Draw a **cartesian system of coordinates** and denote its quadrants.

(Solution:)

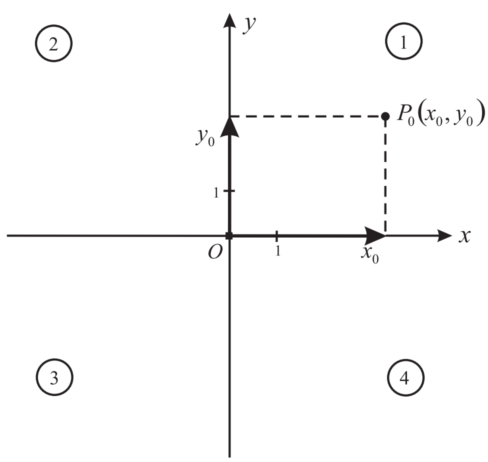


Fig1.5. **1**: An arbitrary point P_o on the plane is given by its Cartesian coordinates (x_o, y_o). The plane is divided into four quadrants, counted in the mathematically positive sense. When a point is given a name, e.g. P_o, its Cartesian coordinates are given as a 2-tuple, e.g. (x_o, y_o), written after its name.
Cartesian coordinates are defined only after a unit of length (e.g. cm or inches) are chosen or, alternatively, points with coordinates (0,1) and (1,0), both denoted by 1, are chosen.

REM : Cartesian coordinates are named after the French mathematician Descartes (lat: Cartesius).

1.**Q 6: Sine and cosine as projections**
Give the (geometrical) definition of sine (**sin**) and cosine (**cos**) as the **projection** p and side-projection s in a **right triangle**[$\stackrel{\mathbf{G}}{=}$ `rechtwinkliges Dreieck`].

(Solution:)

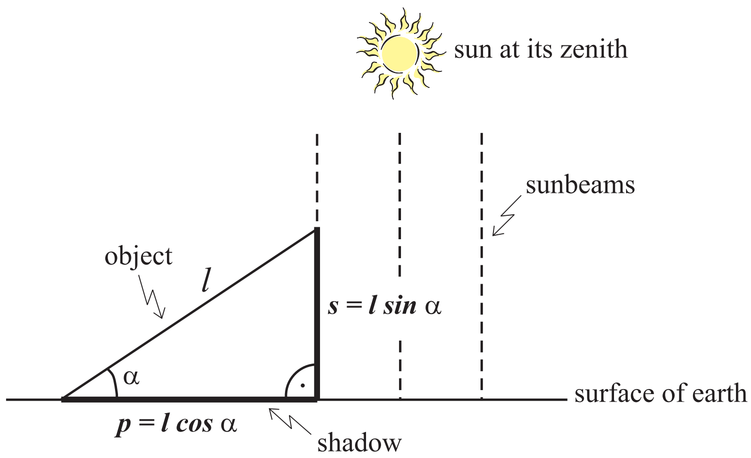


Fig1.6. **1**: The projection p of a line l is obtained by cos, the side projection s by sin. The sun is very far away, therefore its beams are nearly parallel.

$$\boxed{p = l\cos\alpha} \tag{1}$$

$$\boxed{s = l\sin\alpha} \tag{2}$$

MNEMONIC:

$$\boxed{\begin{array}{l}\textbf{Projection is cosine}\\ \textbf{Side-projection is sine}\end{array}} \tag{3}$$

CAUTION: one must use an *orthogonal* projection (also called a *normal* projection), i.e. a projection under a right angle.

REM: As seen in (1) and (2), it is usual to omit argument brackets, when the argument consists of a single symbol only, e.g. $\cos\alpha = \cos(\alpha)$.
In other words: functional binding, i.e. applying the function to its argument has

higher priority than multiplication. Thus we have the following interpretation when brackets are omitted:

$$\cos\alpha\beta = \cos(\alpha)\beta = \beta\cos(\alpha) = \beta\cos\alpha \tag{4}$$

1.**Q 7: Sine and cosine in a right triangle**
Give the geometrical definition of sin and cos in a right triangle using the **hypotenuse**, the **base** [$\stackrel{\mathbf{G}}{=}$ `An-Kathete`] and the **perpendicular** [$\stackrel{\mathbf{G}}{=}$ `Gegen-Kathete`]. (**perpendicular** [$\stackrel{\mathbf{G}}{=}$ `senkrecht`])

(Solution:)

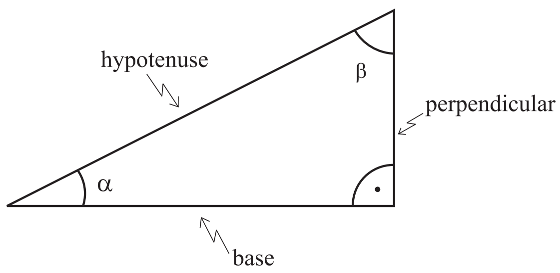


Fig1.7. **1**: Definition of the trigonometric functions in a right triangle

$$\boxed{\sin\alpha = \frac{\textbf{perpendicular}\,[\stackrel{\mathbf{G}}{=}\ \texttt{Gegen-Kathete}]}{\text{hypotenuse}}} \tag{1}$$

$$\boxed{\cos\alpha = \frac{\textbf{base}\,[\stackrel{\mathbf{G}}{=}\ \texttt{An-Kathete}]}{\text{hypotenuse}}} \tag{2}$$

Rem 1: Observe that according to the definition (1)(2) $\sin\alpha$ and $\cos\alpha$ are independent of the size of the triangle, but depend only on α.

Rem 2: In the English language there is no common word for base and perpendicular. Sometimes the word **leg** [$\stackrel{\mathbf{G}}{=}$ `Schenkel`] or **side** are used. However, these terms are also used in case of an **equilateral** [$\stackrel{\mathbf{G}}{=}$ `gleichschenklig`] triangle.

Rem 3:'Hypothenuse' is unique. 'Perpendicular' and 'base' are relative to the chosen angle (α). Taking the other angle ($\beta = \pi/2 - \alpha$), perpendicular and base get interchanged.

1.Q 8: Graph, zeroes, domain, range, period

1.8. **a)** Draw the **graph** of the **function** $y = \sin x$.

(Solution:)

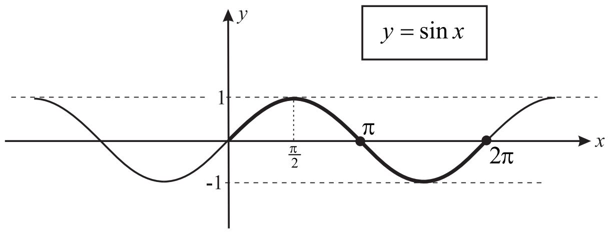


Fig1.8. **1**: Graph of $y = \sin x$

REM 1: When an angle is interpreted as a rotation, negative angles and angles greater than 2π are meaningful. However, purely geometric angles have to been taken **modulo** 2π. Then only the **bold**[$\stackrel{\mathbf{G}}{=}$ `fett`] part of the sin curve in fig. 1 is meaningful.

REM 2: The symbol y is used in 3 different meanings:

- as a name for an axis of the coordinate system (the **ordinate**[$\stackrel{\mathbf{G}}{=}$ `Ordinate`])
- as the **dependent** variable; x is the **independent** variable
- as the **value**[$\stackrel{\mathbf{G}}{=}$ `Wert`] of the function (e.g of the function sin) for a special value of the argument x

1.8. **b)** Give its **zeroes**[$\stackrel{\mathbf{G}}{=}$ `Nullstellen`].

(Solution:)

$$\boxed{\sin x = 0 \quad \Rightarrow \quad x = n\pi, \;\; n \in \mathbb{Z}} \tag{1}$$

REM: $\mathbb{Z}$ denotes the **set**[$\stackrel{\mathbf{G}}{=}$ `Menge`] of **integers**[$\stackrel{\mathbf{G}}{=}$ `ganze Zahlen`].

$$\mathbb{Z} = \{\cdots - 2, -1, 0, 1, 2, \ldots\} \tag{2}$$

1.8. **c**) What is the **domain**[$\overset{\mathbf{G}}{=}$ `Definitionsbereich`] $\mathcal{D}$ of that function?

(Solution:)

$$\mathcal{D} = (-\infty, \infty) \tag{3}$$

REM: An **interval** is denoted by the pair of its end-point. If an end-point does *not* belong to the interval (**open interval**) **round brackets** ( ) are used. When an end-point does belong to the interval (**closed interval**) then **square brackets** [ ] are used.

1.8. **d**) What is its **range**[$\overset{\mathbf{G}}{=}$ `Wertebereich`]?

(Solution:)

$\sin \mathcal{D} = [-1, 1]$ (end-points of interval inclusive)

1.8. **e**) Is it a unique function?

(Solution:)

yes, the function is unique.

REM: In mathematical language a function is always unique. In physics the word function is also used to denote **multiple valued functions**. E.g. $\sqrt{4} = \pm 2$, i.e $\sqrt{x}$ is a **double valued function**.

1.8. **f**) What is its (primitive) period T?

(Solution:)

$$T = 2\pi \tag{4}$$

REM 1: When for a function $y = f(t)$, it holds

$$f(t+T) = f(t) \quad \text{for all} \quad t \in \mathcal{D} \tag{5}$$

with $T \neq 0$ T is called a **period** of that function.

(We must exclude $T = 0$ because otherwise (5) is always valid, and every function is periodic.)

When T is a period, then nT ($n \in \mathbb{Z}$, when we include $T = 0$ as a the trivial period) is also a period.

There is the following theorem for periodic functions: There exists a so called **primitive** period T ($T > 0$) so that every period is a multiple of T.

REM 2: In Rem 1 we have used t (instead of x) for the independent variable, since the every-day meaning of the word 'period' refers to time t.

REM 3: In (5), as is usual, a general, i.e. unspecified, function is denoted by f. In our case we have $f = \sin$.

1.Q 9: Inverse function

What is the **inverse function**[$\stackrel{\mathbf{G}}{=}$ `Umkehrfunktion`] of $y = \sin x$? Draw its graph and give its name.

(Solution:)

$$y = \sin x \quad \Rightarrow \quad x = \arcsin y \tag{1}$$

REM 1: ‘arcsin x’ means ‘arcus (= angle) whose sin is x.

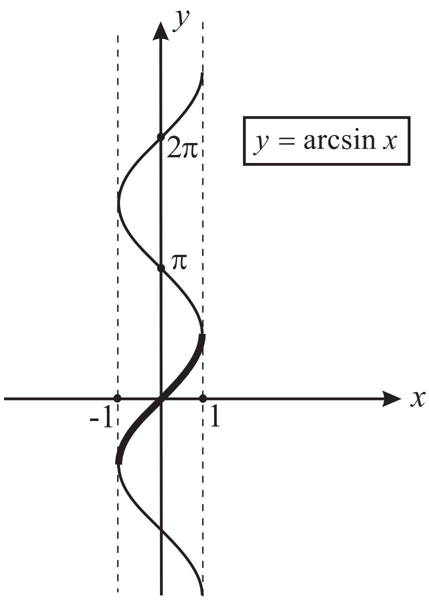


Fig1.9. **1**: Graph of the arc sin function, which is the inverse function to the sine function. Restriction to the fat branch of the graph makes the arcsin function a unique function.

REM 2: $y = \arcsin x$ is not a unique function, but it is **multivalued**[$\stackrel{\mathbf{G}}{=}$ `vieldeutig`], in fact **infinitely multivalued**[$\stackrel{\mathbf{G}}{=}$ `unendlich vieldeutig`]. It can be made a unique function by restricting the graph to one **branch**[$\stackrel{\mathbf{G}}{=}$ `Ast`], e.g. by requiring for the domain $\mathcal{D} = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

REM 3: In mathematical terminology a function is by definition a unique function. So arcsin without a restriction to a branch is not a function in the mathematical sense of the word. In physical terminology a function may also be multivalued.

REM 4: The inverse function is **in essence**[$\stackrel{\mathbf{G}}{=}$ `im Wesentlichen`] the same function as the original one, except that the role of independent (x) and dependent (y)

variables are interchanged, since the pairwise **allocation**[$\stackrel{\mathbf{G}}{=}$ Zuordnung] of an x to an y is the same for the function and the inverse function. Only what is considered to be given *at first* (i.e. **arbitrarily**[$\stackrel{\mathbf{G}}{=}$ willkürlich]) (= independent variable) and what *then* is fixed (possibly multivalued) by the function is different in the case of the original and the inverse function.
Corresponding to these new roles of x and y, the names are interchanged ($x \leftrightarrow y$) so x and y have again their usual roles: x= independent vaible, y = dependent variable.

REM 5: The graph of the inverse function is obtained from the graph of the original function by a **mirror symmetry**[$\stackrel{\mathbf{G}}{=}$ Spiegelsymmetrie] at the **bisection of angles**[$\stackrel{\mathbf{G}}{=}$ Winkelhalbierende] of the $x-$ and $y-$axes.

REM 6: The inverse function of the inverse function is the original function.

1.**Q 10: Cosine**
Draw the graph of $y = \cos x$.

(Solution:)

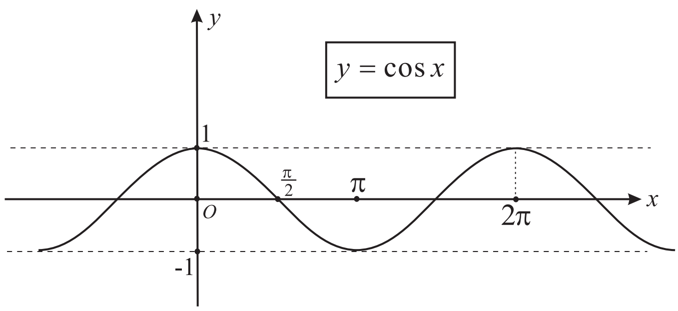


Fig1.10. **1**: Graph of $y = \cos x$. It is identical to the graph of the sine function, but shifted by $\frac{\pi}{2}$.

REM: $\sin x$ and $\cos x$ are identical, but only shifted along the x-axis.

1.**Ex 11: Rope around the earth**
A **rope**[$\stackrel{\mathbf{G}}{=}$ Seil] is laid around the **equator**[$\stackrel{\mathbf{G}}{=}$ Äquator] of the earth. Now, the rope is extended by $l = 1$m and streched again to a circle (**dotted**[$\stackrel{\mathbf{G}}{=}$ punktiert] circle in figure).

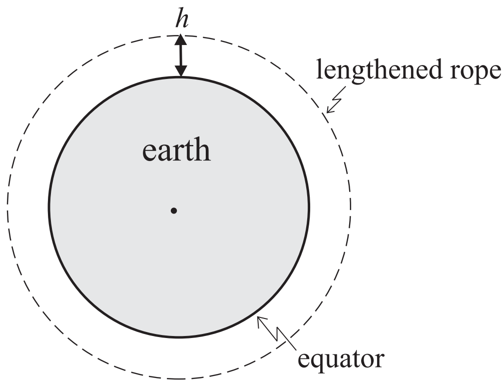


Fig1.11. **1**: A rope around the earth is l =1 m too long, so it will have a heigth h above the earth.

What is the heigth h of the rope above the earth?
HINT: Let R be the radius of the earth. Calculate the length of the equator and then the length of the dotted circle.
RESULT: $h = 15.9$ cm

(Solution:)

Original length of the rope $L = 2\pi R$, $R =$ radius of the earth. Length of extended rope is:

$$L + l = 2\pi(R + h) \qquad (\text{ = length of rope}) \tag{1}$$

$$l = 2\pi h, \qquad h = \frac{l}{2\pi} = 15.9 \text{ cm} \tag{2}$$

(Astonishingly, the radius of the earth **cancels**[$\stackrel{\mathbf{G}}{=}$ `herausfallen`] itself out.)

1.Ex 12: Transforming radians into degrees

1.12. **a**) Give the following angles in radians:

$$\alpha_1 = 13^\circ, \qquad \alpha_2 = 12', \qquad \alpha_3 = 1'' \tag{1}$$

HINT: One degree (1°) is divided into $60'$, $1'$ is divided into $60''$
RESULTS:

$$\alpha_1 = 0.2269, \qquad \alpha_2 = 0.0035, \qquad \alpha_3 = 4.85 \cdot 10^{-6} \tag{2}$$

(Solution:)

$$\alpha_1 = 13^\circ = 13^\circ \frac{\pi}{180^\circ} = \frac{13\pi}{180} = 0.2269 \tag{3}$$

$$\alpha_2 = 12' = \frac{12^\circ}{60} = \frac{12^\circ}{60}\frac{\pi}{180^\circ} = \frac{12\pi}{60 \cdot 180} = 0.0035 \tag{4}$$

$$\alpha_3 = 1'' = \frac{1^\circ}{3600} = \frac{\pi}{3600 \cdot 180} = 4.85 \cdot 10^{-6} \tag{5}$$

1.12. **b**) Give the following angles in degrees:

$$\alpha_4 = \frac{\pi}{4}, \qquad \alpha_5 = 3 \tag{6}$$

RESULTS:

$$\alpha_4 = 45^\circ, \qquad \alpha_5 = 171.89^\circ \tag{7}$$

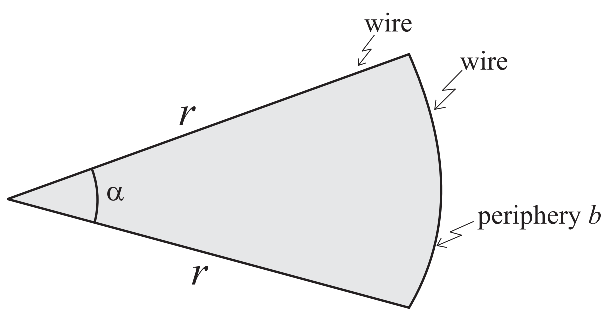
(Solution:)

$$\alpha_4 = \frac{\pi}{4} = 45^\circ \tag{8}$$

$$\alpha_5 = 3 = 3 \cdot \frac{180^\circ}{\pi} = 171.89^\circ \tag{9}$$

1.**Ex 13: Folding wire into a sector**

A child has a piece of **wire**[$\overset{\mathbf{G}}{=}$ `Draht`] of length l and folds it into a sector (shaded area of the figure) after selecting the length b for the periphery in the middle of the wire.



`Fig`1.13. **1**: A fixed length is folded into a sector

Calculate α. In particular calculate α in degrees for $l = 3b, b = 10$ cm.
RESULTS:

$$\alpha = \frac{2b}{l-b} = 57.30^\circ \tag{1}$$

(Solution:)

$$b = \alpha r \tag{2}$$

$$b + 2r = l \qquad \Rightarrow \qquad r = \frac{l-b}{2} \tag{3}$$

$$\alpha = \frac{b}{r} = \frac{2b}{l-b} = \frac{20}{20} = 1 = \frac{180^\circ}{\pi} = 57.30^\circ \tag{4}$$

1.**Ex 14: Folding to a cylinder**

1.14. **a**) A sheet of paper (l = length, b = breadth) is folded into a cylinder.

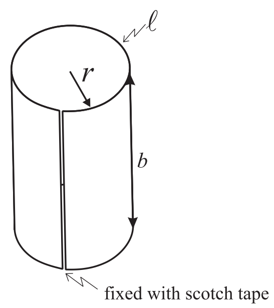


Fig1.14. **1**: Length l is folded to a circle of radius r

What is the radius r of the resulting cylinder?
RESULT:

$$r = \frac{l}{2\pi} \tag{1}$$

(Solution:)

$$l = 2\pi r \tag{2}$$

$$r = \frac{l}{2\pi} \tag{3}$$

1.14. **b**) Look up the formula for the volume of a cylinder and calculate the volume if a DIN A4 sheet is used.
RESULT:

$$V = \frac{bl^2}{4\pi} = 1474.08 \text{ cm}^3 \tag{4}$$

(Solution:)

$$V = \pi r^2 b = \pi b \frac{l^2}{4\pi^2} = \frac{bl^2}{4\pi} \tag{5}$$

For DIN A4:

$$l = 29.7 \text{ cm}, \qquad b = 21 \text{ cm} \tag{6}$$

$$V = 1474.08 \text{ cm}^3 \tag{7}$$

1.Ex 15: Application of trigonometric functions in a triangle

1.15. **a**)

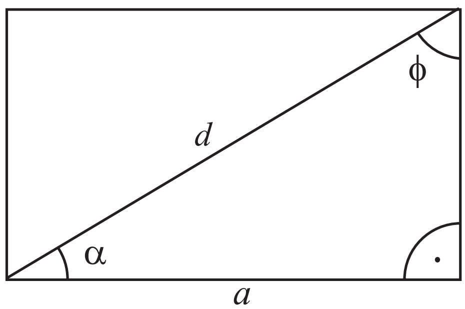


Fig1.15. **1**: d is the diagonal in a rectangle with side length a.

The diagonal in the above rectangle is $d = 13$ cm and $\phi = 72°$. Calculate a.
RESULT:

$$a = 12.36 \text{ cm} \tag{1}$$

(Solution:)

$$a = d \sin\phi = 13 \text{ cm} \cdot \sin 72^\circ = 12.36 \text{ cm} \tag{2}$$

1.15. **b**) The same situation as above, except $d = 14$ cm and $a = 12$ cm. Calcualte α in degrees.
RESULT:

$$\alpha = 31^\circ \tag{3}$$

(Solution:)

$$a = d \cos\alpha \tag{4}$$

$$\cos\alpha = \frac{a}{d} = \frac{12}{14} \tag{5}$$

$$\alpha = \arccos\frac{a}{d} = \arccos\frac{12}{14} = 31^\circ \tag{6}$$

1.**Ex 16: Addition of rotations**
The time is 1:07. However, a clock which is slightly too fast shows that it is 1:09. By what angle φ does the clock's **big hand**[$\stackrel{\mathbf{G}}{=}$ `großer Zeiger`] have to rotate to set the clock to the correct time? Give your answer in radians and pay attention to the sign of φ.
RESULT:

$$\varphi = \frac{\pi}{15} \tag{1}$$

(Solution:)
One minute corresponds to the angle $\frac{2\pi}{60}$; since we have to correct 2 minutes we have to rotate by the angle

$$\varphi = \frac{2 \cdot 2\pi}{60} = \frac{\pi}{15} \tag{2}$$

Since our clock is too fast we have to rotate its big hand counterclockwise. Thus, φ is positive.

1.**Ex 17: Sign of rotations. Angles greater than 2π**

Since Monday 1:00 the big hand of an (exact) clock has rotated by the angle

$$\varphi = -170.1696 \qquad (1)$$

What time is it now and what day of the week is it?
HINT: Since the hands of a clock rotate clockwise, φ is negative. $\varphi = -2\pi$ would mean that one hour had passed. First calculate the number of complete hours which have passed.
RESULT: Tuesday 4:05

(Solution:)

$$\frac{-\varphi}{2\pi} = 27.0833 \qquad (2)$$

i.e. 27 complete hours (i.e. one day and three hours) have passed, so it is shortly after 1+3 = 4 o'clock. We are left with the clockwise angle

$$-\varphi - 27 \cdot 2\pi = 0.5236 \qquad (3)$$

Clockwise, one minute corresponds to the angle $\frac{2\pi}{60}$. Thus we have an additional

$$0.5236 \cdot \frac{60}{2\pi} = 5 \text{ minutes} \qquad (4)$$

1.**Ex 18: Graphical construction of trigonometric functions**

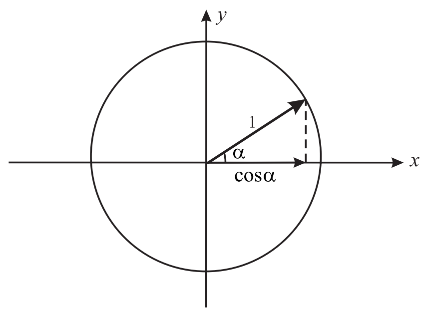
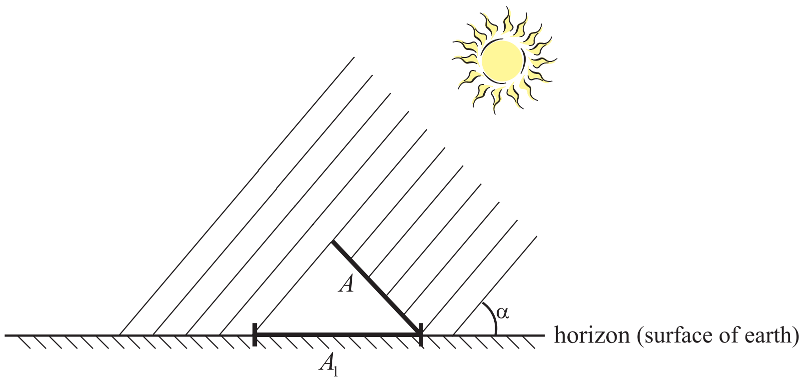


Fig1.18. **1**: Projecting the unit radius onto the x-axis gives $\cos\alpha$

Draw a circle with radius $r = 1$ (e.g. $r = 10$ cm, i.e. unity = 10 cm) and insert a radius (**bold**[$\stackrel{G}{=}$ `fett`] line in the above figure) for $\alpha = 0°, 10°, 20°, \ldots 360°$.

Measure the corresponding values for $\sin\alpha$ and sketch the graph of $y = \sin\alpha$. Check some values (e.g. for $\alpha = 0°, \alpha = 50°, \alpha = 120°$) with your **calculator**[$\stackrel{G}{=}$ `Taschenrechner`].

1.Ex 19: ☻ Photon flux density on the earth



`Fig`1.19. **1**: Photons from the sun hit the earth at angle α

Let the sun have an angular height α above the horizon. 10^{20} photons (= energy quants) from the sun hit the area $A = 1$ m^2 per second.

1.19. **a**) Calculate the area A_1 onto which the same photons fall if A were removed. Take the special value $\alpha = 30°$.
RESULT:

$$A_1 = \frac{A}{\sin\alpha} = 2 \text{ m}^2 \tag{1}$$

(Solution:)

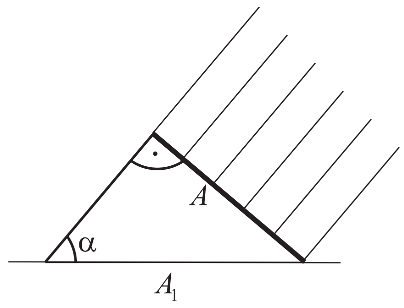


Fig1.19. **2**: Angle α can also be found inside the triangle

$$A = A_1 \sin\alpha, \qquad \sin 30^\circ = \frac{1}{2}, \qquad A_1 = 2A = 2\ \mathrm{m}^2 \tag{2}$$

1.19. **b**) What is the **photon number flux density** [$\stackrel{\mathbf{G}}{=}$ `Photonenzahlflussdichte`] n (= number of photons per square meter and per second) hitting the (horizontal) surface of the earth? Compare it with the original flux density $n_\perp = 10^{20}\ \mathrm{m}^{-2}\mathrm{s}^{-1}$ for perpendicular incidence.

RESULT:

$$n = 0.5 \cdot 10^{20} \text{ photons per square meter and per second, } n = 0.5 n_\perp \tag{3}$$

(Solution:)

10^{20} photons fall at $A_1 = 2\ \mathrm{m}^2$ per second, i.e

$$n = 0.5 \cdot 10^{20}\ \mathrm{m}^{-2}\mathrm{s}^{-1} \tag{4}$$

2 Harmonic oscillator, tangent, Pythagoras

2.**Q 1: Harmonic oscillator**
Consider the following function

$$y = y_0 \sin(\omega t + \alpha_0) \tag{1}$$

which is a generalization of

$$y = \sin x \tag{2}$$

with **constants** y_0, ω, α_0.

Rem 1: An index 0 is often used to **qualify**[$\stackrel{\mathbf{G}}{=}$ `näher bestimmen`] a symbol as a constant.

Rem 2: Physically (1) gives the motion of a so called **harmonic oscillator**[$\stackrel{\mathbf{G}}{=}$ `Schwinger`], e.g. a mass-point with a **spring**[$\stackrel{\mathbf{G}}{=}$ `Feder`] **attached**[$\stackrel{\mathbf{G}}{=}$ `befestigt`] to the earth.

Rem 3: 'harmonic' means sine or cosine with a fixed frequency ω. In acoustics tones with ω's being simple multiples of a ground tone give the impression of a harmonic sound.

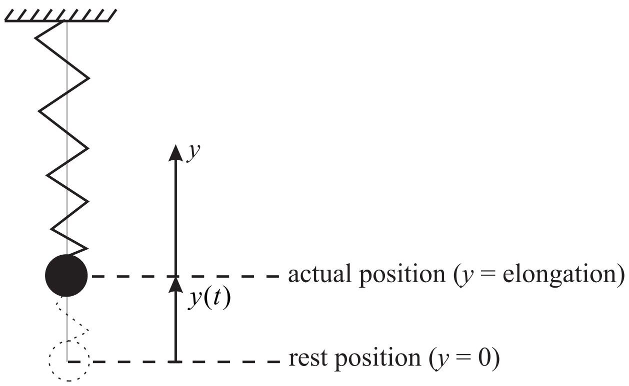


Fig2.1. **1**: Simple physical realization of a harmonic oscillator with a spring. (The elongation y of the harmonic oscillator is counted from its rest-position.)

For the function (1) answer the following questions:

2.1. **a**) What symbol represents the **value of the function**[$\stackrel{\mathbf{G}}{=}$ `Funktionswert`]? (What is its physical significance ?)

(Solution:)

y **elongation**[$\stackrel{\mathbf{G}}{=}$ `Auslenkung`] from the **rest-position**[$\stackrel{\mathbf{G}}{=}$ `Ruhelage`] of the harmonic oscillator

---

2.1. **b**) What is the **amplitude** (significance)?

(Solution:)

y_0 (maximum value of y, maximum elongation)

2.1. **c**) What is the **independent variable**[$\stackrel{\mathbf{G}}{=}$ `unabhängige Variable`] (physical significance)?

(Solution:)

t (= time)

2.1. **d**) What is the **phase** (significance)?

(Solution:)

$\alpha = \omega t + \alpha_0$

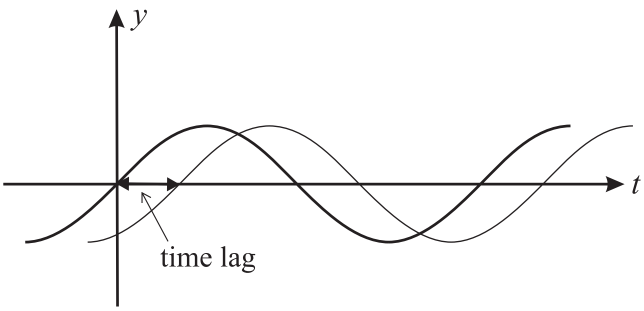
Rem: The word 'phase' is used in many significances in mathematics and physics. Here, 'phase' means argument of a sine (or of a cosine).
(Significance: The phase gives the best information about the momentaneous situation of the oscillator: E.g. when the phase is a multiple of π the oscillator crosses its rest position line (**zero-passage**[$\stackrel{\mathbf{G}}{=}$ `Nulldurchgang`]).

2.1. **e**) **phase-shift**[$\stackrel{\mathbf{G}}{=}$ `Phasenverschiebung`]

(Solution:)

α_0

Rem: Sometimes $-\alpha_0$ is called the phase-shift.



`Fig`2.1. **2**: These two harmonic oscillators are identical (same ω, and y_0) but they have a relative phase-shift (**time-lag**[$\stackrel{\mathbf{G}}{=}$ `Zeitverschiebung`]).

(Two oscillators with different α_0 move identically, but they have a relative time-shift.)

2.1. **f) angular-frequency**[$\stackrel{\mathbf{G}}{=}$ `Kreisfrequenz`]

(Solution:)

ω

Rem: ω is called the 'angular frequency' because it says how often the phase increments by a full angle (2π) (or: how often the oscillator performs a full period, i.e. a full cycle) per unit time.

In every-day language **frequency**[$\stackrel{\mathbf{G}}{=}$ `Häufigkeit`] is the number of **events**[$\stackrel{\mathbf{G}}{=}$ `Ereignisse`] per second.

2.1. **g)** (primitive) period

(Solution:)

$$\boxed{T = \frac{2\pi}{\omega}} \qquad \textbf{(period of motion with angular frequency } \omega\textbf{)} \tag{3}$$

2.1. **h) dependent-variable**[$\stackrel{\mathbf{G}}{=}$ `abhängige Variable`]

(Solution:)

y (the same as a))

2.Q 2: Tangent

2.2. **a)** Give the definition of tan analytically (i.e. with the help of other functions) and geometrically i.e. in a right triangle.).

(Solution:)

$$\boxed{y = \tan x = \frac{\sin x}{\cos x} = \frac{\textbf{perpendicular}[\stackrel{\mathbf{G}}{=} \texttt{Gegen-Kathete}]}{\textbf{base}[\stackrel{\mathbf{G}}{=} \texttt{An-Kathete}]}} \tag{1}$$

Rem 1: In mathematical terminology for $\cos x = 0$ (i.e. for x $= \pi/2 + n\pi, n = \ldots - 2, -1, 0, 1, 2, \ldots$) $\tan x$ is undefined. In physical terminology one says that $\tan x$ is there double-valued having two improper values $\pm\infty$.

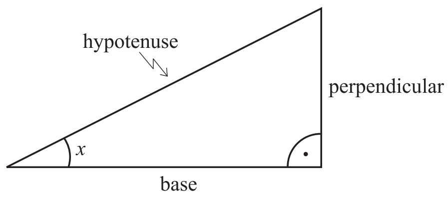


Fig2.2. **1**: The tangent can be defined in a right triangle as the quotient of the perpendicular to the base

REM 2: Instead of tan the older notation **tg** is also used.

2.2. **b**) Draw its graph.

(Solution:)

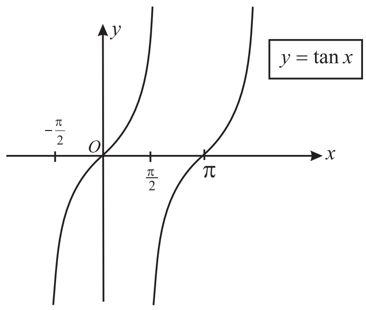


Fig2.2. **2**: Graph of $y = \tan x$

2.2. **c**) Period

(Solution:)

$\boxed{T = \pi}$

2.**Q 3: Cotangent**

The same for $y = \cot x$

(Solution:)

$$\boxed{y = \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x} = \frac{\textbf{base}[\overset{\mathbf{G}}{=} \texttt{An-Kathete}]}{\textbf{perpendicular}[\overset{\mathbf{G}}{=} \texttt{Gegen-Kathete}]}} \qquad (1)$$

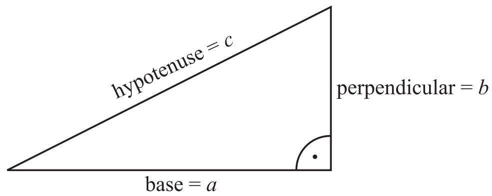
REM 1: For $\sin x = 0$, see Rem 1 for tan.

REM 2: Instead of cot the older notation **ctg** is also used.

2.**Q 4: Pythagoras**

Formulate the **Pythagorean theorem**[$\stackrel{\mathbf{G}}{=}$ `Satz des Pythagoras`].

(Solution:)



`Fig`2.4. **1**: Pythagoras: In a right triangle, the square of the hypotenuse (c) is the sum of the squares of the adjacent legs a and b.

$$\boxed{a^2 + b^2 = c^2} \qquad \textbf{(Pythagoras)} \tag{1}$$

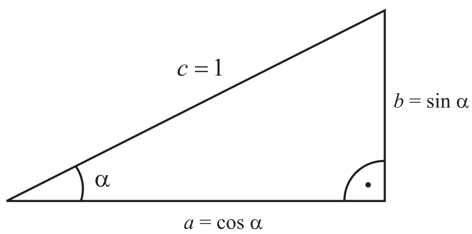
2.**Qx 5: Special values of sine and cosine**

Calculate with the help of a right triangle (i.e. without a calculator)

HINT: Choose a hypotenuse of length 1.

2.5. **a**) $\sin 0$

(Solution:)



`Fig`2.5. **1**: In a right triangle with hypotenuse $c = 1$ the projection (a) is cosine and the side-projection (b) is sine.

$\alpha = 0 \quad \Rightarrow \quad b = \sin 0 = 0$

2.5. **b)** $\sin\frac{\pi}{2}$ $\quad (\frac{\pi}{2} = 90°)$

(Solution:)

$\alpha \to \frac{\pi}{2} \quad \Rightarrow \quad b \to c = 1 = \sin\frac{\pi}{2} \qquad (a \to 0)$

2.5. **c)** $\sin\frac{\pi}{4}$ $(\frac{\pi}{4} = 45°)$ (HINT: Use Pythagoras. We have a **unilateral**[$\stackrel{\mathbf{G}}{=}$ `gleichschenklig`] right triangle.)

(Solution:)

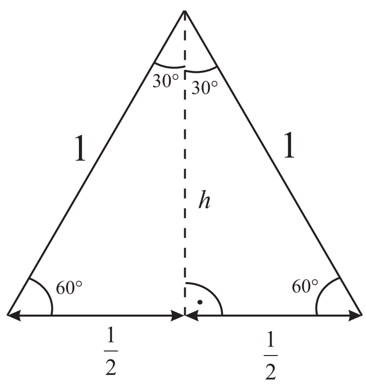
$a = b \Rightarrow a^2 + b^2 = 2b^2 = c^2 = 1 \Rightarrow$

$b^2 = \frac{1}{2} \qquad \Rightarrow \qquad b = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \sin(\frac{\pi}{4})$

2.5. **d)** $\sin\frac{\pi}{6}$ $\quad (\frac{\pi}{6} = \frac{180°}{6} = 30°)$

HINT: Draw a unilateral triangle with side length unity, and a **height**[$\stackrel{\mathbf{G}}{=}$ `Höhe`] to obtain 30°. Use symmetries and Pythagoras.

(Solution:)



`Fig`2.5. **2**: Half of a unilateral triangle yields 30° to calculate $\sin 30°$.

$\sin 30° = \frac{1/2}{1} = \frac{1}{2}$

$$\boxed{\sin 30° = \frac{1}{2}} \tag{1}$$

2.**Qx 6: Parametric representation of a circle**

2.6. **a)** Give and derive the **parametric representation of a circle**[$\stackrel{\mathbf{G}}{=}$ `Parameterdarstellung eines Kreises`] of radius r.

(Solution:)

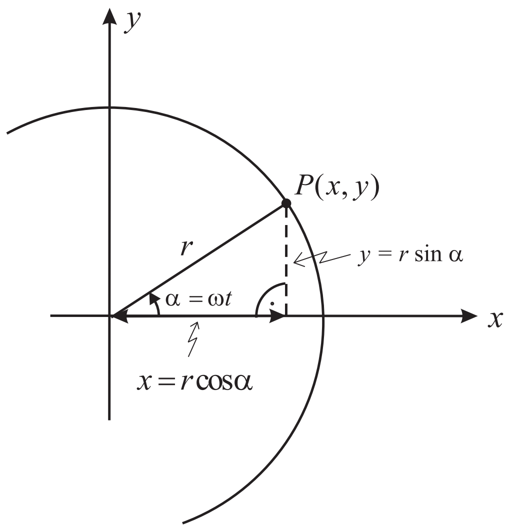


Fig2.6. **1**: Parametric representation of circle with radius r gives (x, y) in terms of the angle α which rotates with angular velocity ω. Either α or t can be called the parameter.

$$\boxed{\begin{aligned} x &= r\cos(\omega t) \\ y &= r\sin(\omega t) \end{aligned}} \qquad \textbf{(parametric representation of a circle)} \qquad (1)$$

$(\alpha = \omega t)$

$$\boxed{\begin{array}{c} \omega = \frac{2\pi}{T} = \text{angular frequency} = \\ \textbf{angular velocity}[\overset{\mathbf{G}}{=} \texttt{Winkelgeschwindigkeit}] \\ T = \text{ period} \end{array}} \qquad (2)$$

Rem 1: Though not a completely correct **notation**[$\overset{\mathbf{G}}{=}$ `Bezeichnungsweise`], in physics it is usual to omit the brackets around ωt in (1).

Rem 2: **Parameter** is just another word for variable. It is used when that variable is of less importance. Here (x, y) are the essential variables for the points of the circle. t or ω are **auxiliary**[$\overset{\mathbf{G}}{=}$ `Hilfs-`] variables not belonging to the circle **proper**[$\overset{\mathbf{G}}{=}$ `eigentlich`].

2.6. **b**) Give and derive the important formula by which $(\cos\alpha)^2$ and $(\sin\alpha)^2$ can be transformed into one another.

(Solution:)

Put $r = 1$ and use Pythagoras:

$$\boxed{(\sin\alpha)^2 + (\cos\alpha)^2 = 1} \qquad \textbf{(Basic trigonometric identity)} \tag{3}$$

mostly written as:

$$\boxed{\sin^2\alpha + \cos^2\alpha = 1} \qquad \textbf{(Basic trigonometric identity)} \tag{3'}$$

Rem 1: Strictly speaking (3′) is not a correct notation for (3), since (3′) means literally $\sin(\sin\alpha) + \cos(\cos\alpha) = 1$ which is wrong.

2.**Qx 7: Zeros of sine with phase shift**

Calculate the zeros of $y = \sin(x - \alpha_0)$

(Solution:)

$x - \alpha_0 = n\pi, \quad n \in \mathbb{Z}$
$x = x_n = \alpha_0 + n\pi$
(α_0 = phase shift)

Rem: Since our problem has several (infinite many) solutions, we have distinguished them by the index n.

2.**Ex 8: Slope of a street**

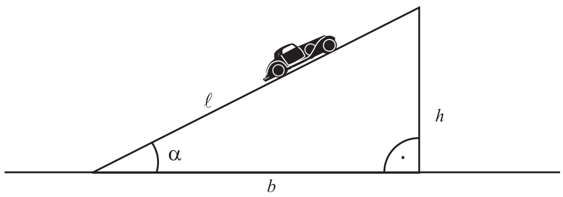


Fig2.8. **1**: slope of a street defined by **inclination**[$\overset{G}{=}$ `Neigung`] angle α or by the **ratio**[$\overset{G}{=}$ `Verhältnis`] height h divided by base length b

The **slope**[$\overset{G}{=}$ `Steigung`] s of a street is defined as the increase in height h divided by the base length b of the street, mostly given in percent:

$$s = \frac{h}{b} = \frac{h}{b}100\% \tag{1}$$

2.8. **a**) Calculate s for $\alpha = 22°$.

Result:

$$s = 40.4\% \tag{2}$$

(Solution:)

$$s = \tan\alpha = \tan 22^\circ = 40.4\% \tag{3}$$

2.8. **b**) Conversely, for $s = 10\%$, calculate α.
RESULT:

$$\alpha = 5.7^\circ \tag{4}$$

(Solution:)

$$\alpha = \arctan 0.1 = 5.7^\circ \tag{5}$$

2.8. **c**) For the length $l = 1$ km, $\alpha = 15^\circ$, calculate b, h.
RESULT:

$$b = 965.9 \text{ m}, \qquad h = 258.8 \text{ m} \tag{6}$$

(Solution:)

$$b = l\cos\alpha, \qquad h = l\sin\alpha \tag{7}$$

$$b = 965.9 \text{ m}, \qquad h = 258.8 \text{ m} \tag{8}$$

2.Ex 9: Trigonometric functions used for calculations of triangles

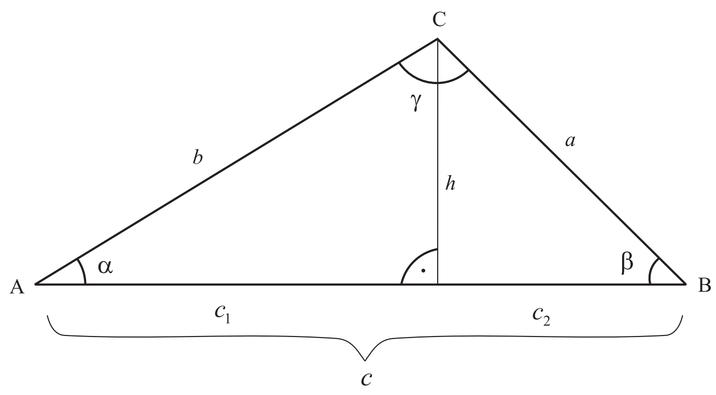


Fig2.9. **1**: Triangle with side lengths a, b, c and opposite angles α, β, γ at corners A, B, C. One height $h = h_c$ is also shown.

An arbitrary triangle has angle (α), the opposite side (a) and a neighboring side (b). Calculate the remaining pieces of the triangle, i.e c, β and γ.
HINT 1: First calculate c_1 and h.
HINT 2: **The sum of the angles in a triangle is π.**
RESULT:

$$c = b\cos\alpha + \sqrt{a^2 - b^2\sin^2\alpha} \tag{1}$$

$$\beta = \arcsin\frac{b\sin\alpha}{a} \tag{2}$$

$$\gamma = \pi - \alpha - \beta \tag{3}$$

(Solution:)

$$c_1 = b\cos\alpha \tag{4}$$

$$h = b\sin\alpha \tag{5}$$

$$c_2 = \sqrt{a^2 - h^2} \tag{6}$$

$$c = c_1 + c_2 \quad = \quad b\cos\alpha + \sqrt{a^2 - b^2\sin^2\alpha} \tag{7}$$

$$\sin\beta = \frac{h}{a}, \quad \beta = \arcsin\frac{h}{a} = \arcsin\frac{b\sin\alpha}{a} \tag{8}$$

$$\gamma = \pi - \alpha - \beta \tag{9}$$

2.Ex 10: Approximate calculation of π

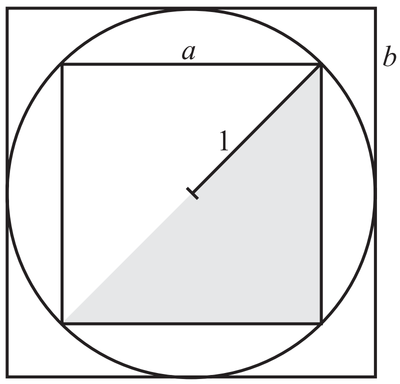


Fig2.10. **1**: Area of inscribed square and circumscribed square for approximating area of circle

In the above figure we see a circle of radius 1, with a **circumscribed**[$\stackrel{\mathbf{G}}{=}$ `umschrieben`] larger **square**[$\stackrel{\mathbf{G}}{=}$ `Quadrat`] (length b) and an **inscribed**[$\stackrel{\mathbf{G}}{=}$ `eingeschrieben`] smaller **square**[$\stackrel{\mathbf{G}}{=}$ `Quadrat`] (side lengths a).
2.10. **a**) Calculate b.

(Solution:)

$$b = 2 \tag{1}$$

2.10. **b**) Calculate a.
HINT: use Pythagoras for the **shaded**[$\stackrel{\mathbf{G}}{=}$ `schattiert`] rectangle.
RESULT:

$$a = \sqrt{2} \tag{2}$$

(Solution:)

The shaded rectangle has hypothenuse = 2 and two identical **adjacent**[$\stackrel{\mathbf{G}}{=}$ `anliegend`] **legs**[$\stackrel{\mathbf{G}}{=}$ `Schenkel`] a and a.
Pythagoras:

$$2^2 = a^2 + a^2 \tag{3}$$

$$4 = 2a^2 \tag{4}$$

$$2 = a^2, \quad a = \sqrt{2} \tag{5}$$

2.10. **c**) Look up the formula for the **area**[$\stackrel{G}{=}$ `Fläche`] A of a **circle**[$\stackrel{G}{=}$ `Kreis`]. Since A is between a^2 and b^2, you can give an approximate value for π.
RESULT:

$$2 \leq \pi \leq 4 \tag{6}$$

REM: By using regular **polygons**[$\stackrel{G}{=}$ `Vieleck`] with n corners instead of squares, π can be calculated to **arbitrary**[$\stackrel{G}{=}$ `beliebig`] **precision**[$\stackrel{G}{=}$ `Genauigkeit`].

(Solution:)

$$A = \pi r^2 = \pi \tag{7}$$

$$a^2 \leq \pi \leq b^2 \tag{8}$$

$$2 \leq \pi \leq 4 \tag{9}$$

2.**Ex 11: Any trigonometric function expressed by any other one**
When one trigonometric function is known (e.g. $\tan\alpha$) every other one (e.g. $\cos\alpha$) can be calculated from it. **Elaborate**[$\stackrel{G}{=}$ `ausarbeiten`] that example, i.e. express $\cos\alpha$ with the help of $\tan\alpha$.

HINT: express $\tan\alpha$ by $\sin\alpha$ and $\cos\alpha$. Express $\sin\alpha$ by $\cos\alpha$ and a **square root**[$\stackrel{G}{=}$ `Quadratwurzel`]. Remove the square root by **squaring**[$\stackrel{G}{=}$ `quadrieren`]. Solve for $\cos\alpha$.
RESULT:

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} \tag{1}$$

Look up that formula (and ones for similar cases) in a **formulary**[$\stackrel{G}{=}$ `Formelsammlung`].

(Solution:)

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha} \tag{2}$$

$$\cos\alpha \ \tan\alpha = \sqrt{1-\cos^2\alpha} \tag{3}$$

$$\cos^2\alpha \ \tan^2\alpha = 1-\cos^2\alpha \tag{4}$$

$$\cos^2\alpha \ (1+\tan^2\alpha) = 1 \tag{5}$$

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} \tag{6}$$

2.**Ex 12: ☻☻ Inverse function**

Consider the function f given by

$$y = f(x) = \frac{1}{2}x - 2 \tag{1}$$

2.12. **a**) Draw the graph of that function.

2.12. **b**) Calculate the inverse function g, i.e. solve (1) for x and **interchange**[$\stackrel{\mathbf{G}}{=}$ `vertauschen`] $x \leftrightarrow y$.

REM: The inverse function is also denoted by $g = f^{-1}$

RESULT:

$$y = g(x) = 2x + 4 \tag{2}$$

(Solution:)

$$\frac{1}{2}x = y + 2 \tag{3}$$

$$x = 2y + 4 \tag{4}$$

interchanging $x \leftrightarrow y$ yields

$$y = 2x + 4 = g(x) \tag{5}$$

2.12. **c**) Draw the graph of g and check that both graphs have a **mirror symmetry**[$\stackrel{\mathbf{G}}{=}$ `Spiegelsymmetrie`], where the mirror is the **bisectrix of the angle**[$\stackrel{\mathbf{G}}{=}$ `Winkelhalbierende`] of the x and y-axis.

2.12. **d**) Using a lot of pressure draw the graph of f and the symbols x and y of these axes. Look at the sheet of paper from the opposite side (with the graph shining through the sheet) with the x-axis upwards. Check that you can see the graph of g when x is interchanged with y. (In other words: the inverse function gives the same relation between x and y but the independent and dependent variables are interchanged.)

2.12. **e**) Check

$$f^{-1} \circ f = id \qquad \text{and} \qquad f \circ f^{-1} = id \tag{6}$$

i.e.

$$g(f(x)) = x \qquad \text{and} \qquad f(g(x)) = x \tag{7}$$

(In other words: applying the function and the inverse function (in both orders) in succession gives the identity, i.e. both applications cancel each other out.)

REM: Explanation of the **notation**[$\stackrel{G}{=}$ `Bezeichnungsweise`] in (6): The inverse function is denoted by $g = f^{-1}$. $\circ$ denotes **composition of functions**[$\stackrel{G}{=}$ `Zusammensetzung von Funktionen`], i.e. applying one after the other, where the rigth most is the innermost building site. E.g. $h = g \circ f$ denotes the function $y = h(x) = g(f(x))$.
id denotes the identical function (= identity): $y = id(x) = x$. Here, we have the **peculiarity**[$\stackrel{G}{=}$ `Besonderheit`] that the function name (usually f, g, h, etc.) is a two-letter string id.

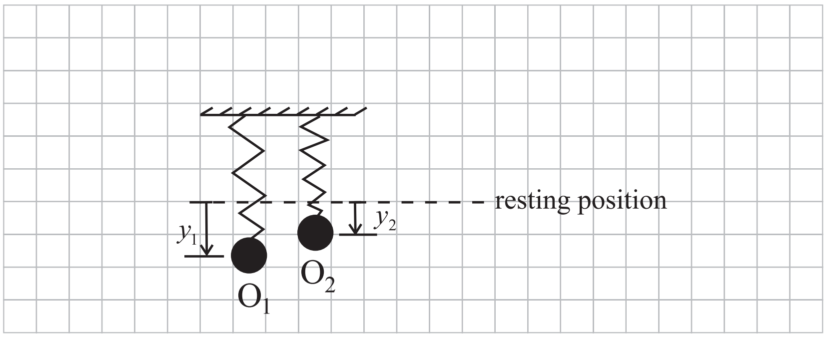
(Solution:)

$$g(f(x)) = 2f(x) + 4 = 2[\frac{1}{2}x - 2] + 4 = x - 4 + 4 = x \tag{8}$$

$$f(g(x)) = \frac{1}{2}g(x) - 2 = \frac{1}{2}[2x + 4] - 2 = x + 2 - 2 = x \tag{9}$$

2.**Ex 13: Phase shifts of two harmonic oscillators**

In the following figure you see the motion of two harmonic oscillators. y_1 (solid line) is the elongation of oscillator O_1, and y_2 (dotted line) is the elongation of oscillator O_2.
[One square of the sheet is 1 cm for y and 1 sec for t, t = time.]



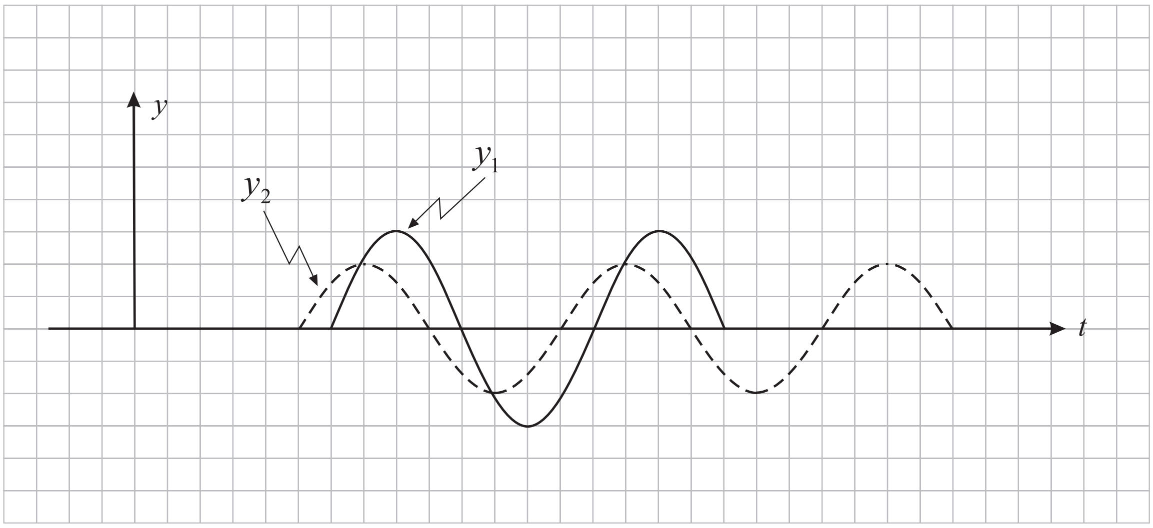


Fig2.13. **1**: Phase shifts of two identical harmonic oscillators having elongation $y_1 = y_1(t)$ and $y_2 = y_2(t)$.

2.13. **a**) At what time t_{1i} (i = initial) did O_1 start oscillating, and at what time t_{1f} (f = final) did O_1 stop oscillating?
RESULTS: $t_{1i} = 6$ sec, $t_{1f} = 18$ sec

2.13. **b**) How many periods did O_1 oscillate?
RESULT: 1.5 periods

2.13. **c**) What was the approximate elongation of O_1 at time $t = 7$ sec?
RESULT: $y = 2$ cm

2.13. **d**) When was the the first **zero crossing**[$\stackrel{\mathbf{G}}{=}$ `Nulldurchgang`] (i.e. $y_1 = 0$) of oscillator O_1 after its start?
RESULT: $t = 10$ sec

2.13. **e**) What are the amplitudes y_{o1} and y_{o2} of both oscillators?
RESULTS: $y_{o1} = 3$ cm, $y_{o2} = 2$ cm

2.13. **f**) What are their (primitive) periods, T_1 and T_2?
RESULT: $T_1 = T_2 = 8$ sec

2.13. **g**) Consider the **onset** [$\stackrel{\mathbf{G}}{=}$ `Beginn`] of oscillation of O_1 as phase $\varphi = 0$; what is the phase of O_1 at time $t = 10$ sec?
RESULT: $\varphi = \pi$

2.13. **h**) At what phase did it stop?
RESULTS: $\varphi = 3\pi$

2.13. **i**) Taking the same **convention** [$\stackrel{\mathbf{G}}{=}$ `Verabredung`] (for the **origin** [$\stackrel{\mathbf{G}}{=}$ `Nullpunkt`] of the phase, i.e. for $\varphi = 0$ as in g): at what phase did O_2 start?
RESULT: $\varphi = -\frac{1}{4}\pi$

2.13. **j**) What is the phase shift of O_2 relative to O_1?
RESULT: also $\varphi = -\frac{1}{4}\pi$

2.13. **k**) Using $\omega = \frac{2\pi}{T}$, what are the angular frequencies of O_1 and O_2?
RESULT:

$$\omega_1 = \omega_2 = \frac{2\pi}{8 \text{ sec}} = 0.785 \text{ s}^{-1} = 0.785 \text{ Hz} \tag{1}$$

REM: **Hz** is an abbreviation of Hertz and means s^{-1}.

2.13. **l**) Give the **analytical** [$\stackrel{\mathbf{G}}{=}$ `formelmäßig`] expressions for y_1 and y_2.
RESULTS:

$$y_1 = \begin{cases} 3 \text{ cm} \sin[\frac{2\pi}{8 \text{ sec}}(t - 6 \text{ sec})] & \text{for } 6 \text{ sec} \le t \le 18 \text{ sec} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$y_2 = \begin{cases} 2 \text{ cm} \sin[\frac{2\pi}{8 \text{ sec}}(t - 5 \text{ sec})] & \text{for } 5 \text{ sec} \le t \le 25 \text{ sec} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

2.13. **m**) Check for the above result that at time $t = 10$ sec the phase of sin of y_1 is π.

3 Formulae for trigonometric functions. Absolute value

3.Q 1: Fundamental formulae for sin and cos

3.1. **a**)

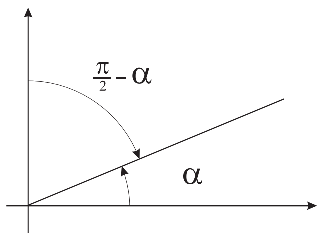


Fig3.1. **1**: Two angles whose sum is $\frac{\pi}{2} = 90°$ are called **complementary** to each other: $\frac{\pi}{2} - \alpha$ is the complementary angle of α and, vice versa, α is the complementary angle of $\frac{\pi}{2} - \alpha$.

for complementary angles

(Solution:)

$$\boxed{\sin(\tfrac{\pi}{2}-\alpha) = \cos\alpha} \tag{1}$$

$$\boxed{\cos(\tfrac{\pi}{2}-\alpha) = \sin\alpha} \tag{2}$$

Rem: $\frac{\pi}{2} = 90°$, $\frac{\pi}{2} - \alpha$ is called the **complementary angle**[$\stackrel{\mathbf{G}}{=}$ `Komplementärwinkel`] to α

sin is cos of complementary angle, and vice versa

3.1. **b**) for negative arguments (**even**[$\stackrel{\mathbf{G}}{=}$ `gerade`] or **odd**[$\stackrel{\mathbf{G}}{=}$ `ungerade`] function?)

(Solution:)

$$\boxed{\sin(-\alpha) = -\sin\alpha} \qquad \boxed{\textbf{sin is an odd function}[\stackrel{\mathbf{G}}{=} \texttt{ungerade Funktion}]} \tag{3}$$

$$\boxed{\cos(-\alpha) = \cos\alpha} \qquad \boxed{\textbf{cos is an even function}[\stackrel{\mathbf{G}}{=} \texttt{gerade Funktion}]} \tag{4}$$

3.1. **c**) period

(Solution:)

$$\boxed{\sin(\alpha \pm 2\pi) = \sin\alpha} \qquad \text{(period } 2\pi\text{)} \tag{5}$$

$$\boxed{\cos(\alpha \pm 2\pi) = \cos\alpha} \qquad \text{(period } 2\pi\text{)} \tag{6}$$

3.1. **d**) 'half-period'

(Solution:)

$$\boxed{\sin(\alpha + \pi) = -\sin\alpha} \qquad \text{('half-period' is } \pi\text{)} \tag{7}$$

$$\boxed{\cos(\alpha + \pi) = -\cos\alpha} \qquad \text{('half-period' is } \pi\text{)} \tag{8}$$

REM: We have **coined**[$\overset{\mathbf{G}}{=}$ `geprägt`] here the term 'half-period'. It is the period up to a sign.

3.Q 2: Absolute value

3.2. **a**) Say in words what is the **absolute value**[$\overset{\mathbf{G}}{=}$ `absoluter Betrag`] and how it is denoted.

(Solution:)

The absolute value of a (real) number x, denoted by $\mid x \mid$, is that number without its sign. So, the absolute value is always a positive number.

3.2. **b**) Give the fundamental calculation rule for the absolute value.

(Solution:)

$$\boxed{\mid ab \mid = \mid a \mid \mid b \mid} \tag{1}$$

3.2. **c**) $\mid 5 \mid = ?$

(Solution:)

$\mid 5 \mid = 5$

3.2. **d**) $\mid -5 \mid = ?$

(Solution:)

$\mid -5 \mid = 5$

3.2. **e**) Let be $a < 0$ (e.g. $a = 5$) Calculate $\mid a \mid = ?$

(Solution:)

$\mid a \mid = -a$ (for $-a$ is positive)

3.2. **f**) Give the solution of the equation $\mid a \mid = 5$

(Solution:)

$\mid a \mid = 5 \Rightarrow a = \pm 5$

3.2. **g**) Prove $\mid -a \mid = \mid a \mid$

(Solution:)

$$\mid -a \mid \quad = \quad \mid (-1)\, a \mid \quad \stackrel{(1)}{=} \quad \mid -1 \mid \mid a \mid \quad = \quad 1 \mid a \mid \quad = \quad \mid a \mid$$

3.**Q 3: Zeros and poles of** tan

Rem: $\sin x$ and $\cos x$ are not **simultaneously**[$\stackrel{\mathbf{G}}{=}$ `gleichzeitig`] (i.e. for the same x) zero

3.3. **a**) Give the solution of the equation $\tan x = 0$ (zeros of tan)

(Solution:)

$\tan x = \frac{\sin x}{\cos x} = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi, \quad n\epsilon\mathbb{Z}$

3.3. **b**) Give the solution of the equation $\tan x = \pm\infty$ **poles**[$\stackrel{\mathbf{G}}{=}$ `Polstellen`] of tan.

(Solution:)

$\tan x = \frac{\sin x}{\cos x} = \pm\infty \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2} + n\pi, \quad n\varepsilon\mathbb{Z}$

3.**Qx 4: Tangens and cotangens are odd functions**

Prove that tan and cot are odd functions.

(Solution:)

1) $\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x$

2) $\cot(-x) = \frac{1}{\tan(-x)} = -\frac{1}{\tan x} = -\cot x$

3.**Qx 5: Period of tangens and cotangens is π**

Prove that tan and cot both have the period π

(Solution:)

1) $\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \frac{\sin x}{\cos x} = \tan x$

2) $\cot(x+\pi) = \frac{1}{\tan(x+\pi)} = \frac{1}{\tan x} = \cot x$

3.**Qx 6: Tangens and cotangens for complementary angles**

Calculate tan and cot of the complementary angle.

(Solution:)

1) $\tan(\frac{\pi}{2}-\alpha) = \frac{\sin(\frac{\pi}{2}-\alpha)}{\cos(\frac{\pi}{2}-\alpha)} = \frac{\cos\alpha}{\sin\alpha} = \cot\alpha$

2) $\cot(\frac{\pi}{2}-\alpha) = \frac{1}{\tan(\frac{\pi}{2}-\alpha)} = \frac{1}{\cot\alpha} = \tan\alpha$

3.Qx 7: Addition theorem for trigonometric functions

3.7. **a**) Give or look up the formula for sin and cos of a sum.

(Solution:)

$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha} \tag{1}$$

$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta} \tag{2}$$

3.7. **b**) Derive from them the double angle formulae.

(Solution:)

From (1) putting $\alpha = \beta$:

$$\boxed{\sin(2\alpha) = 2\sin\alpha\cos\alpha} \tag{3}$$

From (2) putting $\alpha = \beta$:

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = \cos^2\alpha - (1-\cos^2\alpha) = -1 + 2\cos^2\alpha \tag{4}$$

$$\boxed{\cos(2\alpha) = -1 + 2\cos^2\alpha} \tag{5}$$

Rem: This result can be generalized and is quite important:

Powers of trigonometric functions, e.g.
$\sin^n\alpha, \quad \cos^n\alpha, \quad \sin^n\alpha\ \cos^m\alpha$
can be reduced to sums of trigonometric functions of multiple angles:
$\sin(k\alpha)$ and $\cos(k\alpha) \quad (k = 0, \cdots n)$

(5')

3.7. **c**) Derive the formula for the cos of a difference.

(Solution:)

In (2) replace $\beta \mapsto -\beta$:

$$\cos(\alpha-\beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{6}$$

$$\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta} \tag{7}$$

3.Ex 8: Graphical significance of absolute value

3.8. **a**) For a point $P(x, y)$ we have the information

$$|x| = 5, \quad |y - 2| = 3 \tag{1}$$

In a cartesian coodinate system draw all possibilities for P.
RESULT:

$$P_1(5, 5), \quad P_2(-5, 5), \quad P_3(-5, -1), \quad P_4(5, -1) \tag{2}$$

(Solution:)

$$x = \pm 5, \quad y - 2 = \pm 3, \quad y = \pm 3 + 2, \quad y = 5 \text{ or } y = -1 \tag{3}$$

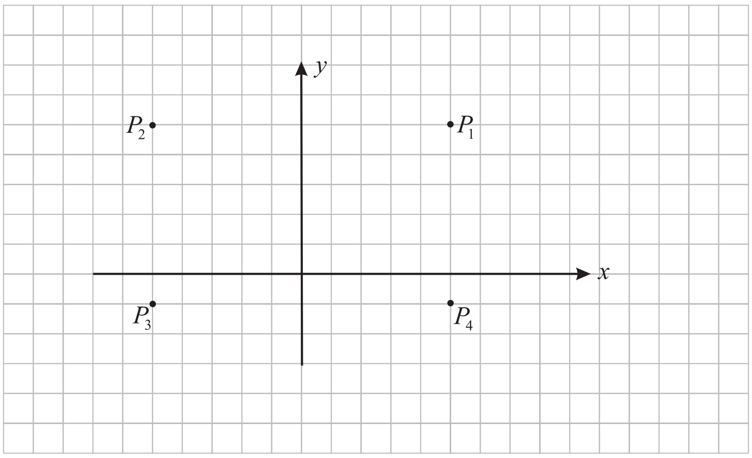


Fig3.8. **1**: Points P_1, P_2, P_3, P_4 defined by equations for their x- and y-coordinates.

3.8. **b**) If we have the additional information that P lies in the second quadrant, what is P?
RESULT:

$$P = P_2(-5, 5) \tag{4}$$

3.**Ex 9: Simplification of trigonometric functions**
Without using a **calculator**[$\stackrel{G}{=}$ `Taschenrechner`], express everything by $\varepsilon = \sin 13° \approx 0.2250$.

REM: To save space we have introduced the abbreviation ε. All results should be expressed in terms of ε.

3.9. **a)** $\sin 373°$
RESULT: ε

(Solution:)

$$\sin 373° = \sin(360° + 13°) = \sin 13° = \varepsilon \tag{1}$$

3.9. **b)** $\sin 347°$
RESULT: $-\varepsilon$

(Solution:)

$$\sin 347° = \sin(360° - 13°) = \sin(-13°) = -\sin 13° = -\varepsilon \tag{2}$$

3.9. **c)** $\cos 13°$
HINT: $\cos^2 + \sin^2 = 1$
RESULT: $\cos 13° = \sqrt{1 - \varepsilon^2}$

(Solution:)

$$\cos 13° = \sqrt{1 - \sin^2 13°} = \sqrt{1 - \varepsilon^2} \tag{3}$$

3.9. **d)** $\cot 13°$
RESULT: $\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}$

(Solution:)

$$\cot 13° = \frac{\cos 13°}{\sin 13°} = \frac{\sqrt{1 - \varepsilon^2}}{\varepsilon} \tag{4}$$

3.9. **e)** $\sin 77°$
RESULT: $\sqrt{1-\varepsilon^2}$

(Solution:)

$$\sin 77° = \sin(90° - 13°) = \cos(13°) = \cos 13° = \sqrt{1-\varepsilon^2} \tag{5}$$

3.9. **f)** $\cos 77°$
RESULT: ε

(Solution:)

$$\cos 77° = \cos(90° - 13°) = \sin 13° = \varepsilon \tag{6}$$

3.9. **g)** $\cos 103°$
RESULT: $-\varepsilon$

(Solution:)

$$\cos 103° = \cos(90° + 13°) = \sin(-13°) = -\sin 13° = -\varepsilon \tag{7}$$

3.9. **h)** $\sin 26°$
HINT: Use the double angle formula.
RESULT: $\sin 26° = 2\varepsilon\sqrt{1-\varepsilon^2}$

(Solution:)

$$\sin 26° = \sin(2 \cdot 13°) = 2\sin 13° \cos 13° = 2\varepsilon\sqrt{1-\varepsilon^2} \tag{8}$$

3.9. **i)** $\sin(-103°)$
RESULT: $-\sqrt{1-\varepsilon^2}$

(Solution:)

$$\begin{aligned} &\sin(-103°) = -\sin(103°) = -\sin(90° + 13°) = -\cos(-13°) = -\cos 13° = \\ &= -\sqrt{1-\varepsilon^2} \end{aligned} \tag{9}$$

3.9. **j)** $\cos(-26^\circ)$
RESULT: $1 - 2\varepsilon^2$

(Solution:)

$$\cos(-26^\circ) = \cos(26^\circ) = \cos(2 \cdot 13^\circ) = -1 + 2\cos^2 13^\circ \tag{10}$$

$$= -1 + 2(1-\varepsilon^2) = 1 - 2\varepsilon^2 \tag{11}$$

3.9. **k)** $\sin(193^\circ)$
RESULT: $-\varepsilon$

(Solution:)

$$\sin 193^\circ = \sin(180^\circ + 13^\circ) = -\sin 13^\circ = -\varepsilon \tag{12}$$

3.9. **l)** $\cos(-167^\circ)$
RESULT: $-\sqrt{1-\varepsilon^2}$

(Solution:)

$$\cos(-167)^\circ = \cos 167^\circ = \cos(180^\circ - 13^\circ) = -\cos(-13^\circ) = -\cos 13^\circ = \tag{13}$$
$$= -\sqrt{1-\varepsilon^2}$$

3.9. **m)** $\sin 43^\circ$
HINT: Use $\sin 30^\circ = \frac{1}{2}$, $\cos 30^\circ = \sqrt{3}/2$ and the addition theorem for cos.
RESULT: $\sin 43^\circ = \frac{1}{2}\sqrt{1-\varepsilon^2} + \frac{\sqrt{3}\varepsilon}{2}$

(Solution:)

$$\sin 43^\circ = \sin(30^\circ + 13^\circ) = \sin 30^\circ \cos 13^\circ + \cos 30^\circ \sin 13^\circ = \frac{1}{2}\sqrt{1-\varepsilon^2} + \frac{\sqrt{3}\varepsilon}{2} \tag{14}$$

3.9. **n)** $\cos(1001.5\pi + 13^\circ)$
RESULT: ε

(Solution:)

$$\cos(1001.5\pi + 13°) = \cos(500 \cdot 2\pi + \pi + \frac{1}{2}\pi + 13°) \tag{15}$$

$$= \cos(\pi + \frac{1}{2}\pi + 13°) = -\cos(\frac{1}{2}\pi + 13°) = -\sin(-13°) \tag{16}$$

$$= \sin(13°) = \varepsilon \tag{17}$$

3.Ex 10: ☻ Multiple values of the arcus function

A calculator yields

$$\sin 13° = 0.2250 \tag{1}$$

and conversely

$$\arcsin 0.2250 = 13° \tag{2}$$

However 13° is only one possible value for

$$\arcsin 0.2250 \tag{3}$$

(since arcsin is a multiple valued function).
3.10. **a**) In the graph for

$$y = \sin x \tag{4}$$

indicate all possible values of

$$\arcsin 0.2250 \tag{5}$$

In other words: give all solutions of

$$\sin x = 0.2250 \tag{6}$$

(Solution:)

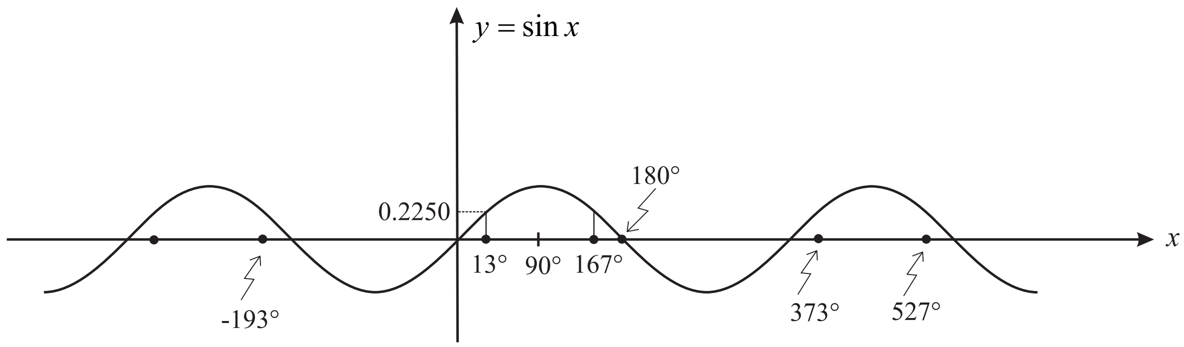


Fig3.10. **1**: $\sin(13°) = \sin(167°) = 0.2250$.
Thus, $\arcsin 0.2250 = 13°$ or $= 167°$ or $= 373°, \ldots$

$\sin(167°) = \sin(180° - 13°) = -\sin(-13°) = \sin 13° = 0.2250$ and all multiples of $2\pi = 180°$ can be added (or subtracted) from these solutions $x = 13°$ and $x = 167°$ i.e. all solutions are given by

$$x = 13° + n \cdot 360° \qquad n \in \mathbb{Z} \tag{7}$$

and

$$x = 167° + n \cdot 360° \qquad n \in \mathbb{Z} \tag{8}$$

3.10. **b**) Give the value of $\arcsin 0.2250$, when it is known that it must be in the interval

$$[\frac{\pi}{2}, \pi] \tag{9}$$

RESULT:

$$\arcsin 0.2250 = 167° \tag{10}$$

3.10. **c**) Give all solutions to the equations

$$\left| \begin{array}{l} |\sin x| = \frac{1}{2} \\ \\ |x| < \frac{\pi}{2} \end{array} \right. \tag{11}$$

RESULT:

$$x = \pm 30° = \pm\frac{\pi}{6} \tag{12}$$

(Solution:)

$$\sin 30° = \frac{1}{2} \tag{13}$$

$$\sin(-30°) = -\frac{1}{2} \tag{14}$$

$$|\sin(-30°)| = \frac{1}{2} \tag{15}$$

3.Ex 11: ☻ Ellipses

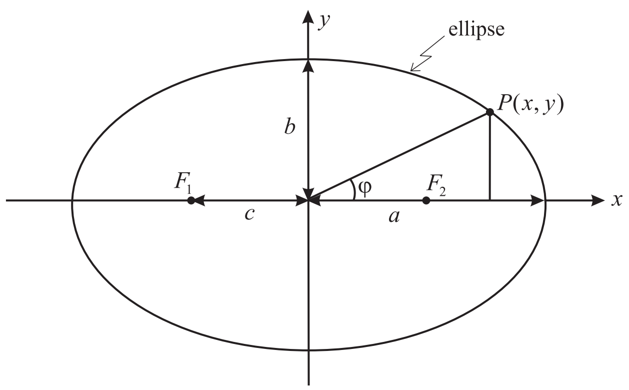


Fig3.11. **1**: Ellipse with half axis a and b and focal distance $2c$

$$\boxed{\left| \begin{array}{ll} x = a\cos\varphi & \text{(parametric representation of} \\ y = b\sin\varphi & \text{an ellipse)} \end{array} \right.} \tag{1}$$

is the equation of an ellipse. (a = **great diameter**[$\stackrel{\mathbf{G}}{=}$ `große Halbachse`], b = **small diameter**[$\stackrel{\mathbf{G}}{=}$ `kleine Halbachse`].)

3.11. **a**) Draw an ellipse for

$$a = 5 \text{ cm}, \quad b = 3 \text{ cm} \tag{2}$$

by constucting the points $P(x, y)$ for

$$\varphi = 0, \quad \varphi = 30°, \quad \varphi = 45°, \quad \varphi = 60°, \quad \varphi = 90° \tag{3}$$

according to the above parametric representation.

(Solution:)

φ	$\cos\varphi$	$\sin\varphi$	x	y
0	1	0	5 cm	0
30°	0.866	0.5	4.33 cm	1.5 cm
45°	0.7071	0.7071	3.5 cm	2.12 cm
60°	0.5	0.866	2.5 cm	2.6 cm
90°	0	1	0	3 cm

(4)

3.11. **b**) Show that the x-axis is an axis of **mirror symmetry**[$\stackrel{G}{=}$ `Spiegelsymmetrieachse`].
HINT: If a point P with φ is at $P(x, y)$, the point P' with $-\varphi$ is at

$$P'(x, -y) \tag{5}$$

i.e., they arise from each other by mirror symmetry with the x-axis as the mirror.

(Solution:)

$$\left| \begin{array}{ll} x = a\cos\varphi & P(x,y)\text{is the point for } \varphi \\ y = b\sin\varphi & \end{array} \right. \tag{6}$$

$$\left| \begin{array}{l} x' = a\cos(-\varphi) = a\cos\varphi = x \\ y' = b\sin(-\varphi) = -b\sin\varphi = -y \end{array} \right. \tag{7}$$

$$P'(x', y') \text{ is the point for } \varphi' = -\varphi \tag{8}$$

3.11. **c**) The same for the y-axis.
HINT: Consider

$$\varphi \quad \text{and} \quad \pi - \varphi \tag{9}$$

(Solution:)

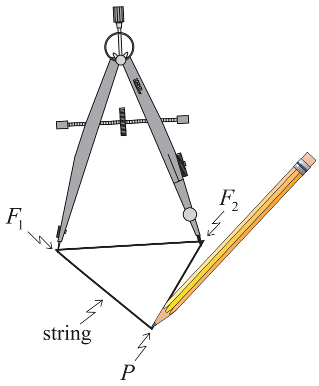
$$x' = a\cos(\pi - \varphi) = -a\cos(-\varphi) = -a\cos\varphi = -x \tag{10}$$

$$y' = b\sin(\pi - \varphi) = -b\sin(-\varphi) = +b\sin\varphi = y \tag{11}$$

3.11. **d**) An ellipse has the following geometric property: there are two **focal points** [$\stackrel{\mathbf{G}}{=}$ `Brennpunkte`] F_1 and F_2 so that an arbitrary point P of the the ellipse has a constant sum of distances to F_1 and F_2:

$$|F_1P| + |F_2P| = \text{ const.} \tag{12}$$

Draw an ellipse by using the tips of a **compass** [$\stackrel{\mathbf{G}}{=}$ `Zirkel`] as the fixed focal



`Fig`3.11. **2**: An ellipse is the set of all points (pencil) having the same sum of distance from two fixed focal points F_1 and F_2.

points F_1, F_2. Use a closed **string** [$\stackrel{\mathbf{G}}{=}$ `Faden`] for the constant length.

3.11. **e**) In fig. 1 the focal points F_1, F_2 lie on the x-axis and have the distance

$$c = \sqrt{a^2 - b^2} \tag{13}$$

from the center of the ellipse and const $= 2a$. **Check** [$\stackrel{\mathbf{G}}{=}$ `überprüfen`] that **statement** [$\stackrel{\mathbf{G}}{=}$ `Behauptung`] for the **vertices** [$\stackrel{\mathbf{G}}{=}$ `Scheitel`] of an ellipse, i.e. for the points $S_1(a, 0)$ and $S_2(0, b)$.

(Solution:)

For S_1:

$$\text{const.} = (c + a) + (a - c) = 2a \tag{14}$$

For S_2 (using Pythagoras):

$$\text{const.} = 2\sqrt{b^2 + c^2} = 2\sqrt{b^2 + a^2 - b^2} = 2\sqrt{a^2} = 2a \tag{15}$$

Rem 1: In (12)(14)(15) we have assumed that 'const.' denotes the same constant. Such usage of 'const.' is objectionable, since at least according to one view, 'const.' should not be used as a

constant variable, as done here, but only as a **predicate** (= property). I.e. 'const.' in (12) simply says that the left hand side is constant with respect to some variable (the point P, in our case), and 'const.' in different formulae cannot be identified as the *same* constant.
According to that view we should write instead of (12)

$$|F_1P| + |F_2P| = l = \text{ const.} \tag{12'}$$

(l = string length) and 'const.' in (14)(15) and in 'const = $2a$' should be replaced by l.
However, our usage of 'const.', i.e. at the same time as a prediacte and as a constant, is widely used in physics, and when a second, different constant variable is required, e.g. 'konst.' instead of 'const.' is used.

REM 2: We can distinguish between **absolute constants** like 2, 3.1, π and **constant variables**, e.g. l in (12'), which is constant for a fixed ellipse, but may differ from ellipse to ellipse.

3.Ex 12: ☻ Addition theorem for tangens

3.12. **a**) Derive the addition theorem for the tangens

$$\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}} \quad \textbf{(addition theorem for tangens)} \tag{1}$$

REM: As is usual in physics in this and similar cases, we do not mention explicitely that (1) should not be applied when a denominator is zero (e.g. when $\alpha + \beta = \pi/2$), or if a function (e.g. tan) is undefeined (e.g. when $\alpha = \pi/2$).
Sometimes, even in these exceptional cases, (1) is valid in the sense that both sides are $\pm\infty$.

HINT: $\tan = \frac{\sin}{\cos}$, and use the addition theorem for sin and cos

(Solution:)

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \tag{2}$$

Division by $\cos\alpha\cos\beta$ yields

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{3}$$

3.12. **b**) Look up an analogous formula for cot.

3.Ex 13: ☻ Addition of sines expressed as a product

3.13. **a)** Derive the following formula

$$\boxed{\sin\alpha + \sin\beta \quad = \quad 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} \tag{1}$$

HINTS: Use the addition theorem for sin and cos by writing

$$\frac{\alpha\pm\beta}{2} \quad = \quad (\frac{\alpha}{2}\pm\frac{\beta}{2}) \tag{2}$$

Use the formula for $(\sin 2\alpha)$,

$$\sin^2 + \cos^2 = 1 \tag{3}$$

(Solution:)

$$\sin(\frac{\alpha}{2}+\frac{\beta}{2}) \quad = \quad \sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2} \tag{4}$$

$$\cos(\frac{\alpha}{2}-\frac{\beta}{2}) \quad = \quad \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2} \tag{5}$$

$$\begin{aligned} 2\sin\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2} &= 2\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\cos^2\tfrac{\beta}{2} + 2\sin^2\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}+ \\ &\quad +2\cos^2\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2} + 2\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\sin^2\tfrac{\beta}{2} = \\ &= \sin\alpha(\cos^2\tfrac{\beta}{2}+\sin^2\tfrac{\beta}{2}) + \sin\beta(\sin^2\tfrac{\alpha}{2}+\cos^2\tfrac{\alpha}{2}) = \\ &= \sin\alpha+\sin\beta \end{aligned} \tag{6}$$

3.13. **b)** Look up similar formulas for cos, tan, and cot.

3.Ex 14: Frequency ν, angular frequency ω and period T

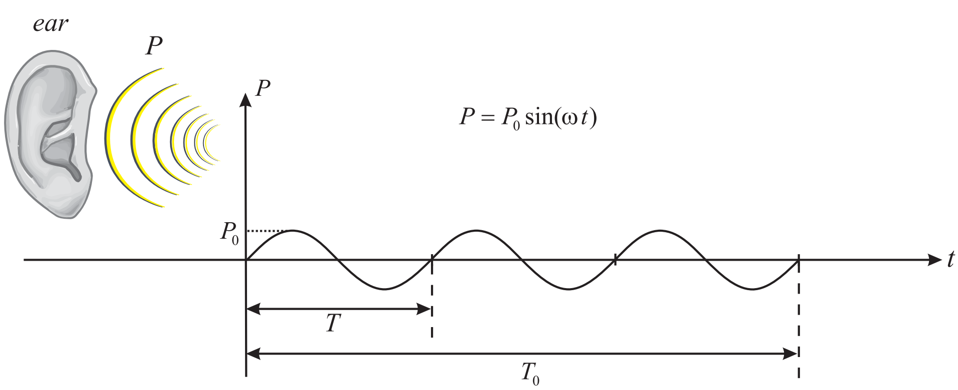


Fig3.14. **1**: A sound wave, represented by a sine, hitting the ear

Let the sound pressure

$$p = p(t) \tag{1}$$

at the ear be

$$p = p_0 \sin(\omega t) \qquad [\omega = \text{ angular frequency}] \tag{2}$$

3.14. **a**) Calculate the period T.
HINT: For what $t = T$ does the phase of $\sin(\omega t)$ have the value 2π?
RESULT:

$$\boxed{T = \frac{2\pi}{\omega}} \quad , \quad \boxed{\omega = \frac{2\pi}{T}} \quad \text{(relation between angular frequency and period.)} \tag{3}$$

(Solution:)

$$\omega T = 2\pi \qquad \Rightarrow \qquad T = \frac{2\pi}{\omega} \quad \text{or} \quad \omega = \frac{2\pi}{T} \tag{4}$$

3.14. **b**) The following table gives the number n of periods T which **fit** [$\overset{\mathbf{G}}{=}$ passen] into a given time interval T_0.

n	T_0	
3	$3T$	(L1)
1	$1T$	(L2)
$\frac{1}{2}$	$\frac{1}{2}T$	(L3)
?	1	(L4)

(5)

Check each line of the table. E.g. for line (L1) (see figure): $n = 3$ periods T fit into the time interval $T_0 = 3T$. Any line can be found from a given one by mutiplying by a factor, e.g. if we multiply line (L2) by the factor 3, we get line (L1).

3.14. **c**) The frequency ν is the number of periods (T) which fit into unit time ($T_0 = 1$). [Or in slightly different words: the frequency is the number of periods per unit time.] By completing line (L4) calculate the frequency ν expressed by the period T and give all remaining relations between $\omega, \nu,\ T$.
RESULTS:

$$\boxed{\nu = \frac{1}{T}} \quad T = \frac{1}{\nu} \ \textbf{(Relation between period } T \textbf{ and frequency } \nu\textbf{)} \tag{6}$$

$$\boxed{\omega = 2\pi\nu} \quad \nu = \frac{\omega}{2\pi} \ \textbf{(Relation between angular frequency } \omega \textbf{ and freq. } \nu\textbf{)} \tag{7}$$

3.Ex 15: ☻ Superposition of waves

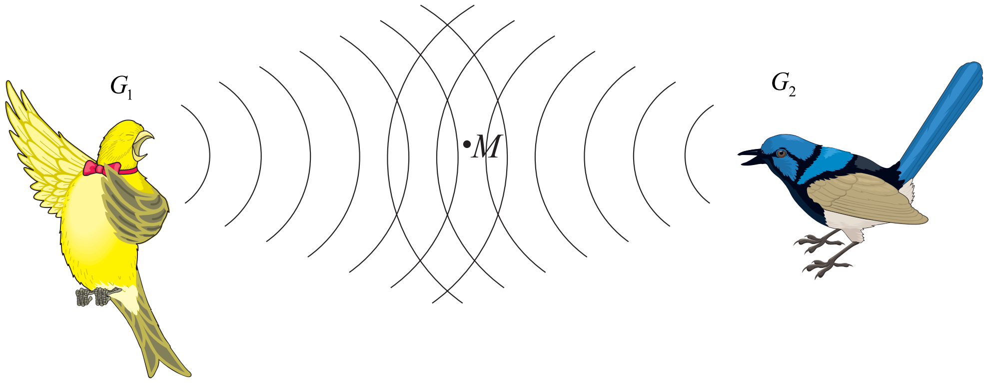


Fig3.15. **1**: Microphone M hears the **superposition**[$\stackrel{G}{=}$ `Überlagerung`] of the **sound waves**[$\stackrel{G}{=}$ `Schallwellen`] produced by generators G_1 and G_2.

If there are two generators of waves the elongations are added together.

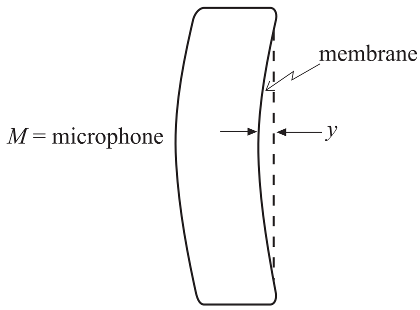


Fig3.15. **2**: At the microphone M its membrane is **shifted**[$\stackrel{G}{=}$ `verschieben`] by $y = y(t)$ because of the sound pressure.

So, when the shift y of microphone M's membrane is

$$y_1 = y_{10} \sin(\omega_1 t + \alpha_1) \qquad \text{from generator } G_1 \tag{1}$$

$$y_2 = y_{20} \sin(\omega_2 t + \alpha_2) \qquad \text{from generator } G_2 \tag{2}$$

the total signal at the microphone (i.e. when both generators are operating) is

$$y = y_1 + y_2 \tag{3}$$

3.15. **a**) Take the special case

$$y_{10} = y_{20} = 0.1 \text{ mm}, \qquad \alpha_1 = \alpha_2 = 0 \tag{4}$$

$$\omega_1 = 10\,002 \text{ Hertz }, \qquad \omega_2 = 10\,000 \text{ Hertz} \tag{5}$$

$$\boxed{[\text{ Hz} = \text{Hertz } = \frac{1}{\text{sec}} \text{]}} \tag{6}$$

and calculate the signal at the microphone using the **previous**[$\stackrel{\mathbf{G}}{=}$ `vorhergehend`] exercise.

RESULT:

$$y = 0.2 \text{ mm } \cos(1 \text{ Hz} \cdot t) \sin(10\,001 \text{ Hz} \cdot t) \tag{7}$$

3.15. **b**) **Sketch**[$\stackrel{\mathbf{G}}{=}$ `skizzieren`] this function qualitatively [using **suitable**[$\stackrel{\mathbf{G}}{=}$ `geeignet`] units].

HINT: Consider ± 0.2 mm cos (1 Hz$\cdot t$) as the amplitude of sin(10 001 Hz$\cdot t$). That amplitude is approximately constant during one period of the fast sin(10 001 Hz$\cdot t$) oscillation.

(Solution:)

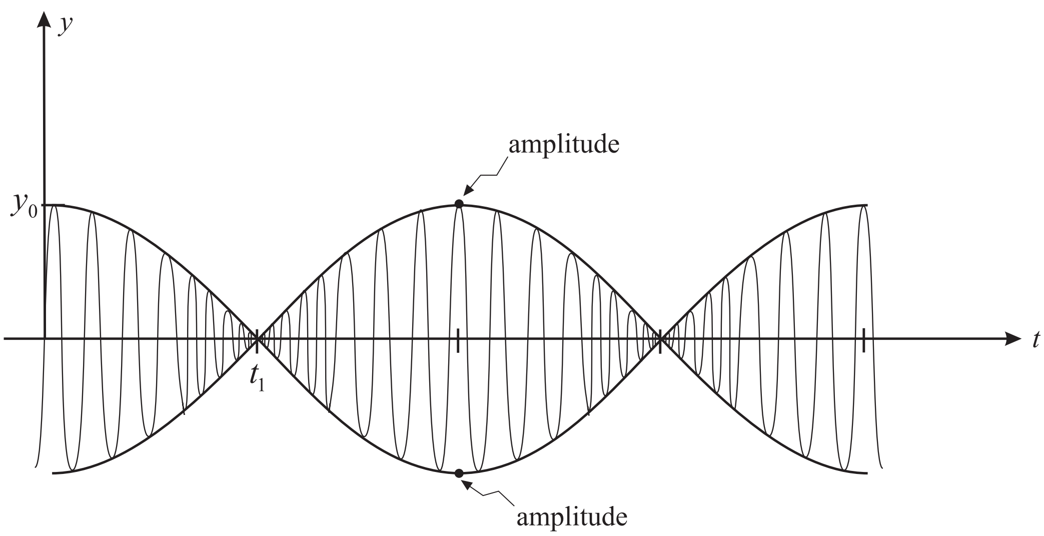


Fig3.15. **3**: Superposition of two sound waves with nearly equal frequencies (beating)

Answer the following questions about the above qualitative sketch.

3.15. **c**) What is the value of y_0?
RESULT:

$$y_0 = 0.2 \text{ mm} \tag{8}$$

3.15. **d**) What is $y(0)$?
RESULT:

$$y_0 = 0 \tag{9}$$

3.15. **e**) What is the time t_1?
RESULT:

$$t_1 = \frac{\pi}{2} \text{ sec } \approx 1.57 \text{ sec} \tag{10}$$

(Solution:)

$$1 \text{ Hz} \cdot t_1 = \frac{\pi}{2} \quad \Rightarrow \quad t_1 = \frac{\pi}{2} \text{ sec} \tag{11}$$

3.15. **f**) What is $y(t_1)$?
RESULT:

$$y(t_1) = 0 \;, \qquad \text{because } \cos = 0 \tag{12}$$

3.15. **g**) Calculate $y(2t_1)$.
RESULT:

$$y(2t_1) = 0 \tag{13}$$

(Solution:)

$$\begin{aligned} y(2t_1) &= 0.2 \text{ mm } \cos(1 \text{ Hz} \cdot 2t_1) \sin(10\,001 \text{ Hz} \cdot 2t_1) \\ &= 0.2 \text{ mm } \underbrace{\cos(\pi)}_{-1} \underbrace{\sin(10\,001 \cdot \pi)}_{(-1)^{10\,001} \sin 0} \\ &= 0 \end{aligned} \tag{14}$$

3.15. **h**) In the best case the human ear can hear (depending on age) in the frequency interval 16 Hz ... 20 000 Hz. Let an older person be able to hear in the interval 20 Hz ... 5 000 Hz, what is the frequency ν_1 of generator G_1? Can it be heard by that person?
HINT: ω_1 is an angular frequency
RESULT:

$$\nu_1 = 1\,591 \text{ Hz} \;;\; \textit{yes} \tag{15}$$

(Solution:)

$$\omega = 2\pi\nu, \quad \nu_1 = \frac{\omega_1}{2\pi} = \frac{10\,000}{2\pi} \text{ Hz } = 1\,591 \text{ Hz} \tag{16}$$

3.15. **i**) The slow oscillation $\cos(1 \text{ Hz} \cdot t)$ is called a **beating (of oscillations)** [$\stackrel{G}{=}$ `Schwebung`] which can be heard. What is the beating-frequency ν_b, and can it be heard in our case?
RESULT:

$$\nu_b = 0.16 \text{ Hz} \;;\; \textit{no} \tag{17}$$

(Solution:)

$$\omega_b = 1 \text{ Hz}, \quad \nu_b = \frac{\omega_b}{2\pi} = \frac{1}{2\pi} \text{ Hz } = 0.16 \text{ Hz} \tag{18}$$

3.15. **j**) Calculate the zeros of $y(t)$ in the interval $[0, t_1)$. What is the number N of zeros in that interval?
HINTS: t_1 does not belong to the interval. A product is zero only if one of its factors is zero.
RESULTS:

$$t = \frac{n\pi}{10\,001} \text{ sec}, \quad n = 0, 1, \ldots 5000 \tag{19}$$

$$N = 5001 \tag{20}$$

(Solution:)

In the interval $[0, t_1)$ $\cos(1 \text{ Hz}{\cdot}t)$ is not zero. At $t = t_1$ both sin and cos are zero. So in that interval the zeros of $y(t)$ are the zeros of

$$\sin(10\,001 \text{ Hz} \cdot t). \tag{21}$$

$$\sin(10\,001 \text{ Hz} \cdot t) = 0 \quad \Rightarrow \quad 10\,001 \text{ Hz} \cdot t = n\pi \quad (n \in \mathbb{Z}) \tag{22}$$

$$t = \frac{n\pi}{10\,001} \text{ sec}, \quad n = 0, 1, \dots 5000 \tag{23}$$

(The range for n was chosen so that $t \in [0, t_1)$) $N = 5001$

3.15. **k**) Describe what will be observed in our case.

(Solution:)

One hears the tone $\nu \approx \nu_1 \approx \nu_2 \approx 1\,591$ Hz with an intensity varying with the frequency $\nu_b = 2 \cdot 0.16$ Hz $= 3.2$ Hz. [The intensity corresponds to the absolute value of $\cos(1 \text{ Hz} \cdot t)$ having twice its frequency.]

4 Powers, roots and exponential functions

4.**Q 1: Powers**

What's the meaning of **powers**[$\stackrel{\mathbf{G}}{=}$ `Potenzen`] such as a^n for $n = 5, 1, 0, -5$?

(Solution:)

$$a^5 = a \cdot a \cdot a \cdot a \cdot a \tag{1}$$

$$\boxed{a^1 = a} \tag{2}$$

$$\boxed{a^0 = 1} \qquad (\text{ if } a \neq 0) \tag{3}$$

$$\boxed{0^0 \text{ is undefined}} \tag{4}$$

$$a^{-5} = \frac{1}{a^5} = \frac{1}{a \cdot a \cdot a \cdot a \cdot a} \tag{5}$$

REM: (3) and (5) are very reasonable definitions, since they will lead to beautiful theorems. On the other hand it is impossible to **devise**[$\stackrel{\mathbf{G}}{=}$ `ausdenken`] a reasonable definition for 0^0.

4.**Q 2: Square roots**

4.2. **a**) What is a **square root**[$\stackrel{\mathbf{G}}{=}$ `Quadratwurzel`], and in particular $\sqrt{2}$

(Solution:)

The square root of a number x (x is called the **radicand**) is a number when multiplied by itself gives the radicand:

$$\sqrt{2}\sqrt{2} = 2 \tag{1}$$

4.2. **b**) Give $\sqrt{2}$ as an approximate decimal number.

(Solution:)

$$\boxed{\sqrt{2} \approx \pm 1.41421356 \cdots} \tag{2}$$

REM: The square root is a **double valued**[$\stackrel{\mathbf{G}}{=}$ `doppeldeutig`] symbol:

$$(-\sqrt{2})(-\sqrt{2}) = \sqrt{2}\sqrt{2} = 2 \tag{3}$$

4.2. **c**) What is the meaning of ${}_{+}\sqrt{2}$

(Solution:)

The subscripted + denotes the positive square root, e.g.

$$ {}_{+}\sqrt{2} \approx 1.41421356 \cdots \tag{4} $$

$$ {}_{+}\sqrt{x} = \mid \sqrt{x} \mid \tag{5} $$

$$ {}_{-}\sqrt{x} = - \mid \sqrt{x} \mid \tag{6} $$

$$ \sqrt{x} = \pm {}_{+}\sqrt{x} \tag{7} $$

REM: Sometimes the symbol $\sqrt{}$ is understood to mean ${}_{+}\sqrt{}$, i.e. the positive square root is **implied**[$\overset{\mathbf{G}}{=}$ `impliziert, unterstellt, angenommen`]. E.g.

$$ \sqrt{2} \approx 1.4142136 \tag{8} $$

4.**Q 3: General roots**

What is the meaning of the n-**th root**, e.g. $\sqrt[n]{5}$ for $(n = 2, 3, 4)$

(Solution:)

$$ \sqrt[2]{5} = \sqrt{5} \quad \text{the square root is the same as the second root} \tag{1} $$

$$ \sqrt[3]{5}\ \sqrt[3]{5}\ \sqrt[3]{5} = 5 \tag{2} $$

REM: The third root (in general: the n-th root, with $n =$ **odd**[$\overset{\mathbf{G}}{=}$ `ungerade`]) is a *unique* symbol (in the domain of real numbers).

$$ \sqrt[4]{5}\ \sqrt[4]{5}\ \sqrt[4]{5}\ \sqrt[4]{5} = 5 \tag{3} $$

REM: Since it also holds:

$$ (-\sqrt[4]{5})(-\sqrt[4]{5})(-\sqrt[4]{5})(-\sqrt[4]{5}) = 5 \tag{4} $$

the fourth root (in general: the n-th root with $n =$ **even**[$\overset{\mathbf{G}}{=}$ `gerade`]) is again a double valued symbol.

4.**Q 4: General powers**

What is the meaning of a^b

4.4. **a**) for $b = \frac{1}{n}$ $\quad (n = 1, 2, 3...)$

(Solution:)

$$\boxed{a^{\frac{1}{n}} = \sqrt[n]{a}} \tag{1}$$

4.4. **b**) for $b = \frac{n}{m}$ $\quad (n, m = 1, 2, 3, ...)$

(Solution:)

$$\boxed{a^{\frac{n}{m}} = \sqrt[m]{a^n} = (\sqrt[m]{a})^n} \tag{2}$$

4.4. **c**) What is the meaning of a^b for arbitrary real numbers a, b $\quad (a > 0)$

(Solution:)

Since every real number b can be approximated by a rational number: $b \approx \frac{n}{m}$ the general power a^b can be approximated by $a^{\frac{n}{m}}$, which is defined by the above formula.

4.4. **d**) What are the names for a, b, a^b

(Solution:)

$a^b =$ **power**[$\stackrel{\mathbf{G}}{=}$ `Potenz`] $=$ b-th power of a, $\quad a =$ **basis**, $b =$ **exponent** (from lat. exponent = the outstanding)

4.**Q 5: Calculation rules for powers**

$(a, b, n, m \in \mathbb{R})$

4.5. **a**) $a^{-n} = ?$

(Solution:)

$$\boxed{a^{-n} = \frac{1}{a^n}} \tag{1}$$

4.5. **b**) $a^n a^m = ?$ (Proof for $n = 2, m = 3$)

(Solution:)

$$\boxed{a^n a^m = a^{n+m}} \tag{2}$$

Proof (for $n = 2, m = 3$):

$$a^2 a^3 = aa \cdot aaa = a^5 \tag{3}$$

4.5. **c)** $(a^n)^m = ?$ (Proof for $n = 2, m = 3$)

(Solution:)

$$\boxed{(a^n)^m = a^{nm}} \tag{4}$$

REM about operator priority:
Because of the outstanding position of the exponent, it is clear that it represents the inner-most **building site**[$\stackrel{\mathbf{G}}{=}$ `Baustelle`], i.e.

$$a^{nm} := a^{(nm)} \qquad [\text{and not } := (a^n)m] \tag{5}$$

PROOF of (4) for $n = 2, m = 3$:

$$(a^2)^3 = aa\ aa\ aa = a^6 = a^{2 \cdot 3} \tag{6}$$

4.5. **d)** Prove

$$\boxed{a^{n-m} = \frac{a^n}{a^m}} \tag{7}$$

(Solution:)

$$a^{n-m} = a^{n+(-m)} = a^n a^{-m} = a^n \frac{1}{a^m} = \frac{a^n}{a^m} \tag{8}$$

4.5. **e)** $(ab)^n = ?$ (Proof for $n = 3$)

(Solution:)

$$\boxed{(ab)^n = a^n b^n} \tag{9}$$

PROOF of (9) for $n = 3$:

$$(ab)^3 = ab\ ab\ ab = aaabbb = a^3 b^3 \tag{10}$$

4.Q 6: ☻ Operator priority

Write with superfluous **brackets**[$\stackrel{\mathbf{G}}{=}$ `Klammern`] and formulate the applied priority rule.

4.6. **a)** $a + bc$

(Solution:)

$$a + bc := a + (bc) \qquad [\text{and not } := (a+b)c] \tag{1}$$

(multiplication or division) have higher priority than (addition or subtraction)

4.6. **b)** $a - b/c$

(Solution:)

$$a - b/c := a - (b/c) \qquad [\text{and not } := \frac{a-b}{c}] \tag{2}$$

same rule as in a)

4.6. **c)** a/bc

(Solution:)

$$a/bc := (a/b)c \qquad [\text{and not } := a/(bc)] \tag{3}$$

Multiplication and division have the same priority.
With equal priority the order decides: left comes before right.

4.6. **d)** $a - b + c$

(Solution:)

$$a - b + c := (a - b) + c \qquad [\text{and not } := a - (b + c)] \tag{4}$$

$\boxed{\textbf{addition and subtraction have the same priority} \text{ and rule c)}}$

4.6. **e)** $\frac{a+b}{c+d}$

(Solution:)

$$\frac{a+b}{c+d} := \frac{(a+b)}{(c+d)} \qquad [\text{and not } := \frac{a}{(c+d)} + b] \tag{5}$$

line of the fraction[$\stackrel{G}{=}$ `Bruchstrich`] **involves brackets**

4.6. **f)** ab^c

(Solution:)

$$ab^c := a(b^c) \qquad [\text{and not} := (ab)^c] \tag{6}$$

$\boxed{\textbf{Exponentiation (powers) have higher priority than multiplication (or division)}}$

4.6. **g)** a/b^c

(Solution:)

$$a/b^c := \frac{a}{b^c} = a/(b^c) \qquad [\text{and not} := (\frac{a}{b})^c] \tag{7}$$

same as f)

4.Q 7: The (natural) exponential function $y = e^x$

4.7. **a)** Give its representation as a **power series**[$\overset{\mathbf{G}}{=}$ `Potenzreihenentwicklung`] and give an alternative notation for e^x.

$$\boxed{y = e^x = \exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}} \tag{1}$$

Rem 1: $n!$ are the **factorials**[$\overset{\mathbf{G}}{=}$ `Fakultäten`]:

$$\boxed{n! = 1 \cdot 2 \cdots n} \tag{2}$$

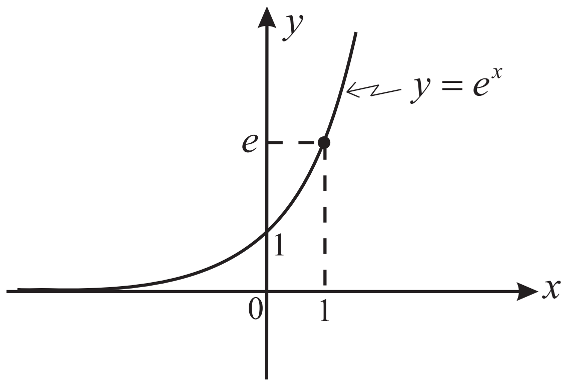
$$\boxed{1! = 1} \tag{3}$$

$$\boxed{0! = 1} \tag{4}$$

Rem 2: Again later on, it will turn out that (4) is a reasonable definition because it will lead to simple theorems.
In particular it allows the elegant notation in (1) as an infinite sum.

4.7. **b**) Give its graph (qualitatively).

(Solution:)



`Fig`4.7. **1**: Graph of the (natural) exponential function

4.7. **c**) Ex: From a) calculate **Euler**'s number e to within some decimals.

(Solution:)

Set $x = 1$

$e^1 = e = 1 + 1 + \frac{1}{2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots = 2.71828 \cdots$

$$\boxed{e = 2.71828 \cdots} \qquad \boxed{e \approx 2.7} \tag{5}$$

Rem 1: Like π there are a lot of occasions in mathematics where the Eulerean number e occurs naturally. Therefore, we call e^x the natural exponential function. Here, we can motivate the number e only by the simple form of the power series (1). Later, we will also see that only with the basis e, the exponential function has the property that it is identical with its derivative, i.e. it satisfies the differential equation $y' = y$.

Rem 2: $\exp x$ *is defined* by its power series. That it is e to the power of x is a non-trivial theorem, not proved here.

4.**Q 8: General exponential function**

Give the formula for the general exponential function and give the names for the constants occuring in it.

(Solution:)

$$y = y(x) = ab^{cx} \tag{1}$$

a = **prefactor** [$\stackrel{G}{=}$ `Vorfaktor`]
b = Basis
c = **growth-constant** [$\stackrel{G}{=}$ `Wachstumskonstante`] ($-c$ = **decay-constant** [$\stackrel{G}{=}$ `Zerfallskonstante`]

4.**Ex 9: General powers on a calculator**
With a calculator calculate:
4.9. **a**) 2^3 (RESULT: 8)

4.9. **b**) 2^{-3} (RESULT: 0.1250)

4.9. **c**) $(-2)^3$ (RESULT: -8)

4.9. **d**) $3.54^{7.28}$ (RESULT: 9 925.3024)

4.9. **e**) $\pi^{\sin 13°}$ (RESULT: 1.2937)

4.**Ex 10: Simplification of general powers**

Calculate the following without using a calculator:
(Here, square roots are always understood to be positive.)

4.10. **a**) $(0.351)^0$ (RESULT: 1)

4.10. **b**) $(\pi^{\frac{1}{7}})^0$ (RESULT: 1)

4.10. **c**) $(0.5)^3 \cdot (0.5)^{-4} \cdot (0.5)^0$ (RESULT: 2)

(Solution:)

$$(0.5)^3 \cdot (0.5)^{-4} \cdot (0.5)^0 = 0.5^{3-4+0} = 0.5^{-1} = \frac{1}{0.5} = 2 \tag{1}$$

4.10. **d**) $4^{\frac{3}{2}}$
HINT: Write as $(4^{\frac{1}{2}})^3$.
RESULT: 8

4.10. **e**) $(2^2)^{1.5}$
HINT: Multiply the exponents.
RESULT: 8

(Solution:)

$$(2^2)^{1.5} = 2^{2 \cdot 1.5} = 2^3 = 8 \tag{2}$$

4.10. **f**) $32^{\frac{1}{5}}$
HINT: Try an integer
RESULT: 2

4.10. **g**) $\sqrt{18}\,\sqrt{2}$
HINT: Multiply bases.
RESULT: 6

(Solution:)

$$\sqrt{18}\,\sqrt{2} = 18^{\frac{1}{2}}2^{\frac{1}{2}} = 36^{\frac{1}{2}} = \sqrt{36} = 6 \tag{3}$$

4.10. **h**) Write $\sqrt{32}$ in a form with a radicand as small as possible.
HINT: Break down 32 into its factors.
RESULT: $4\sqrt{2}$

(Solution:)

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16}\sqrt{2} = 4\sqrt{2} \tag{4}$$

4.10. **i**) $4^{-\frac{1}{2}}$ (RESULT: $\frac{1}{2}$)

(Solution:)

$$4^{-\frac{1}{2}} = (4^{\frac{1}{2}})^{-1} = (\sqrt{4})^{-1} = 2^{-1} = \frac{1}{2} \tag{5}$$

4.Ex 11: Space diagonal in a cube

4.11. **a**) Calculate the length d of a **(space-) diagonal**[$\stackrel{\mathbf{G}}{=}$ `Raumdiagonale`] of a **cube**[$\stackrel{\mathbf{G}}{=}$ `Würfel`] with side lengths a. In particular for $a = 2$ m.

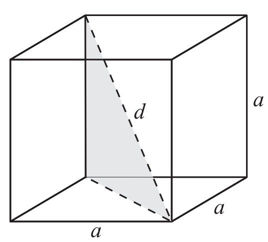


Fig4.11. **1**: Length d of space diagonal in a cube with sides lengths a.

HINT: Determine all sides with length a and all right angles. Use Pythagoras twice, first use it to calculate the **dotted**[$\stackrel{\mathbf{G}}{=}$ `punktiert`] surface diagonal.
RESULT:

$$d = \sqrt{3}a = 3.4641 \text{ m} \quad (1)$$

(Solution:)

$$d_1 = \text{ surface diagonal} \quad (2)$$

$$d_1^2 = a^2 + a^2 = 2a^2 \quad (3)$$

$$d^2 = d_1^2 + a^2 = 3a^2 \quad \Rightarrow \quad d = \sqrt{3}a = 3.4641 \text{ m} \quad (4)$$

4.11. **b**) The volume V of the cube is given. Calculate the area of its **surface**[$\stackrel{\mathbf{G}}{=}$ `Oberfläche`]. In particular for $V = 5$ cm^3.
RESULT:

$$A = 6V^{\frac{2}{3}} = 17.54 \text{ cm}^2 \quad (5)$$

(Solution:)

$$V = a^3 \quad a = V^{\frac{1}{3}}, \qquad A = 6a^2 = 6(V^{\frac{1}{3}})^2 = 6V^{\frac{2}{3}} = 17.54 \text{ cm}^2 \quad (6)$$

4.Ex 12: Mathematical properties of the square root function

Consider the function $y = \sqrt{x}$
(Here, a square root is understood to be a double valued symbol.)

4.12. **a**) Draw the graph of that function by constructing points for

$$x = 0, \quad x = 1, \quad x = 4, \quad x = 9, \quad x = 16. \quad (1)$$

4.12. **b**) Try $x = -1$ with your calculator.

4.12. **c**) What is the domain of that function?
RESULT:

$$\mathcal{D} = [0, \infty) \quad (2)$$

4.12. **d**) What is the range of that function?
RESULT:

$$(-\infty, \infty) \tag{3}$$

4.12. **e**) Is it a unique function?
RESULT: no, it is double valued:

$$y = \pm_{+}\sqrt{x} \tag{4}$$

4.12. **f**) Calculate its zeros.
HINT: Remove the square root by squaring.
RESULT: x = 0

(Solution:)

$$0 = \sqrt{x} \quad \Rightarrow \quad 0 = |x| \quad \Rightarrow \quad 0 = x \tag{5}$$

4.12. **g**) Show that it is not a periodic function.
HINT: Assume that it has period T. Remove square roots by squaring. Assume

$$x \geq 0, \quad T \geq 0. \tag{6}$$

Show that T = 0.

(Solution:)

$$\sqrt{x+T} = \sqrt{x} \quad \Rightarrow \quad |x+T| = |x| \quad \Rightarrow \quad x+T = x \quad \Rightarrow \quad T = 0 \tag{7}$$

4.**Ex 13: Calculation rules for powers**
Simplify.

REM: In general there is a matter of taste what is the simplest form, since there is no **unambiguous**[$\stackrel{G}{=}$ `unzweideutig`] definition of simplicity.

4.13. **a**) $(t^4)^3$ RESULT: t^{12}

4.13. **b)** $(\sqrt{a}\, e^{\frac{bx}{2}})^2$ RESULT: $a\, e^{bx}$

4.13. **c)** $x^{3+t}x^{-t}$ RESULT: x^3

4.13. **d)** $x^{2.5}x^{3.5}$ RESULT: x^6

4.13. **e)** $(x^2t^6)^{\frac{1}{2}}$ RESULT: xt^3

4.13. **f)** $(c\sqrt[3]{a})^9$ RESULT: c^9a^3

4.**Ex 14: Power series to calculate function values (trivial case)**
Using the power series calculate *exp* 0.
RESULT: $e^0 = 1$

4.**Ex 15: Power series to calculate function values (numeric example)**
In a formulary look up the power series for $\sin\varphi$ and calculate

$$y = \sin 1 = \sin 57.3^\circ \tag{1}$$

within a few decimal places.
RESULT:

$$\sin 1 \approx 0.8417 \tag{2}$$

(Solution:)

$$y = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots \tag{3}$$

$$\sin 1 \approx 1 - \frac{1}{6} + \frac{1}{120} = 0.8417 \tag{4}$$

4.**Ex 16: ☻☻ Reflecting the graph of a function**
Using the graph of $y = e^x$ derive the graph of $y = e^{-x}$

(Solution:)

It is obtained by a **mirror-symmetry**[$\stackrel{\mathbf{G}}{=}$ `Spiegelung`] at the y-axis.

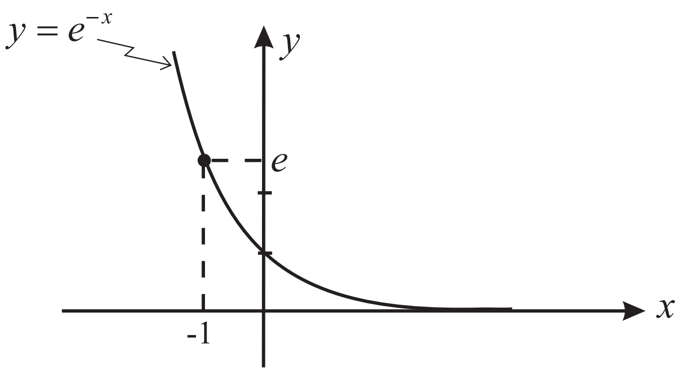


Fig4.16. **1**: Graph of $y = e^{-x}$

4.Ex 17: Evaluating a symbolic infinite sum

The following power series is valid:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{1}$$

Write out explicitly the first four (non-vanishing) terms of that infinite sum (i.e. for $n = 0, 1, 2, 3$).

(Solution:)

$$\begin{aligned} \cos x &= (-1)^0 \tfrac{x^0}{0!} + (-1)^1 \tfrac{x^2}{2!} + (-1)^2 \tfrac{x^4}{4!} + (-1)^3 \tfrac{x^6}{6!} - + \cdots \\ &= 1 - \tfrac{x^2}{2} + \tfrac{x^4}{24} - \tfrac{x^6}{720} + \cdots \end{aligned} \tag{2}$$

4.Ex 18: Power series used to prove an inequality

Using the power series prove $e^5 > e^{-5}$.

(Solution:)

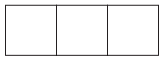
$$e^5 = 1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \cdots \tag{1}$$

$$e^{-5} = 1 - 5 + \frac{5^2}{2!} - \frac{5^3}{3!} + \cdots \tag{2}$$

$$5 > -5, \quad \frac{5^3}{3!} > -\frac{5^3}{3!} \quad \text{thus } e^5 > e^{-5} \tag{3}$$

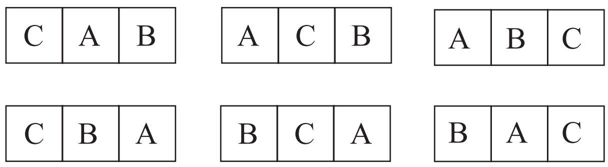
4.Ex 19: Permutations

4.19. **a**) Let A, B, and C be three people, and we have three rooms for them.



`Fig`4.19. **1**: Three rooms in a hotel waiting for three guests A, B, C.

Give all possible arrangements for them. What is the number N of these arrangements? In other words, give all **permutations** of the three elements A, B, C and what is the number N of permutations of the 3 elements.
HINT: First find all permutations of the elements, A, B, then find all possible places for C.
RESULT:



`Fig`4.19. **2**: All possibilities of **distributing**[$\stackrel{\mathbf{G}}{=}$ `verteilen`] the three guests A, B and C into the three rooms.

$$N = N_3 = 6 \tag{2}$$

(Solution:)

All possible **arrangements**[$\stackrel{\mathbf{G}}{=}$ `Anordnungen`] for two people

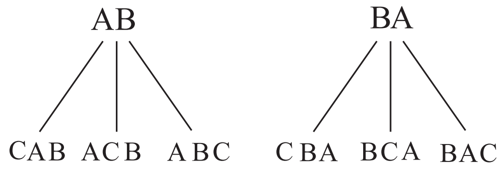


Fig4.19. **3**: For two people in two rooms, we have two cases: AB (upper line) and BA (lower line). C can be to the left, in the middle or to the right of them.

$$N_3 = N_2 \cdot 3 = 2 \cdot 3 = 3! = 6 \tag{4}$$

4.19. **b**) Using the same hint, find the number N of permutations of 4 elements ($N = N_4$) and generally for n elements ($N = N_n$).
RESULT:

$$N_4 = 4!, \qquad N_n = n! \tag{5}$$

(Solution:)

When

$$N_{n-1} = (n-1)! \tag{6}$$

we can arrange, for each case, the last ($= n^{th}$) element at n places, i.e.

$$N_n = N_{n-1} n \tag{7}$$

In particular

$$N_1 = 1! = 1 \tag{8}$$

$$N_2 = 2! = 1!\,2 = 2! = 2 \tag{9}$$

$$N_3 = 3! = 2!\,3 = 3! = 6 \tag{10}$$

$$N_4 = 3!\,4 = 4! = 24 \tag{11}$$

$$N_5 = 5! \tag{12}$$

$\dots \; \dots$

$$N_n = n! \tag{13}$$

4.**Ex 20: Indexed quantities arranged as matrices**

With

$$A_{ij} = 2i + j; \quad i = 1, 2, 3; \quad j = 1, 2, 3 \tag{1}$$

we have defined quantities A_{ij}.

4.20. **a**) Calculate A_{23}

RESULT: $A_{23} = 7$

(Solution:)

$$A_{23} = 2 \cdot 2 + 3 = 7 \tag{2}$$

4.20. **b**) Calculate all quantities A_{ij} and write them in **matrix** form.

$$A = (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \tag{3}$$

The first index (i) of the matrix distinguishes the **rows**[$\stackrel{\mathbf{G}}{=}$ `Zeilen`] and the second (j) the **columns**[$\stackrel{\mathbf{G}}{=}$ `Spalten`], i.e.

$$A = (A_{ij}) = \begin{array}{cc} & \xrightarrow{\quad j \quad} \\ i\big\downarrow & \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \end{array} \tag{3'}$$

RESULT:

$$A = \begin{pmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{pmatrix} \tag{4}$$

4.20. **c**) How many quantities A_{ij} do we have?

RESULT:

$$N = 3^2 = 9 \tag{5}$$

4.20. **d**) How many quantities does B_{ijk} have?

$$i, j, k = 1, 2, 3; \quad i, j, \text{ and } k \text{ run independently from 1 to 3.} \tag{6}$$

RESULT: $N = 27$

(Solution:)

$$N = 3 \cdot 3 \cdot 3 = 3^3 = 27 \tag{7}$$

4.20. **e**) The same question for

$$C_{ijkl}\ ; \qquad i,j,k,l = 1, \dots , m \tag{8}$$

RESULT:

$$N = m^4 \tag{9}$$

(Solution:)

$$N = m \cdot m \cdot m \cdot m = m^4 \tag{10}$$

4.20. **f**) The same question for

$$D_{i_1, \dots i_n}\ ; \qquad i_1, \dots i_n = 1, \cdots , p \tag{11}$$

RESULT: p^n

(Solution:)

$$N = \underbrace{p \cdot p \cdots p}_{n-times} = p^n \tag{12}$$

5 Approximations

Only very few problems can be treated exactly. Therefore, approximative methods are a very important branch in physics.

5.**Q 1: Approximation of numbers**
$\pi = 3.14159\ 26535\cdots$
5.1. **a**) Give π **approximately**[$\stackrel{\mathbf{G}}{=}$ `näherungsweise`] where all decimals are **truncated**[$\stackrel{\mathbf{G}}{=}$ `abgeschnitten`] except the first four.

(Solution:)

$\pi \approx 3.1415$ (**truncation**)

5.1. **b**) Give the best decimal approximation of π to four decimal digits.

(Solution:)

$\pi = 3.1416$ (**rounding**)

5.1. **c**) What is the **absolute error**[$\stackrel{\mathbf{G}}{=}$ `absoluter Fehler`] and what is the **relative error**[$\stackrel{\mathbf{G}}{=}$ `relativer Fehler`] when π is approximated by $\pi_0 = 3$?

Rem 1: In a) b) c) we have used 3 different notations to denote an approximation:

a) $\approx$ instead of $=$

b) $=$ because it is known from the context, that $=$ is only an approximative equality.

c) Using a new symbol (e.g. π_0) for the approximative value.

Rem 2: As usual in physics, errors itself are calculated only approximatively.

(Solution:)

absolute error: $\Delta = \pi - \pi_0 = 0.14159\cdots$
relative error: $\varepsilon = \frac{\Delta}{\pi}100\% = \frac{14.159}{\pi}\% = 4.5\%$

5.1. **d**) Write π in the form

$$\pi = 3\cdot(\frac{1}{10})^0 + 1\cdot(\frac{1}{10})^1 + 4\cdot(\frac{1}{10})^2 + 1\cdot(\frac{1}{10})^3 + 5\cdot(\frac{1}{10})^4 + \cdots$$

and consider $x = \frac{1}{10} = 0.1$ as a **small quantity of first order**.
What is π in **second order of approximation** (inclusive)?

(Solution:)

$\pi = 3.14$

5.1. **e**) The same in **linear approximation** ($\equiv$ first order approximation, inclusive).

(Solution:)

$\pi = 3.1$

5.1. **f**) The same in zeroth order approximation (inclusive).

(Solution:)

$\pi = 3$

5.**Q 2: Approximation of functions**

Let $f(x) = 3 + 2x + x^3$

5.2. **a**) Calculate f(0.1)

(Solution:)

$3 + 0.2 + 0.001 = 3.201$

5.2. **b**) Calculate f(0.01)

(Solution:)

$3 + 0.02 + 0.000\,001 = 3.020\,001$

5.2. **c**) Consider x to be a **small quantitiy of first order** (which symbolically is written as $x \ll 1$).

Calculate f(x) in **linear approximation** (i.e. in first order, inclusive).

REM: For large x, $(x \gg 1)$ x^3 is **dominant**.

For small x, $(x \ll 1)$ x^3 can be neglected.

(Solution:)

$f(x) = 3 + 2x$

5.2. **d**) What is the relative error in case of $x = 0.1$ if the linear approximation of $f(x)$ is used?

(Solution:)

$$\varepsilon = \frac{f(0.1) - f_{\text{approx}}(0.1)}{f(0.1)} = \frac{3.201 - 3.20}{3.201} \approx \frac{0.001}{3} \approx 0.0003 = 0.3‰$$

5.2. **e**) Give $f(x)$ in second order (inclusive).

(Solution:)

$f(x) = 3 + 2x$, i.e. the same as linear approximation since second order contributions **vanish**[$\stackrel{\mathbf{G}}{=}$ `verschwinden`], i.e. are absent.

5.2. **f**) Give the zeroth order of approximation.

(Solution:)

$f(x) = 3$

5.2. **g**) Give the lowest (non-vanishing) approximation for $f(x)$.

(Solution:)

$f(x) = 3$

5.2. **h**) For $g(x) = 2x + x^3$ give the lowest (non-vanishing) order of approximation. What order is that?

(Solution:)

$g(x) = 2x$; it is the first order of approximation.

5.2. **i**) For $h(x) = x^2 + 2x^3$ what is $h(x)$ in linear approximation?

(Solution:)

$h(x) = 0$

5.2. **j**) Why is the first order approximation also called a **linear approximation**?

(Solution:)

The graph of the linear approximation is a straight line. In old fashioned terminology 'line' was a straight line, wheras 'curve' was arbitrary.

5.**Q 3: For sufficiently small x the linear approximation is always valid.**
Let $f(x) = 1 + 1000x^2$. The linear approximation is $f_1(x) = 1$, i.e. we write

$$\boxed{f(x) \approx 1 \quad \text{for } x \ll 1}$$

What is the meaning of $x \ll 1$ when we want an accuracy within 1‰?
(For reasons of simplicity, we restrict ourselves to the domain $\mathcal{D} = [0, \infty)$.)
HINT: First determine the x for which the relative error is 1‰.

(Solution:)

$$0.001 = \varepsilon = \frac{f(x) - f_1(x)}{f(x)} = \frac{1000x^2}{1 + 1000x^2}$$

$$0.001 + x^2 = 1000x^2$$

$$0.001 = 999x^2$$

$$x^2 = \frac{0.001}{999} \approx \frac{0.001}{1000} = (0.001)^2$$

Thus in our case $x \ll 1$ means $0 \leq x < 0.001$.

REM: In calculating errors, the lowest (non-vanishing) approximation is used, e.g. 999 is replaced by 1000.

FACIT: $x \ll 1$ means 'sufficiently small'. What this means concretely depends on the particular case.

5.**Q 4: Approximations can save calculation time**
Let $f(x) = (3x + 1)^3$ with x being a small quantity (of first order, $x \ll 1$).
5.4. **a**) Calculate f(x) exactly.

(Solution:)

$$f(x) = (3x+1)(9x^2+6x+1) = 27x^3 + 18x^2 + 3x + 9x^2 + 6x + 1$$

$$\boxed{f(x) = 27x^3 + 27x^2 + 9x + 1}$$

TERMINOLOGY: 1 is called the zeroth order term (or contribution). $9x$ is called the first order term (or contribution), ... , $27x^3$ is called the third order term (or contribution).

5.4. **b**) Calculate $f(x)$ directly in linear approximation.

(Solution:)

After having calculated

$$f(x) = (3x+1)(9x^2+6x+1)$$

we can omit the second order term $9x^2$ since it will give second order or third order contributions in the result for $f(x)$, i.e.

$$f(x) \approx (3x+1)(6x+1) = 6x + 1 + 3x$$

where we have immediately omitted the second order contribution $3x \cdot 6x$.
RESULT: $f(x) \approx f_1(x) = 1 + 9x$

5.Q 5: Several small quantities

Let

$$f(x,y) = (3x+1)(2y+1)^2 \tag{1}$$

be a **function of two variables** x and y.

A function f of two (independent) variables x and y is a prescription which when x and y are given uniquely specifies a function value $f(x,y)$, e.g. that one given by (1).

For sufficiently small x and y (i.e. $x \ll 1, y \ll 1$) we would like to calculate $f(x,y)$ in linear approximation.
REM: Both x and y are small quantities of the first order, i.e. xy is already a contribution of second order.

(Solution:)

$$f(x,y) = (3x+1)(\underline{4y^2} + 4y + 1) \qquad \text{(The underlined term can be neglected.)}$$

$$f(x,y) \approx (3x+1)(4y+1) = 3x + 4y + 1 \qquad \text{(neglecting } 3x \cdot 4y\text{)}.$$

5.Ex 6: Power series for some important cases

Look up the following cases in a formulary.

$$\boxed{\sin x = x - \frac{1}{6}x^3 + \cdots} \tag{1}$$

$$\boxed{\cos x = 1 - \frac{1}{2}x^2 + \cdots} \tag{2}$$

$$\boxed{\tan x = x + \frac{1}{3}x^3 + \cdots} \qquad (|x| < \frac{\pi}{2}) \tag{3}$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots} \qquad (-1 < x \leq 1) \tag{4}$$

$$\boxed{\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots} \qquad (|x| < 1) \tag{5}$$

$$\boxed{\frac{1}{1+x} = 1 - x + x^2 - \cdots} \qquad (|x| < 1) \tag{6}$$

$$\boxed{(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots} \qquad (|x| < 1) \tag{7}$$

5.**Ex 7: Fast calculations using approximations**
Calculate

$$f(x) = (3x+1)^3 \tag{1}$$

in zeroth order of approximation of the small quantity x (i.e. x is by definition a quantity of first order small).
RESULT: $f(x) \approx 1$

(Solution:)

$$f(x) = (3x+1)(3x+1)(3x+1) \tag{2}$$

In each factor $(3x+1)$ we can omit the first order **contribution**[$\stackrel{G}{=}$ `Beitrag`] $3x$ since, in the result, it would lead to a first order contribution (or higher). Thus,

$$f(x) \approx \overset{(0)}{f}(x) = 1 \cdot 1 \cdot 1 = 1 \tag{3}$$

REM: The superscript (0) indicates that we have the zeroth-order contribution of the quantity.

5.**Ex 8: Linear approximation in a simple case**

Calculate $f(x) = (1+x)^{100}$ in a linear approximation for $x \ll 1$.
HINT: First solve the problem $(1+x)^n$ for $n = 2, 3, \cdots$. Check that for $n = 3$ the result for $n = 2$ in linear approximation was sufficient.
RESULT: $f(x) = 1 + 100x$

(Solution:)

$$f(x) = \underbrace{\underbrace{(1+x)(1+x)}_{1+2x}(1+x)}_{1+3x}\cdots(1+x) \tag{1}$$

Where in each intermediate step we have omitted quadratic (i.e. second order) contributions. Thus

$$f(x) \approx f_1(x) = 1 + 100x \tag{2}$$

5.**Ex 9: Linear approximation of transcendental functions**

5.9. **a**) For $x \ll 1$ calculate $f(x) = \sin x\, e^x$ in first order approximation.
HINT: Use the power series

$$\sin x = x - \frac{1}{6}x^3 + \cdots\,, \tag{1}$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots\,. \tag{2}$$

RESULT:

$$f(x) \approx x \tag{3}$$

(Solution:)

$$f(x) = (x - \frac{1}{6}x^3 + \cdots)(1 + \underline{x + \frac{1}{2}x^2 + \cdots}) \tag{4}$$

The third order contribution $-\frac{1}{6}x^3$ gives, in the result, third order contributions or higher, i.e. they can be omitted. The underlined terms x and $\frac{1}{2}x^2$ both give second order terms or higher, i.e. they can be omitted. Thus

$$f(x) \approx \overset{(1)}{f}(x) = x \tag{5}$$

5.9. **b**) The same in second order approximation.
RESULT:

$$f(x) \approx x + x^2 \tag{6}$$

(Solution:)

$$f(x) \approx (x) \cdot (1 + x) = x + x^2 \tag{7}$$

5.Ex 10: ☻ Different meanings of 'very small'

In the following table you will see

$$\varphi, \quad \sin\varphi, \quad \frac{\varphi}{\sin\varphi} \tag{1}$$

for some small values of φ.
(For reasons of simplicity, we assume φ to be a non-negative quantity: $\varphi \geq 0$.)

φ	$\sin\varphi$	$\frac{\varphi}{\sin\varphi}$
$0° = 0$	0	?
$1° = 0.0175$	0.0175	1
$2° = 0.0349$	0.0349	1
$3° = 0.0524$	0.0523	1
$4° = 0.0698$	0.0698	1
$5° = 0.0873$	0.0872	1.0011
$6° = 0.1047$	0.1045	1.0019
$7° = 0.1222$	0.1219	1.0025
$8° = 0.1396$	0.1392	1.0029
$9° = 0.1571$	0.1564	1.0045
$10° = 0.1745$	0.1736	1.0052
$11° = 0.1920$	0.1908	1.0063
$12° = 0.2094$	0.2079	1.0072
$13° = 0.2269$	0.2250	1.0084
$14° = 0.2443$	0.2419	1.0099
$15° = 0.2618$	0.2588	1.0116
$16° = 0.2793$	0.2756	1.0134
$17° = 0.2967$	0.2924	1.0147
$18° = 0.3142$	0.3090	1.0168
$19° = 0.3316$	0.3256	1.0184
$20° = 0.3491$	0.3420	1.0208
$21° = 0.3665$	0.3584	1.0226
$22° = 0.3840$	0.3746	1.0251
$23° = 0.4014$	0.3907	1.0274
$24° = 0.4189$	0.4067	1.0300
$25° = 0.4363$	0.4226	1.0324
$26° = 0.4538$	0.4384	1.0351
$27° = 0.4712$	0.4540	1.0379
$28° = 0.4887$	0.4695	1.0409
$29° = 0.5061$	0.4848	1.0439
$30° = 0.5236$	0.5	1.0472
$31° = 0.5411$	0.5150	1.0507
$32° = 0.5585$	0.5299	1.0540
$33° = 0.5760$	0.5446	1.0577
$34° = 0.5934$	0.5592	1.0612
$35° = 0.6109$	0.5736	1.0650
$36° = 0.6283$	0.5878	1.0689
$37° = 0.6458$	0.6018	1.0731
$38° = 0.6632$	0.6157	1.0771
$39° = 0.6807$	0.6293	1.0817
$40° = 0.6981$	0.6428	1.0860

In the literature, for *small values* of φ the following approximation is recommended:

$$\boxed{\sin\varphi \approx \varphi} \qquad \text{for } \varphi \ll 1 \tag{2}$$

5.10. **a**) If you want an *accuracy within 1%* what is the meaning of 'small φ' (i.e. what is the meaning of '$\varphi \ll 1$')?

HINT: A relative error less than ε (e.g. $\varepsilon = 1\% = 0.01$) means $\frac{\varphi - \sin\varphi}{\sin\varphi} = \frac{\varphi}{\sin\varphi} - 1 < \varepsilon$ i.e. $\frac{\varphi}{\sin\varphi} < 1 + \varepsilon$.
Consult the previous table.

RESULT:

$$\varphi \ll 1 \quad \text{means} \quad \varphi \leq 14^\circ \tag{3}$$

5.10. **b**) The same for 5%.
RESULT:

$$\varphi \ll 1 \quad \text{means} \quad \varphi \leq 30^\circ \tag{4}$$

5.Ex 11: Area of a ring in linear approximation of width

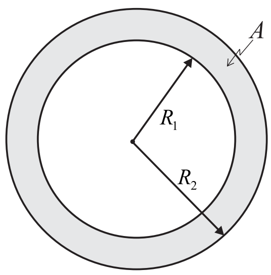


Fig5.11. **1**: Area of ring with radii R_1 and R_2

Calculate the area A of the shaded ring with inner radius $R_1 = R$ and outer radius $R_2 = R + h$ in linear approximation in the small quantity h.

HINT: Calculate the area of the circles, e.g. $A_1 = \pi R_1^2$.
RESULT:

$$A = 2\pi R h \tag{1}$$

(Solution:)

$$A = \pi R_2^2 - \pi R_1^2 = \pi[(R+h)^2 - R^2] = \pi[R^2 + 2Rh + h^2 - R^2] \tag{2}$$

Neglecting h^2 yields:

$$A = \pi 2Rh \tag{3}$$

5.**Ex 12: Propagation of error**
(**propagation of errors**[$\stackrel{G}{=}$ `Fehler-Fortpflanzung`]) In a laboratory there is a rectangle and a student is asked to measure the area of the rectangle. He/she measures length (a) and width (b) and calculates the area using the formula

$$A = ab \tag{1}$$

The exact values (unknown to the student) are

$$a_0 = 1 \text{ m}, \quad b_0 = 1 \text{ m} \tag{2}$$

Instead he/she measures

$$a_0 = 1.001 \text{ m}, \quad b_0 = 1.002 \text{ m} \tag{3}$$

What are the absolute errors Δa, Δb, and the relative errors ε_a, ε_b? What is the absolute error ΔA and the relative error ε_A in the result (ΔA and ε_A should be calculated in lowest order (non-vanishing) approximation (in the small quantities Δa, Δb)).
RESULT:

$$\Delta a = 1 \text{ mm}, \quad \Delta b = 2 \text{ mm}, \quad \varepsilon_a = \frac{1 \text{ mm}}{1 \text{ m}} = 1‰ \tag{4}$$

$$\varepsilon_b = 2‰, \quad \Delta A = 30 \text{ cm}^2, \quad \varepsilon_A = 3‰ \tag{5}$$

(Solution:)

The student calculates

$$A = (1 \text{ m} + \underbrace{0.001 \text{ m}}_{\Delta a})(1 \text{ m} + \underbrace{0.002 \text{ m}}_{\Delta b}) \tag{6}$$

In linear approximation (in the small quantities $\Delta a, \Delta b$):

$$A = 1 \text{ m}^2 + \underbrace{0.001 \text{ m}^2 + 0.002 \text{ m}^2}_{\Delta A} \tag{7}$$

$$\Delta A = 0.003 \text{ m}^2 = 30 \text{ cm}^2 \tag{8}$$

$$\varepsilon_A = \frac{\Delta A}{A} = \frac{0.003 \text{ m}^2}{1 \text{ m}^2} 100\% = 0.3\% = 3‰ \tag{9}$$

FACIT:

$$\boxed{\begin{array}{l}\text{When a quantity } (A) \text{ is the product of two quantities } (a \\ \text{and } b) \text{ the relative errors are additive:} \\ \qquad\qquad\qquad \varepsilon_A = \varepsilon_a + \varepsilon_b\end{array}} \tag{10}$$

REM: This is the worst case. In particular cases errors can cancel each other out and, by **coincidence**[$\overset{\mathbf{G}}{=}$ `Zufall`], lead to a better result.

5.**Ex 13: ☻☻ Properties of the exponential function proved approximately**
Using its power series representation, calculate $y = e^x$ in several orders of approximation (inclusive) for $x \ll 1$.
5.13. **a**) 4^{th} order
RESULT:

$$y = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 \tag{1}$$

5.13. **b**) Linear approximation (i.e. first order)
RESULT:

$$y = 1 + x \tag{2}$$

5.13. **c**) 0^{th} order
RESULT:

$$y = 1 \tag{3}$$

5.13. **d**) Lowest (non-vanishing) order
RESULT:

$$y = 1 \tag{4}$$

5.13. **e**) Prove

$$\boxed{e^{2x} = (e^x)^2} \tag{5}$$

in second order approximation.

(Solution:)

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2} \tag{6}$$

On the other hand

$$(e^x)^2 = (1 + x + \frac{1}{2}x^2 + \boxed{?}x^3 + \cdots)(1 + x + \frac{1}{2}x^2 + \boxed{?}x^3 + \cdots) = \tag{7}$$

($\boxed{?}$ means these values are irrelevant.)

$$= \underbrace{1 + x + \frac{1}{2}x^2} + \underbrace{x + x^2} + \frac{1}{2}x^2 \tag{8}$$

This is the same as (6). q.e.d.

5.13. **f**) Let x and y both be small quantities of first order small. Prove

$$\boxed{e^x e^y = e^{x+y}} \tag{9}$$

in second order approximation.

REM: (5) and (9) are valid exactly. However, we did prove them only in certain approximations.

(Solution:)

$$e^x e^y = (1 + x + \frac{1}{2}x^2 + \boxed{?}x^3 + \cdots)(1 + y + \frac{1}{2}y^2 + \boxed{?}y^3 + \cdots) = \tag{10}$$

$$= \underbrace{1 + y + \frac{1}{2}y^2} + \underbrace{x + xy} + \frac{1}{2}x^2 \qquad (+) \tag{11}$$

On the other hand

$$e^{x+y} = 1 + (x + y) + \frac{1}{2}(x + y)^2 = 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 \tag{12}$$

Which is the same as (11) q.e.d.

5.Ex 14: Pseudoprobability in the decimal expansion of π (probability [$\stackrel{\mathbf{G}}{=}$ Wahrscheinlichkeit])

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69399\ 37510 \cdots \tag{1}$$

For each decimal digit (0, 1, 2, …) (after the decimal point) determine the **frequency** [$\stackrel{\mathbf{G}}{=}$ Häufigkeit] in which it occurs within the first 50 decimals of π. Give the answer in the form of a table and as a histogram. For each decimal digit also calculate the probability for its occurence within the first 50 decimals.

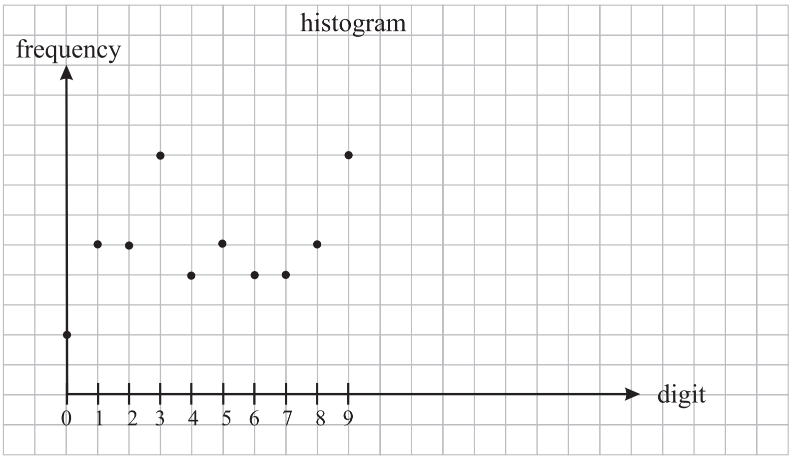
REM 1: These are *pseudo*-probabilities since they are mathematically fixed i.e. forseeable. An example of a true probability is roulette.

REM 2: Another term for 'pseudo-probability' is '**deterministic chaos**'.

REM 3: In the irrational number π each decimal occurs with the same probability, i.e. 10%. Our deviations from 10% are the natural variations (**variance**[$\stackrel{G}{=}$ `Streuung`]) since we have only considered a finite **sample**[$\stackrel{G}{=}$ `Stichprobe`].

(Solution:)

digit	frequ.	probability
0	2	2/50=4%
1	5	5/50=10%
2	5	5/50=10%
3	8	8/50=16%
4	4	4/50=8%
5	5	5/50=10%
6	4	4/50=8%
7	4	4/50=8%
8	5	5/50=10%
9	8	8/50=16%
total	50	100%



`Fig`5.14. **1**: Histogram for the frequency of occurrence of decimal digits in the first 50 decimals of π

5.**Ex 15: A power series defined by the digits of π**

Consider a function $f(x)$ given by the following power series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

where a_n are the decimal digits of

$$\pi = 3.14159\cdots, \tag{2}$$

$$\text{e.g.} \quad a_0 = 3, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 1, \quad a_5 = 5, \quad a_6 = 9, \cdots \tag{3}$$

Calculate $f(0.1)$.
RESULT:

$$f(0.1) = \pi \tag{4}$$

6 Logarithms

6.**Q 1: Examples for logarithms**

Give the solution of the following equations and give the mathematical **notations**[$\stackrel{G}{=}$ `Bezeichnungsweisen`] (in each case in 5 versions) for the **obtained**[$\stackrel{G}{=}$ `erhalten`] quantity.

6.1. **a**) $1000 = 10^n$

(Solution:)

$n = 3 = \log_{10}\ 1000 = \lg 1000 =$
$=$ **decimal logarithm**[$\stackrel{G}{=}$ `Zehner-Logarithmus`] of $1000 =$
$=$ **decadic logarithm**[$\stackrel{G}{=}$ `dekadischer Logarithmus`] of $1000 =$
$=$ logarithm to the base 10 of $1000 =$
$=$ logarithmus decimalis of 1000

6.1. **b**) $16 = 2^n$

(Solution:)

$n = 4 = \log_2\ 16 = \mathrm{ld}\ 16 =$
$=$ **dual logarithm**[$\stackrel{G}{=}$ `Zweier-Logarithmus`] of $16 =$
$=$ logarithm to the base 2 of $16 =$
$=$ logarithmus dualis of 16

6.1. **c**) $1 = e^n$

(Solution:)

$n = 0 = \log_e 1 = \ln 1 =$
$=$ natural logarithm of $1 =$
$=$ logarithm to the base e of $1 =$
$=$ logarithmus naturalis of 1

Rem: The notation $\log x$, though widely used, is **ambiguous**[$\stackrel{G}{=}$ `zweideutig`]. In mathematical texts it means $\ln x$, in technical texts it can also mean $\lg x$.

6.**Q 2: General logarithms**

6.2. **a**) What is the meaning of $\log_b x$ (in words and in formulae)

We restrict ourselves to the case that the base b is positive: $b > 0$

(Solution:)

$\log_b x$ (the logarithm to base b of x) is that number when taken as the exponent to the base b gives the **numerus** x, in formulae: see (3).

6.2. **b**) Draw its graph for $b = e$

(Solution:)

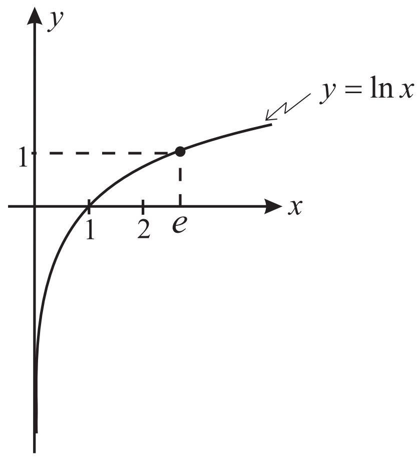


Fig6.2. **1**: Graph of the (natural) logarithm

6.2. **c**) Give the limits for $x \to 0_+$ and for $x \to \infty$.
REM: 0_+ means means that x goes to 0 with the restriction $x > 0$ (approximation from the right hand side).

(Solution:)

$$\lim_{x \to 0_+} \log_b x = -\infty \tag{1}$$

$$\lim_{x \to \infty} \log_b x = \infty \tag{2}$$

6.2. **d**) Write down the formula which says that taking the logarithm is the inverse of **raising to a power**[$\stackrel{\mathbf{G}}{=}$ `potenzieren`].

(Solution:)

$$\boxed{b^{\log_b x} = x} \qquad \boxed{\log_b(b^x) = x} \tag{3}$$

6.2. **e**) Show $\boxed{\log_b 1 = 0}$

(Solution:)

$b^0 = 1 \qquad (b \neq 0)$

6.2. **f**) Show $\boxed{\log_b b = 1}$

(Solution:)

$b^1 = b$

6.**Q 3: Calculation rules for logarithms**

Let $\log x$ be the logarithm to an arbitrary (but within a **particular**[$\stackrel{\mathbf{G}}{=}$ `festgelegt`] formula fixed) basis b:

$$\log x := \log_b x \tag{1}$$

Give the the formulae for:

6.3. **a**) $\log(xy) = ?$

(Solution:)

$$\boxed{\log(xy) = \log x + \log y} \tag{2}$$

The logarithm of a product is
the sum of the logarithms of the factors

6.3. **b**) $\log(x^y) = ?$

(Solution:)

$$\boxed{\log x^y = y\ \log x} \tag{3}$$

6.3. **c**) Derive the formula for $\log \frac{x}{y}$

(Solution:)

$$\log \frac{x}{y} = \log(xy^{-1}) = \log x + \log y^{-1} = \tag{4}$$

$$\boxed{\log \frac{x}{y} = \log x - \log y} \tag{5}$$

The logarithm of a quotient is the difference
of the logarithms of nominator and denominator

6.3. **d**) Derive the formula for $\log \sqrt[y]{x}$

(Solution:)

$$\log \sqrt[y]{x} = \log x^{\frac{1}{y}} = \frac{1}{y} \log x \tag{6}$$

$$\boxed{\log \sqrt[y]{x} = \frac{1}{y} \log x} \tag{7}$$

HIST 1: The rules (2)(3)(5)(7) have been important at former times to reduce multiplication, division and exponentiation to the much simpler process of addition and subtraction. To calculate xy in (2) with the help of a so called logarithmic table, one had to look up $\log x$ and $\log y$ and to add them. Then one had to use the table inversely (which is simple because log is a monotonic function) to look up the result xy.

HIST 2: The word 'logarithm' comes from the Greak 'logos' and 'arithmos' = number. Among the different meanings of 'logos' the translation '$\lambda o\gamma o\sigma$' = 'intrinsic meaning' seems most appropriate here. Therefore, the logarithm is a second, *intrinsic* number living inside the original number (= numerus).

6.**Q 4: Decay laws**

$$\boxed{N(t) = N_0\, e^{-\lambda t}} \tag{1}$$

is the law for **radioactive decay** [$\stackrel{\mathbf{G}}{=}$ `radioaktiver Zerfall`], with
$N(t)$ = number of radioactive atoms at time t,
λ = **decay** [$\stackrel{\mathbf{G}}{=}$ `Zerfall`] constant (decay **rate** [$\stackrel{\mathbf{G}}{=}$ `Rate`])

6.4. **a**) What's the number of radioactive atoms at the **initial** [$\stackrel{\mathbf{G}}{=}$ `Anfangs-`] time $t = 0$?

(Solution:)

$t = 0: \quad N(0) = N_0$

6.4. **b**) What's the meaning of the **half life time** [$\stackrel{\mathbf{G}}{=}$ `Halbwertszeit`], and express it by λ

(Solution:)

half decay time $T = T_{\frac{1}{2}} =$
= time **lapse** [$\stackrel{\mathbf{G}}{=}$ `Zeit-Spanne`] until the number of radioactive atoms is only half its initial number:

$$N(t+T) = \frac{1}{2}\, N(t) \tag{2}$$

$$N_0\ e^{-\lambda(t+T)} = \frac{1}{2}\ N_0\ e^{-\lambda t} \tag{3}$$

$$e^{-\lambda T} = \frac{1}{2} \tag{4}$$

Taking ln on both sides of that equation:

$$\ln e^{-\lambda T} = \ln \frac{1}{2} \tag{5}$$

$$-\lambda T = -\ln 2 \tag{6}$$

$$\boxed{T = T_{\frac{1}{2}} = \frac{\ln 2}{\lambda}} \tag{7}$$

REM: This half decay time T at a later time t is the same, i.e. it is independent of the time t which was chosen as the initial time in (2).

6.**Q 5: ☻ Transforming logarithms to a different base**
Express $\log_a x$ by $\log_b x$, and express the result in words.

(Solution:)

$$\log_a x = \log_a (b^{\log_b x}) = \log_b x\ \ \log_a b \tag{1}$$

$$\boxed{\log_a x = \log_b x\ \ \log_a b} \tag{2}$$

Logarithms of different bases differ only by a factor ($k = \log_a b$)

6.**Ex 6: Logarithms calculated with a calculator**
With a calculator calculate:
HINT: On most calculators lg is the key 'log'.

6.6. **a**) $\lg 100$ RESULT: 2

6.6. **b**) $\lg 110$ RESULT: 2.0414

6.6. **c**) e HINT: Calculate $e^1 = \exp(1)$ RESULT: 2.7183

6.6. **d**) $\ln \pi$ RESULT: 1.1447

6.6. **e**) $10^{\lg 13}$ RESULT: 13

6.6. **f**) $\lg 10^{13}$ RESULT: 13

6.**Ex 7: Simplification of logarithms (numeric arguments)**
Calculate without using a calculator:
6.7. **a**) $\lg 10\,000$ RESULT: 4

6.7. **b**) $\lg \frac{1}{10\,000}$ RESULT: -4

6.7. **c**) $\lg 1$ RESULT: 0

6.7. **d**) $\log_2 16$ RESULT: 4

6.7. **e**) $\ln e^{\frac{5}{2}-2}$ RESULT: 0.5

6.7. **f**) $\frac{(\lg 1000)}{\ln e^3}$ RESULT: $\frac{3}{3} = 1$

6.**Ex 8: Simplification of logarithms (algebraic arguments)**
Simplify.
6.8. **a**) $e^{\ln \pi}$ RESULT: π

6.8. **b**) $\ln e^{\sqrt{\pi}}$ RESULT: $\sqrt{\pi}$

6.8. **c**) $\ln \left(\frac{a}{b^2}\right)^4 - 4\ln a + \ln b^8$ RESULT: 0

(Solution:)

$$\begin{aligned} \ln \left(\tfrac{a}{b^2}\right)^4 - 4\ln a + \ln b^8 &= 4\ln \left(\tfrac{a}{b^2}\right) - 4\ln a + 8\ln b = \\ &= 4\ln a - 4\underbrace{\ln b^2}_{2\ln b} - 4\ln a + 8\ln b = 0 \end{aligned}$$

6.8. **d**) $\frac{2\ln \sqrt{\pi}}{\ln \pi}$ RESULT: 1

(Solution:)

$$\frac{2 \cdot \frac{1}{2} \ln \pi}{\ln \pi} = 1$$

6.**Ex 9: Equations involving logarithms**
Solve the following equations for x.
6.9. **a**)

$$\lg x - 3 = 0 \tag{1}$$

What does that equation mean:

$$(\lg x) - 3 = 0 \tag{2}$$

or,

$$\lg(x-3)=0 \tag{3}$$

Why?
HINT: Put $\lg x$ on one side of the equation then raise 10 to the power of each side: $10^{\text{left hand side}} = 10^{\text{right hand side}}$
RESULT: $x = 1000$

(Solution:)

Functional arguments bind higher than addition or multiplication. Thus the equation means

$$(\lg x) - 3 = 0 \tag{4}$$

From that

$$\lg x = 3, \qquad 10^{\lg x} = 10^3, \qquad x = 1000 \tag{5}$$

6.9. **b**)

$$\ln(x-1) = \ln 3 \tag{6}$$

HINT: remove ln by applying exp to both sides.
RESULT:

$$x = 4 \tag{7}$$

(Solution:)

$$e^{\ln(x-1)} = e^{\ln 3}, \qquad x - 1 = 3, \qquad x = 4 \tag{8}$$

6.9. **c**)

$$I = I_0 e^{-\lambda x} \tag{9}$$

HINT: Isolate x on one side of the equation. Remove e by applying ln to both sides.

(Solution:)

$$\frac{I}{I_0} = e^{-\lambda x}, \qquad \ln\frac{I}{I_0} = -\lambda x, \qquad x = -\frac{1}{\lambda}\ln\frac{I}{I_0} \tag{10}$$

7 Number systems, dimensional quantities

7.**Q 1: Number Systems**
Give the notation for the following sets of numbers, and in each case, give some examples (i.e. elements of the set) and distinguish the case the number zero is excluded.

7.1. **a**) natural numbers

(Solution:)

$$\begin{aligned} \mathbb{N}_o &= \{0,1,2,3,\cdots\} \\ \mathbb{N}^* &= \{\ \ 1,2,3,\cdots\} \\ \mathbb{N}\ &= \{0,1,2,3,\cdots\} \end{aligned} \tag{1}$$

REM 1: Nowadays, 0 is counted as a natural number, therefore the simpler notation $\mathbb{N}$ can be used. But this convention is not uniquely obeyed, and especially in older literature 0 is not counted as a natural number. Thus, the more explicit notations $\mathbb{N}_o$ and $\mathbb{N}^*$ are useful.

REM 2: In **set-theoretic notation** we can write

$$\begin{aligned} \mathbb{N}_o &= \mathbb{N}^* \cup \{0\} \\ \mathbb{N}^* &= \mathbb{N}_o - \{0\} \end{aligned} \tag{2}$$

Here, we see the set brackets $\{\cdots\}$ denoting a set, e.g. $\{0\}$ denotes the set consisting of a single element, namely the number 0. $\cup$ denotes the union of sets.
- denotes subtraction of sets (also denoted by \).

HIST: Kronecker: God made the natural numbers. Everything else is the work of man. (At Kronecker's time 0 was not counted as a natural number.)

7.1. **b**) integers

(Solution:)

$\mathbb{Z} = \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$
$\mathbb{Z}^*$ excludes zero: $\mathbb{Z}^* = \mathbb{Z} - \{0\}$

7.1. **c**) rational numbers

(Solution:)

$\mathbb{Q}$: All fractions, i.e. all numbers of the form $\frac{n}{m}$ with $n \in \mathbb{Z}$, $\quad m \in \mathbb{N}^*$

REM: $\mathbb{Q}$ reminds of **q**uotient.

EXAMPLES: $-5/7$, $\quad 5.8541\overline{731}731731\cdots$, indeed every decimal number which at a certain point becomes periodic, can be brought into the form n/m, i.e. is rational.

(Of course, a finite decimal number can be viewed as an infinite decimal number which becomes periodic.)

$\mathbb{Q}^*$ exludes zero.

7.1. **d**) real numbers

(Solution:)

$\mathbb{R}$: All (infinite) decimal numbers (including the periodic and finite ones).

$\mathbb{R}^*$ exludes zero.

REM: The distinction $\mathbb{R}^*$ from $\mathbb{R}$ is important, because division by zero is forbidden, i.e. not defined.

7.1. **e**) irrational numbers

(Solution:)

$\mathbb{R} - \mathbb{Q}$ i.e. all real numbers which are not rational, namely all decimal numbers, which never become periodic.
Examples: $\pi, e, \sqrt{2}$

REM 1: We have the following **subset relations** [$\stackrel{G}{=}$ `Untermengenbeziehungen`]:

$$\boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}} \tag{3}$$

REM 2: The real numbers can be represented geometrically on a straight line, the so called **real axis** [$\stackrel{G}{=}$ `Zahlengerade`].

REM 3: The rationals are everywhere **dense** [$\stackrel{G}{=}$ `dicht`] on the real axis, i.e. there is no interval (of length arbitrarily small, but greater than zero) which does not contain a rational number.
In other words: Every real number can be approximated to arbitrarily high precision by a rational number.

7.Q 2: dimensioned quantities in physics

The length of a **rod** [$\stackrel{G}{=}$ `Stab`] is not a pure number but is a **dimensioned quantity** [$\stackrel{G}{=}$ `dimensionsbehaftete Größe`], e.g.

$$l = 3 \text{ m} = 300 \text{ cm} \tag{1}$$

7.2. **a**) What is the general name for m or cm?

(Solution:)

Unit [$\stackrel{G}{=}$ `Einheit, Maßeinheit`] of the physical quantity. In the present example it is a unit of length.

7.2. **b**) What is the general name for 3 or 300?

(Solution:)

Measure number[$\stackrel{G}{=}$ `Maßzahl`] of the physical quantities (length of the rod) in a particular system of units.

7.2. **c**) What is the **dimension** of a volume?

(Solution:)

In mathematics the answer would be 3, because a volume is a 3-dimensional object.

In physics the answer would be: The dimension of a volume is m^3 (or: third power of a length).

7.**T 3: ☻ Systems of units in physics**

In physics there are several systems to measure physical quantities, among others:

- The **SI-units** (= standard international units). At former times that system was called the MKS-system, because lengths are measured in meters (m), masses are measured in kilogramms (kg) and times are measured in seconds (s).
- The **cgs**-system, so called because the units are centimeters (cm), grams (g) and seconds (s). It is still widely used in theoretical physics.
- To simplify complicated calculations in theoretical physics (e.g. in Einstein's theory of relativity) a system is used where the velocity of light is put to unity:

$$c = 1 \tag{2}$$

  That means that the second is **discarded**[$\stackrel{G}{=}$ `fallenlassen`] as a separate unit and the unit of time is the time light needs to travel 1 cm, which is approximately 30 ps = 30 picoseconds = $30 \cdot 10^{-12}s$. Such an approach is possible since according to Einstein's relativity the velocity of light is unique (in vacuum), i.e. it is always the same, regardless of the colour of the light, its history, i.e. where it does come from or how it was generated.

  The velocity of light can be different from its (unique) vacuum velocity, if it travels in a medium (e.g. in glas) or when it travels in vaccum but if a medium is very nearby, i.e. light travelling in an empty, but small **cavity**[$\stackrel{G}{=}$ `Hohlraum`].
- **Atomic units**. According to the **principle of uniqueness of quantum systems** atomic systems are unique (like the speed of light). E.g. a **hydrogen**[$\stackrel{G}{=}$ `Wasserstoff`] atom (in its **ground state**[$\stackrel{G}{=}$ `Grundzustand`]) always has the same mass, the same size, and its electron needs the same time **orbiting**[$\stackrel{G}{=}$ `umkreisen`] one **cycle**[$\stackrel{G}{=}$ `Umlauf`]. Therefore, the hydrogen atom can be used to measure masses, lengths and times. Taking these units, we adopt atomic units in physics.

  (For technical reasons other atoms than hydrogen are used.)

  With atomic units all physical quantities are pure numbers, i.e. are **dimensionless**[$\stackrel{G}{=}$ `dimensionslos`]

The unit of a physical quantity, i.e. m, can be viewed as a pure number depending on the chosen system, so that

$$l = 3 \text{ m} = 300 \text{ cm} \tag{1}$$

always gives the correct measure number of the rod.

E.g. when we use SI-units, we will have:

$$\text{m} = 1, \text{ cm} = 0.01 \tag{2}$$

then l in (1) gives the correct measure number $l = 3$.

If, on the other hand, we use cgs-units, instead of (2) we will have

$$\text{m} = 100, \text{ cm}{=}1 \tag{2'}$$

and again (1) will give the correct measure number in the chosen system, namely $l = 300$:

	cm means:	m means:
We use cm as units:	1	100
We use m as units:	0.01	1

Money is a natural number, but when considering several systems to count natural numbers, it becomes a dimensioned quantity. E.g. your account M of money could be

$$M = 300 \text{ euro} = 30'000 \text{ cent} \tag{3}$$

When using euro as units we have: euro=1, cent=0.01, and (3) gives correctly $M = 300$. When on the other hand we use cents to measure money, we will have: euro=100, cent=1, and (3) will again give the correct answer: $M = 30'000$.

The same is true for measuring angles, e.g.

$$\alpha = \frac{3.14159}{2} \text{ rad} = 90^\circ \tag{4}$$

When using rad (radians) to measure angles, we have:

rad $= 1, \qquad {}^\circ = \frac{\pi}{180}$

When using degrees as the unit, we have:

rad $= \frac{180}{\pi}, \qquad {}^\circ = 1;$

	rad means:	° means:
We use rad as units:	1	$\pi/180$
We use ° as units:	$180/\pi$	1

7.Ex 4: ☻ Large dimensioned quantities with a calculator

The size of an H-atom (**hydrogen atom**[$\stackrel{G}{=}$ `Wasserstoffatom`]) can be estimated by the Bohr radius r_B given by

$$r_B = \frac{\hbar^2}{me^2} \tag{1}$$

where $2\pi\hbar$ is Planck's constant,

$$\boxed{\hbar = 1.054 \cdot 10^{-27} \text{ g cm}^2 \text{ sec}^{-1}} \qquad (\text{g} = \text{gram}) \tag{2}$$

m is the mass of an electron

$$\boxed{m = 9.108 \cdot 10^{-28} \text{ g}} \tag{3}$$

and e is the electric charge of the electron (in gaussian electrostatic units)

$$\boxed{e = 4.803 \cdot 10^{-10} \text{ g}^{\frac{1}{2}} \text{ cm}^{\frac{3}{2}} \text{ sec}^{-1}} \tag{4}$$

Using these values calculate r_B and give the result in angstroms.

$$\boxed{1\text{Å} = 10^{-10} \text{ m}} \tag{5}$$

Hint: Manipulate dimensions and powers by hand and use a calculator for mantissae (e.g. for 0.4803) only.
Result:

$$r_B = 0.5287 \text{ Å} \tag{6}$$

(Solution:)

$$r_B = \frac{(1.054)^2\, 10^{-54} \text{ g}^2 \text{ cm}^4 \text{ sec}^{-2}}{9.108 \cdot (4.803)^2\, 10^{-20}\, 10^{-28} \text{ g g cm}^3 \text{ sec}^{-2}} \tag{7}$$

$$\left(\frac{1.054}{4.803}\right)^2 \frac{1}{9.108} = 0.005287 \tag{8}$$

$$\frac{10^{-54}}{10^{-20}10^{-28}} = \frac{10^{48}}{10^{54}} = \frac{1}{10^6} = 10^{-6} \tag{9}$$

$$r_B = 0.5287 \cdot 10^{-2} \cdot 10^{-6} \cdot 10^{-2} \text{ m} \tag{10}$$

$$r_B = 0.5287 \text{ Å} \tag{11}$$

7.**Ex 5: Constant velocity**

A tractor starts at time $t = 3$ sec on a straight line at position $x = 2$ m and stops at $t = 11.2$ sec at position $x = 82$ m. It is also observed at two **intermediate**[$\stackrel{\mathbf{G}}{=}$ `dazwischenliegend`] points P_1, P_2 (see the following table).

Point	t[sec]	x[m]
P_0	3	2
P_1	6.2	35
P_2	9.6	64
P	11.2	82

(1)

7.5. **a**) Take a sheet of millimeter squared paper and plot the positions of the tractor according to the above table by choosing the following units ($t = 1$ sec $\widehat{=}$ 1 cm on the horizontal axis, $x = 1$ m $\widehat{=}$1 mm on the vertical axis, left lower corner $\widehat{=}$ origin i.e. $(t = 0, x = 0)$). Number the axes for seconds and meters and indicate the chosen units in **square brackets**[$\stackrel{\mathbf{G}}{=}$ `eckige Klammern`], e.g. [sec], as was done in the table.

(Solution:)

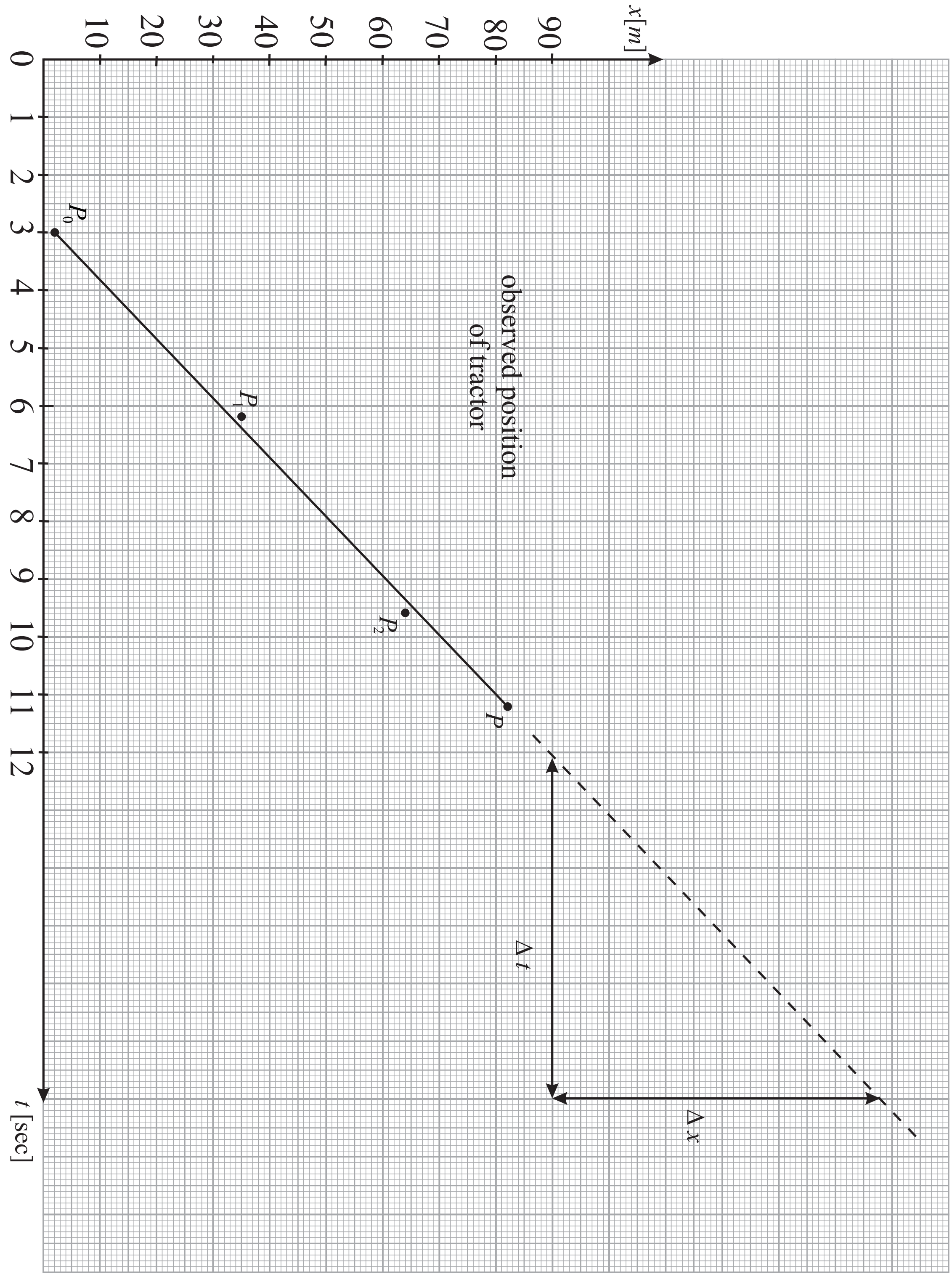


Fig7.5. **1**: Diagram showing position of tractor moving with constant velocity

---

7.5. **b**) From the table calculate the distance travelled Δx, the time travelled Δt and the corresponding **average**[$\stackrel{\mathbf{G}}{=}$ `Durchschnitt`] **velocity**[$\stackrel{\mathbf{G}}{=}$ `Geschwindigkeit`]

$$\bar{v} = \frac{\Delta x}{\Delta t} \tag{2}$$

Give $\bar{v}$ in the units cm sec^{-1}, m sec^{-1} and mph (= miles per hour, 1 mile = 1609.34 m).

(Solution:)

$$\Delta t = 11.2 \text{ sec - } 3 \text{ sec} = 8.2 \text{ sec} \tag{3}$$

$$\Delta x = \ 82 \text{ m - } 2 \text{ m} = 80 \text{ m} \tag{4}$$

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{80 \text{ m}}{8.2 \text{ sec}} = 9.7561 \text{ m sec}^{-1} = 976 \text{ cm sec}^{-1} \tag{5}$$

$$1 \text{ hour} = 1\ h = 3600 \text{ sec} \tag{6}$$

$$1 \text{ sec} = \frac{1}{3600}\, h \tag{7}$$

$$1 \text{ mile} = 1609.34 \text{ m} \tag{8}$$

$$1m = \frac{1}{1609.34} \text{ mile} \tag{9}$$

$$\bar{v} = \frac{80 \cdot 3600}{1609.34 \cdot 8.2} \text{ mph} = 21.82 \text{ mph} \tag{10}$$

7.5. **c**) The tractor's engineer asserts that the engine has moved with a constant velocity v_0. Decide graphically if the points P_1, P_2 support that assertion. Give two possible explanations for the **discrepancies**[$\stackrel{\mathbf{G}}{=}$ `Abweichungen`].

(Solution:)

If the engineer's assertion was correct P_1 and P_2 would lie on the straight line through P_0 and P; this is not the case. Possible explanations are: 1) The tractor has a constant velocity only approximately. 2) The points P_0, P_1, P_2, P have only been approximately measured.

7.5. **d**) Assuming that only the time measurement of P_1 (and P_2) are to blame, determine graphically what the absolute error (Δt_1) and the relative error (ε_1) are in the measurement of the time t_1 of P_1.
RESULT: $\Delta t_1 = 0.2$ sec, $\varepsilon_1 = 3\%$

(Solution:)

From the above figure:

$$\Delta t_1 \widehat{=}\ 2 \text{ mm}, \quad \Delta t_1 \widehat{=}\ 0.2 \text{ sec} \tag{11}$$

$$\varepsilon_1 = \frac{\Delta t_1}{t_1} 100\% = \frac{0.2 \text{ sec}}{6.4 \text{ sec}} 100\% \approx 3\% \tag{12}$$

6.4 sec is the exact value for t_1.

REM: In most cases, errors (e.g. ε_1) can be estimated only approximately since the exact value (e.g. 6.4 sec) is not known. Therefore, it is also correct to write 6.2 sec in the denominator of (12).

7.5. **e**) Assuming that points P_0, P were correctly measured and that the velocity was constant, find the equation for the function $x(t)$.
HINT: Because of the constant velocity, $x(t)$ has to be a linear function

$$x(t) = \alpha + \beta t \tag{13}$$

with constants α, β. Determine the constants α, β with help from the table. Subtract the resulting equation to determine β.
RESULT:

$$\alpha = -27.268 \text{ m}, \qquad \beta = 9.7561 \text{ m sec}^{-1} \tag{14}$$

(Solution:)

According to the table

$$P_0 : \quad x(3 \text{ sec}) = \alpha + \beta \cdot 3 \text{ sec} = 2 \text{ m} \tag{15}$$

$$P \ : \quad x(11.2 \text{ sec}) = \alpha + \beta \cdot 11.2 \text{ sec} = 82 \text{ m} \tag{16}$$

Subtracting

$$\beta(11.2 - 3) \text{ sec} = 82 \text{ m} - 2 \text{ m} \tag{17}$$

$$\beta = \frac{80}{8.2} \text{m sec}^{-1} = 9.7561 \text{ m sec}^{-1} = \bar{v} \tag{18}$$

$$\alpha = 2 \text{ m} - \beta \cdot 3 \text{ sec} = 2 \text{ m} - \frac{80 \cdot 3}{8.2} \text{ m} = -27.268 \text{ m} \tag{19}$$

7.5. **f**) Check to see that each term in equation (13), i.e. $x(t), \alpha, \beta t$ has the same dimension.

(Solution:)

$$\begin{array}{l} \text{dimension of } x(t) = [x(t)] = \text{ m} = \text{meter} \\ [\alpha] = \text{ m}, \quad [\beta] = \text{ m sec}^{-1}, \quad [t] = \text{ sec}, \quad [\beta t] = \text{ m} \quad \text{q.e.d.} \end{array} \tag{20}$$

7.5. **g**) Check to see that equation (13) can also be written as

$$x(t) = x_0 + v_0(t - t_0) \tag{21}$$

with

$$v_0 = \bar{v}, \quad (x_0, t_0) = P_0 \tag{22}$$

Make equation (21) plausible, i.e., derive it directly without the calculation in e).

(Solution:)

$$\begin{array}{l} v_0 = \beta = \bar{v}, \\ x_0 - v_0 t_0 = 2 \text{ m} - \frac{80}{8.2} \text{m sec}^{-1} \; 3 \text{ sec} \; = -27.268 \text{ m} = \alpha \qquad \text{q.e.d.} \end{array} \tag{23}$$

Equation (21) can be directly obtained: it is a linear function. For $t = t_0$, $x(t_0) = x_0$ i.e. it goes through P_0 and it has the correct velocity $v_0 = \bar{v}$.

7.5. **h**) Still assuming that the velocity v is constant but giving all points P_0, P_1, P_2, P equal **credibility**`[`$\stackrel{\mathbf{G}}{=}$ `Glaubwürdigkeit]`, draw a straight line by hand which best fits all points, e.g. the dotted line in the above figure. Choose small increments dx, dt and determine graphically the corresponding best **guess**`[`$\stackrel{\mathbf{G}}{=}$ `Vermutung]` for the velocity $v = \frac{dx}{dt}$.

(Solution:)

$$\Delta x \;\widehat{=}\; 54.7 \text{ mm}, \quad \Delta x = 54.7 \text{ m} \tag{24}$$

$$\Delta t \;\widehat{=}\; 58 \text{ mm}, \quad \Delta t = 5.8 \text{ sec} \tag{25}$$

$$v = \frac{\Delta x}{\Delta t} = \frac{54.7}{5.8} \text{ m sec}^{-1} = 9.43 \text{ m sec}^{-1} \tag{26}$$

7.Ex 6: ☻☻ Logarithmic scaling

7.6. **a**) Take a sheet of a half-logarithmic paper and let the center of the sheet be the origin, i.e. the point $t = 0$, $x = 2$. The horizontal axis (for t) through the origin should have linear **scale**`[`$\stackrel{\mathbf{G}}{=}$` Skalierung = Maßstab]`, the vertical axis (for x) through the origin should have logarithmic scale. On the t-axis attach the values

$$-8, -7, \cdots, -1,\ 0,\ 1,\ 2, \cdots 7,\ 8 \tag{1}$$

and on the x-axis attach the values

$$1,\ 2,\ 3,\ 4,\ 5,\ 10,\ 20,\ 30,\ 40,\ 50,\ 100, \tag{2}$$

$$1.5,\ 2.5,\ 15,\ 25, \tag{3}$$

$$\frac{1}{10}, \ \frac{1}{100} \tag{4}$$

$$0.2,\ 0.3,\ 0.15 \tag{5}$$

HINT: Use the numbering at the edge of the half-logarithmic paper.

7.6. **b**) For the function

$$x = x(t) = 10^t \tag{6}$$

construct the points of its graph for

$$t = 0, 1, 2, -1, -2 \tag{7}$$

and observe that you obtain a straight line.

(Solution:)

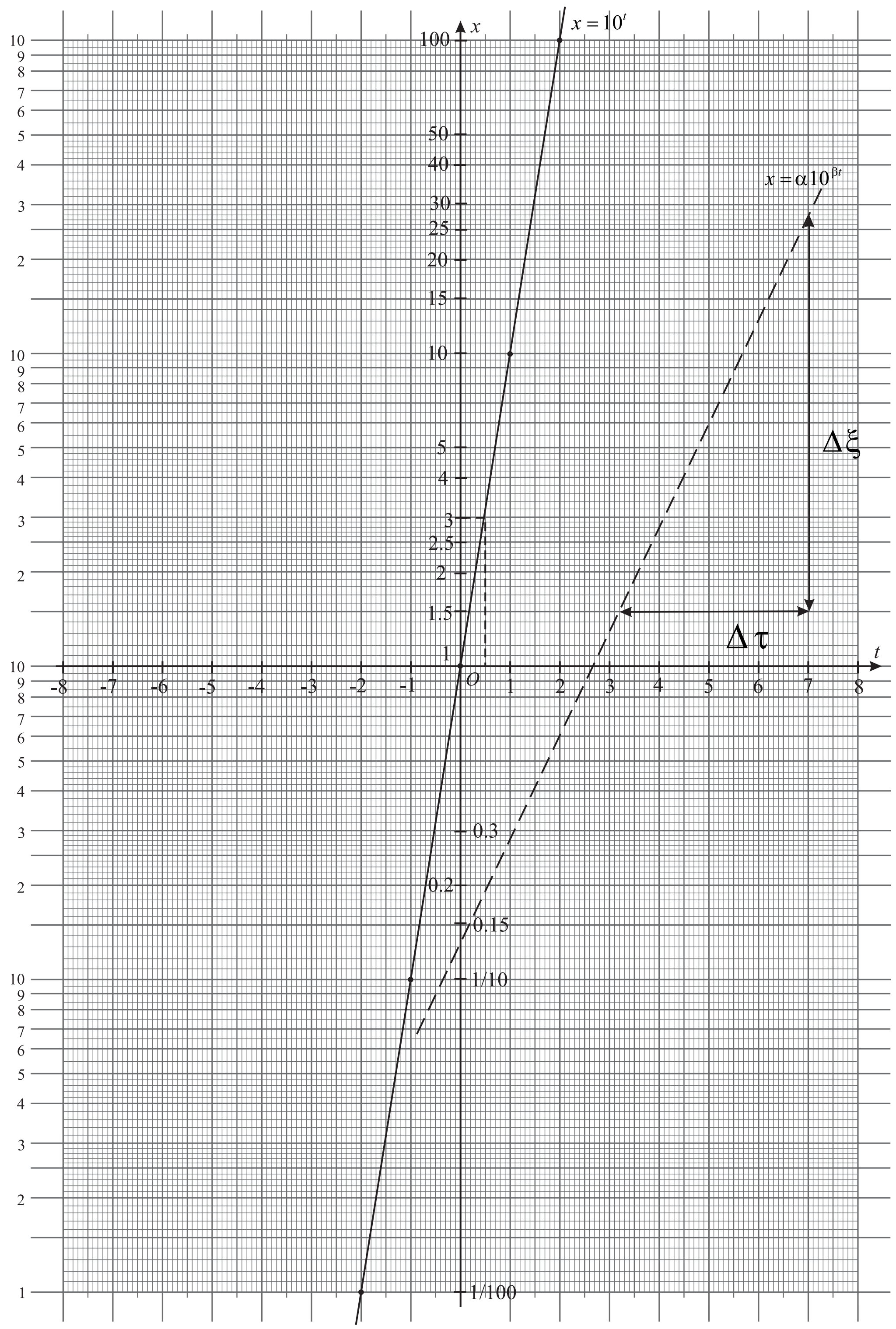


Fig7.6. **1**: Position x of rocket 1 and rocket 2 (dotted line) at time t in a logarithmic scale for x Note that this figure is scaled down by the factor 6/8, compared to a real half-logarithmic paper, where the horizontal units are mm and cm, respectively; 8 units measure only 6 cm on the figure.

7.6. **c**) Graphically and numerically (i.e. by using a calculator) determine $x(0.5)$

(Solution:)

Graphically: see the dotted lines parallel to the axes of the figure: $x(0.5) \approx 3.0$.
Numerically:

$$x(0.5) = 10^{0.5} = \sqrt{10} \approx 3.1623 \tag{8}$$

7.6. **d**) A **rocket**[$\stackrel{\mathbf{G}}{=}$ `Rakete`] is at the position

$$\boxed{x = x(t) = \alpha 10^{\beta t}} \qquad (\alpha, \beta = \text{ const.}) \tag{9}$$

What are the dimensions of x, α and β?
RESULT:

$$[x] = \text{ m}, \quad [\alpha] = \text{ m}, \quad [\beta] = \text{ sec}^{-1} \tag{10}$$

(Solution:)

An exponent like βt (like any argument of a mathematical function) must be a pure number. Since

$$[t] = \text{ sec} \qquad \Rightarrow \qquad [\beta] = \text{ sec}^{-1} \tag{11}$$

Since $10^{\beta t}$ is a pure number α must have the same dimension as x, i.e.

$$[x] = [\alpha] = \text{ m} \tag{12}$$

7.6. **e**) For the special case $\alpha = 1$ m, $\beta = 1$ sec^{-1} give the position of the rocket at time $t = 0$ sec, 1 sec, 2 sec, -1 sec, -2 sec in the form of a table.
RESULT:

t[sec]	x[m]
-2	0.01
-1	0.1
0	1
1	10
2	100

(13)

7.6. **f**) Insert the information from the table onto the half-logarithmic paper by changing the **denotations**[$\stackrel{\mathbf{G}}{=}$ `Bezeichnungen`] of the axes to

$$t[\text{sec}] \quad \text{and} \quad x[\text{m}] \tag{14}$$

i.e. the numbering refers now to the units sec (for t) and m (for x).
RESULT: The same as the old graph for $x = 10^t$.

7.6. **g**) Denote by ξ the (real geometrical) distance on the sheet along the x-axis and by τ the (real geometrical) distance along the t-axis, the numbering on the sheet corresponds to

$$\xi = 6.22 \text{ cm} \lg(1 \text{ m}^{-1} x) \tag{15}$$

$$\tau = 1 \text{ cm sec}^{-1} t \tag{16}$$

Check this for $t = 0$, 1 sec, $x = 1$ m, 10 m.

HINT: We have here the additional problem, that the above figure is not a real half-logarithmic sheet, but to make the latter fit as a figure onto a page of this manuscript we have had it scaled down by a factor 6/8: On a real half-logarithmic paper the larger horizontal units are 1 cm, but in the figure 8 such units measure only 6 cm. So, what we measure on the figure must be multiplied by 8/6 to have a result which would be measured on a real half-logarithmic paper.

REM: Since only one axis has logarithmic scale, the sheet is called **half-logarithmic**.

7.6. **h**) A second rocket with the same law of motion (9) but with different values for α and β is **plotted**[$\stackrel{\mathbf{G}}{=}$ `aufgemalt`] as a dotted line in the above figure. Give the motion of the rocket in terms of ξ and τ.

HINT: Use (1), (2), (9) and eliminate x and t, i.e. use only ξ and τ. Use the rule for log of a product.
RESULT:

$$\xi = \xi(\tau) = 6.22 \text{ cm} \lg(1 \text{ m}^{-1} \alpha) + 6.22 \text{ sec } \beta\tau \tag{17}$$

(Solution:)

$$\xi = 6.22 \text{ cm} \lg(1 \text{ m}^{-1} \alpha 10^{\beta t}) = \tag{18}$$

$$= 6.22 \text{ cm} \lg(1 \text{ m}^{-1} \alpha) + 6.22 \text{ cm } \lg 10^{\beta t} \tag{19}$$

First we calculate

$$\lg 10^{\beta t} = \beta t \underbrace{\lg 10}_{1} = \beta t \tag{20}$$

According to (2)

$$t = \text{ cm}^{-1} \text{ sec } \tau \tag{21}$$

Thus

$$\lg 10^{\beta t} = \ \mathrm{cm}^{-1} \ \sec \ \beta\tau \tag{22}$$

and we obtain

$$\xi = 6.22 \ \mathrm{cm} \ \ \lg(1 \ \mathrm{m}^{-1}\alpha) + 6.22 \ \sec \ \beta\tau \tag{23}$$

This is what really is plotted on the half-logarithmic paper. The slope of the graph of the rocket corresponds to

$$\boxed{6.22 \ \sec \ \beta = \frac{\Delta\xi}{\Delta\tau}} \tag{24}$$

7.6. **i**) Graphically determine α and β for the dotted rocket.

HINT for α: Consider $t = 0$.
HINT for β: Use (24).
RESULTS:

$$\alpha = 0.13 \ \mathrm{m}, \qquad \beta = 0.33 \ \sec^{-1} \tag{25}$$

(Solution:)

$$t = 0: \quad x(0) = \alpha 10^{\beta \cdot 0} = \alpha \tag{26}$$

The dotted line intersects the x-axis at the value $x(0) = \alpha = 0.13$ m.
The increments in the above figure on the dotted line are

$$d\xi = 7.8 \ \mathrm{cm}, \qquad d\tau = 3.8 \ \mathrm{cm} \tag{27}$$

From (24) we obtain

$$\beta = \frac{d\xi}{d\tau}\,\frac{1}{6.22} \ \sec^{-1} = \frac{7.8}{3.8}\,\frac{1}{6.22} \ \sec^{-1} = 0.33 \ \sec^{-1} \tag{28}$$

7.**Ex 7: Periodic decimal as a quotient**

Every decimal number which becomes periodic is a rational number, i.e. equal to

$$\frac{n}{m} \quad \text{with } n \,\epsilon\, \mathbb{Z}, \ m \,\epsilon\, \mathbb{Z} \tag{1}$$

Prove that this is true for

$$x = 15.371\overline{81}\cdots \tag{2}$$

Hint 1: The above notation means $x = 15.371818181\cdots$.
Hint 2: Take $100x$ and subtract x from it.
Intermediate Result:

$$99x = 1521.81 \tag{3}$$

Result:

$$x = \frac{152181}{9900} \tag{4}$$

(Solution:)

$$\begin{array}{rcl} 100x & = & 1537.181\overline{81}\cdots \\ - \quad x & = & -15.371\overline{81}\cdots \\ \hline 99x & = & 1521.810 \end{array} \tag{5}$$

8 Infinite sequences and infinite series

8.T 1: Motivation for infinite sequences

The **ancient** [$\overset{\mathbf{G}}{=}$ `alt`] Greeks had already known that $\sqrt{2}$ and π were irrational, i.e. cannot be represented as a ratio $\frac{n}{m}$ of two integers.

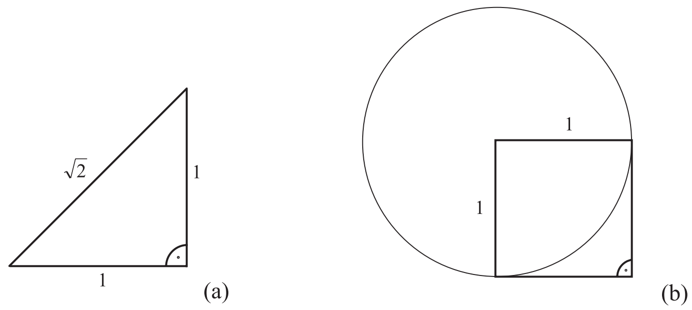


Fig8.1. **1**: The irrational $\sqrt{2}$ is important because it is the hypotenuse of the triangle (a). π is important because it is the ratio of the area of a circle to its radius squared (b).

Since $\sqrt{2}$ and π are obviously important numbers (see Fig 1) one was forced to consider infinite **sequences** [$\overset{\mathbf{G}}{=}$ `Folgen`].

$$
\begin{array}{ll}
a_0 = 1 & b_0 = 3 \\
a_1 = 1.4 & b_1 = 3.1 \\
a_2 = 1.41 & b_2 = 3.14 \\
a_3 = 1.414 & b_3 = 3.141 \\
a_4 = 1.4142 & b_4 = 3.1415 \\
\cdots & \cdots
\end{array}
\tag{1}
$$

And eventually to write

$$
\lim_{n\to\infty} a_n = \sqrt{2} \qquad \lim_{n\to\infty} b_n = \pi \tag{2}
$$

pronounced: '**the limit** [$\overset{\mathbf{G}}{=}$ `Grenzwert`] of a_n [as n goes to ∞] is $\sqrt{2}$ '.

We remind ourselves of how the *members* of a sequence, e.g. a_n, are constructed: $a_0 = 1$ is obviously too small for $\sqrt{2}$, since $1^2 = 1 < 2$, but $a_0 = 2$ is too large, since $2^2 = 4 > 2$. So we take $a_0 = 1$ since it is the largest integer which is still too small. For a_1 we test 1.1, 1.2, 1.3, $\cdots$ 1.9 and have to take $a_1 = 1.4$ since $(1.4)^2 < 2$ but $(1.5)^2 > 2$, etc.

Since their discovery by the ancient Greeks **mankind** [$\overset{\mathbf{G}}{=}$ `Menschheit`] has had to wait almost two **millenniums** [$\overset{\mathbf{G}}{=}$ `Jahrtausende`] until infinite sequences and their limits, including the irrational numbers, could be based on a solid mathematical foundation. We cannot attempt to reproduce that theory here but will merely give some examples so the reader will get some intuitive understanding of it.

In dealing with limits it is necessary (or **convenient** [$\stackrel{\mathbf{G}}{=}$ `bequem`]) to introduce two pseudo-numbers: ∞ and $-\infty$ (**infinity** [$\stackrel{\mathbf{G}}{=}$ `unendlich`] and minus infinity). They are not **ordinary** [$\stackrel{\mathbf{G}}{=}$ `gewöhnlich`] numbers since

$$a + x = a \quad \Rightarrow \quad x = 0 \tag{3}$$

holds for an ordinary number, while for $a = \infty$ we have

$$\infty + 1 = \infty \tag{4}$$

However, many properties of ordinary numbers still hold for $\pm\infty$, such as the possibility of adding an ordinary number to it as we did in (4). However,

$$\infty - \infty = ? \tag{5}$$

has no (definite) meaning.

8.**Q 2: Different notations for limits**
What is the difference between

$$\lim_{n\to\infty} a_n = \sqrt{2} \tag{1}$$

$$\lim a_n = \sqrt{2} \tag{2}$$

$$a_n \quad \stackrel{n\to\infty}{\longrightarrow} \quad \sqrt{2} \tag{3}$$

$$a_n \to \sqrt{2} \tag{4}$$

(Solution:)

They are all synonymous. In (2) and (4) the $n \to \infty$ is implied. The notation lim has the advantage that it can occur in equations as an ordinary number.

8.**Q 3: Definition of a limit**
What does

$$\lim_{n\to\infty} a_n = a \tag{1}$$

mean:
8.3. **a**) intuitively (i.e. in simple words)

(Solution:)

The larger the n, the closer the member a_n comes to a.

REM: Thus, there can be at most one a fulfilling (1) for a given sequence a_n.

8.3. **b)** ☻☻ mathematically precise

(Solution:)

For any number $\varepsilon > 0$ (typically you choose a very small ε but one which is still positive) you can find an n_0, depending on ε, i.e.

$$n_0 = n_0(\varepsilon) \tag{2}$$

so that for all

$$n > n_0 \tag{3}$$

you have

$$|a_n - a| < \varepsilon \tag{4}$$

REM 1: This can be expressed in simpler language: For any $\epsilon > 0$ only a finite number of the a_n are outside the **ϵ-environment**[$\stackrel{G}{=}$ `ϵ-Umgebung`] (4) of a.

REM 2: In mathematics the expression '**almost all**[$\stackrel{G}{=}$ `fast alle`]' means 'all except a finite number of them'.
Using this terminology we can say:

$$\boxed{\begin{array}{c}\lim_{n\to\infty} a_n = a \qquad \text{means:}\\ \text{For any } \epsilon > 0 \text{ almost all } a_n \text{ are inside the}\\ \epsilon - \text{environment of } a.\end{array}} \quad (\textbf{definition of a limit}) \tag{5}$$

8.3. **c)** For the decimal expansion of $\sqrt{2}$ $(a_n \to \sqrt{2}$, see T1(1)) if you let $\varepsilon = \frac{1}{100}$ what can n_0 be?

(Solution:)

$n_0 = 2$
Since $a_3 = 1.414$ it will differ at most by $0.001 < \varepsilon = 0.01$ from $\sqrt{2}$; the same is true for subsequent n's.

8.Ex 4: Simple examples of limits

Find the limit of the following sequences.

8.4. **a)**

$$a_n = \frac{1}{n} \tag{1}$$

(Solution:)

$$\lim a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \qquad \text{(convergent)} \tag{2}$$

8.4. **b)**

$$a_n = (-1)^n \frac{1}{n}, \qquad \text{i.e.} \qquad a_1 = -1,\ a_2 = \frac{1}{2},\ a_3 = -\frac{1}{3},\ \cdots \tag{3}$$

REM: A sequence with limit 0 is also called a **null sequence**[$\stackrel{\mathbf{G}}{=}$ `Nullfolge`].

(Solution:)

$$\lim_{n\to\infty} \left[(-1)^n \frac{1}{n}\right] = 0 \qquad \text{(convergent)} \tag{4}$$

8.4. **c)**

$$a_n = n \tag{5}$$

(Solution:)

$$\lim_{n\to\infty} n = \infty \qquad \text{(definitely divergent)} \tag{6}$$

REM: **Definitely divergent**, e.g.

$$\lim_{n\to\infty} a_n = +\infty$$

(i.e. definitely divergent to $+\infty$, not to $-\infty$ in this case) means:
For any M, as large as we wish, we can find an $n_o = n_o(M)$,
(i.e. n_o may and will depend on M) so that $a_n > M$ for all $n > n_o$.
In a **picturesque**[$\stackrel{\mathbf{G}}{=}$ `bildlich`] way, we can say: a_n comes closer and closer to $+\infty$.

8.4. **d)**

$$a_n = -n \tag{7}$$

(Solution:)

$$\lim_{n\to\infty} (-n) = -\infty \qquad \text{(definitely divergent)} \tag{8}$$

8.4. **e)**

$$a_n = 10^n \tag{9}$$

(Solution:)

$$\lim_{n\to\infty} 10^n = \infty \qquad \text{(definitely divergent)} \tag{10}$$

8.4. **f**)

$$a_n = (-1)^n, \quad \text{i.e.,} \quad a_0 = 1, \; a_1 = -1, \; a_2 = +1, \cdots \tag{11}$$

(Solution:)

$\lim_{n\to\infty}(-1)^n$ does not exist, i.e. $\lim_{n\to\infty}(-1)^n$ is a meaningless expression. Any equation in which it occurs is wrong. (divergent)

8.4. **g**)

$$a_n = 7 \quad (\text{ for all } n) \tag{12}$$

(Solution:)

$$\lim_{n\to\infty} a_n = 7 \qquad \text{(convergent)} \tag{13}$$

8.**Q 5: Convergence and divergence**

In a sequence a_n what does convergent, divergent or definitely divergent mean?

(Solution:)

1) divergent: the limit $\lim_{n\to\infty} a_n$ does not exist, i.e. $\lim_{n\to\infty} a_n$ is a meaningless expression (including the case $\pm\infty$).
2) definitely divergent: $\lim_{n\to\infty} a_n = \pm\infty$
3) convergent: there exists a number a such that $a_n \stackrel{n\to\infty}{\longrightarrow} a$, i.e. $\lim_{n\to\infty} a_n = a$ and $a \neq \pm\infty$.

8.**Ex 6: Insignificant changes in sequences**

8.6. **a**) When you change a finite number of the members in an infinite sequence how does this **affect** [$\stackrel{\mathbf{G}}{=}$ `beeinflussen`] its limit? E.g. define

$$\begin{aligned} c_0 &= 0 \\ c_1 &= 1 \\ c_2 &= 2 \\ c_3 &= 3 \\ c_4 &= 4 \\ c_5 &= 5 \\ c_6 &= a_6 = 1.414213 \\ c_7 &= a_7 = 1.4142136 \\ &\vdots \end{aligned} \tag{1}$$

where a_n is the decimal expansion of $\sqrt{2}$ $(a_n \to \sqrt{2})$ is it still true that

$$c_n \to \sqrt{2} \tag{2}$$

(though, from looking at its first members $c_0, c_1, c_2, c_3, c_4, c_5$, which tend away from $\sqrt{2}$, it would appear that c_n does not approach $\sqrt{2}$).

(Solution:)

Yes.
Changing a finite number of its members does not change its limit since the wording 'a_n comes closer to the number a' does not imply that this must be monotonous. When you change a finite number of its members, from a definite index N on ($N = 5$ in our example, i.e. for all $n > N$) the sequence is unchanged. Only the behaviour of the sequence for $n \to \infty$ is important for the limit and this means that only sufficiently large n's matter.
(☻ In the precise definition always choose $n_0 > N$.)
Simile: To the question as to whether you will get a diploma in physics, it does not matter what you do in the first few semesters, but only if you eventually do something at a later time.

8.6. **b)** **Interchanging**[$\stackrel{G}{=}$ `austauschen`] the members of a sequence by pairs, e.g.

$$\begin{aligned} c_0 &= a_1 = 1.4 \\ c_1 &= a_0 = 1 \\ c_2 &= a_3 = 1.414 \\ c_3 &= a_2 = 1.41 \\ c_4 &= a_5 = 1.41421 \\ &\cdots \quad \cdots \end{aligned} \tag{3}$$

do we still have

$$c_n \to \sqrt{2} \tag{4}$$

(Solution:)

Yes. The **approach**[$\stackrel{G}{=}$ `Annäherung`] of $c_n \to c = \sqrt{2}$ is not monotonic, so it is probably slower than with a_n, but for sufficiently large n, c_n is as near to c as we want.
Simile: If you do your **certificates**[$\stackrel{G}{=}$ `Scheine`] for the first semester in your second semester because you already did your certificates for the second semester in your first, etc, this does not affect your abilities at the end of your studies.

8.**Ex 7: Infinite sums as infinite sequences of its partial sums**
We have defined the exponential function by the infinite sum

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n \tag{1}$$

Re-express the infinite sum with an infinite sequence and re-express (1) with the limit of an infinite sequence.

REM: Instead of infinite sum, the term infinite **series**[$\stackrel{\mathbf{G}}{=}$ `Reihe`] is used.

(Solution:)

An infinite sum is the infinite sequence of its partial sums:

$$a_n = \sum_{m=0}^{n} \frac{1}{m!}\, x^m \tag{2}$$

[In (1) it does not matter which letter is denoted in the summation index. So (1) can also be written as

$$e^x = \sum_{m=0}^{\infty} \frac{1}{m!} x^m \tag{1'}$$

as we have done in (2) since the letter n is already used for the general member of the sequence.]

$$e^x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{m=0}^{n} \frac{1}{m!}\, x^m \tag{3}$$

8.**Q 8: Composite infinite sequences**

What are the rules for the limits of sequences obtained by memberwise addition, multiplication and division of sequences?

(Solution:)

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b \quad \Rightarrow \tag{1}$$

$$\lim_{n\to\infty} (a_n + b_n) = a + b \tag{2}$$

$$\lim_{n\to\infty} (a_n\, b_n) = a \cdot b \tag{3}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b} \qquad (\text{ if } b_n \neq 0,\ b \neq 0) \tag{4}$$

8.**Ex 9: Limits of infinite sequences**

Calculate the limits of the following infinite sequences.

8.9. **a)**

$$a_n = 1 + \frac{1}{n^2}, \quad n = 1, 2, \cdots \tag{1}$$

(Solution:)

$$\lim_{n\to\infty} (1 + \frac{1}{n^2}) = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n^2} = 1 + 0 = 1 \tag{2}$$

8.9. **b)**

$$a_n = \frac{n+1}{n} \tag{3}$$

(Solution:)

$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n}, \quad \lim_{n\to\infty}(\frac{n+1}{n}) = 1 \tag{4}$$

8.9. **c)**

$$a_n = 2^n \tag{5}$$

(Solution:)

$$\lim_{n\to\infty} 2^n = \infty \tag{6}$$

8.9. **d)**

$$a_n = \frac{1}{2}(1 + (-1)^n), \text{ i.e} \tag{7}$$

$$(a_n) = (1, 0, 1, 0, \cdots) \tag{8}$$

(Solution:)

The sequence is divergent, i.e. $\lim_{n\to\infty} a_n$ does not exist.

8.9. **e)**

$$(a_n) = (2,\ 2.1,\ 2.01,\ 2.001,\ 2.0001,\ 2.00001, \cdots) \tag{9}$$

(Solution:)

$$\lim_{n\to\infty} a_n = 2 \tag{10}$$

8.9. **f)**

$$(a_n) = (1.9,\ 1.99,\ 1.999,\ 1.9999,\ 1.99999,\ \cdots) \tag{11}$$

(Solution:)

$$\lim_{n\to\infty} a_n = 2 \tag{12}$$

8.9. **g)**

$$(a_n) = (0.3,\ 0.33,\ 0.333,\ 0.3333,\ 0.33333,\ \cdots) \tag{13}$$

HINT: Consider the sequence

$$b_n = 3a_n \tag{14}$$

(Solution:)

$$(b_n) = (0.9,\ 0.99,\ 0.999,\ 0.9999,\ 0.99999,\ \cdots) \tag{15}$$

$$\lim b_n = 1 = 3 \cdot \lim a_n \tag{16}$$

Thus,

$$\lim a_n = \frac{1}{3} \tag{17}$$

8.Ex 10: Limits of infinite sums

Calculate the following infinite sums

8.10. **a)**

$$\sum_{n=0}^{\infty} \frac{1}{n!} \tag{1}$$

HINT: What is the series for $e^1 = e$?

(Solution:)

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} 1^n = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{1}$$

8.10. **b)**

$$\sum_{n=0}^{\infty} a_n 10^{-n} \tag{3}$$

where a_n are the decimal digits of π, i.e.

$$(a_n) = (3,\ 1,\ 4,\ 1,\ 5,\ 9,\ 2,\ 6,\ 5,\ 3,\ 5,\cdots) \tag{4}$$

RESULT: π

8.10. **c)**

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \tag{5}$$

HINT: Consider the sequence a_n of its partial sums and also consider the sequence $b_n = xa_n$ and $c_n = a_n - b_n = (1-x)a_n$

RESULT:

$$\boxed{\begin{array}{ll} \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} & \textbf{(geometric series)} \\ \text{(convergent for } |x| < 1) & \end{array}} \tag{6}$$

REM: It will not be proved here, however, the restriction $|x| < 1$ in (6) is necessary since otherwise the geometric series is not convergent, i.e. the infinite sum on the left-hand side is meaningless.

(Solution:)

$$\begin{array}{l} a_0 = 1 \\ b_0 = x \\ \hline c_0 = 1 - x \\ \\ a_1 = 1 + x \\ b_1 = x + x^2 \\ \hline c_1 = 1 - x^2 \\ \\ a_2 = 1 + x + x^2 \\ b_2 = x + x^2 + x^3 \\ \hline c_2 = 1 - x^3 \\ \\ a_3 = 1 + x + x^2 + x^3 \\ b_3 = x + x^2 + x^3 + x^4 \\ \hline c_3 = 1 - x^4 \\ \quad \cdots \end{array} \tag{7}$$

$$c_n = 1 - x^{n+1} \tag{8}$$

$$\lim c_n = 1 = (1-x) \lim a_n \tag{9}$$

$$\lim a_n = \frac{1}{1-x} \tag{10}$$

8.10. **d**)

$$\sum_{n=0}^{\infty} n \tag{11}$$

RESULT: ∞, definitely divergent

8.10. **e**)

$$\sum_{n=0}^{\infty} 1 \tag{12}$$

RESULT: ∞, definitely divergent

8.10. **f**)

$$\sum_{n=0}^{\infty} a_n \quad \text{with } a_n = (-1)^n \tag{13}$$

RESULT: divergent

(Solution:)

The partial sums are

$$1,\ 1-1=0,\ 1-1+1=1,\ 0,\ 1,\ 0,\ 1\cdots \tag{14}$$

i.e. they are divergent since the members approach neither 0 nor 1.

8.Ex 11: Limits of composite sequences and series

8.11. **a**) In a formulary the following limit can be found

$$\boxed{\lim_{n\to\infty} (1+\frac{1}{n})^n = e} \tag{1}$$

Calculate the limit of

$$a_n = \frac{n}{n+2}(1+\frac{1}{n})^n \tag{2}$$

HINT: Write

$$a_n = \frac{(1+\frac{1}{n})^n}{\frac{n+2}{n}} \tag{3}$$

and apply the rules for the limit of a composite sequence.
RESULT:

$$\lim_{n\to\infty} a_n = e \tag{4}$$

(Solution:)

$$(1+\frac{1}{n})^n \to e \tag{5}$$

$$\frac{n+2}{n} = 1 + \frac{2}{n} \to 1 \tag{6}$$

Thus,

$$a_n \to 1 \cdot e = e \tag{7}$$

8.11. **b**) In a formulary the following can be found

$$\boxed{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}} \tag{8}$$

Calculate

$$\lim_{n\to\infty} \left[(1+\frac{1}{n})^n \sum_{m=1}^{n} \frac{1}{m^2}\right] \tag{9}$$

RESULT:

$$\frac{e\pi^2}{6} \tag{10}$$

(Solution:)

$$a_n = (1+\frac{1}{n})^n \quad \to \quad e \tag{11}$$

$$b_n = \sum_{m=1}^{n} \frac{1}{m^2} \quad \to \quad \frac{\pi^2}{6} \tag{12}$$

since these are the partial sums of (8). Thus by the rule for the limit of a composite sequence

$$a_n b_n \quad \to \quad e \cdot \frac{\pi^2}{6} \tag{13}$$

9 Continuity and limits of functions

9.Q 1: Continuous functions

9.1. **a**) What does it mean that a function $f(x)$ is continuous at $x = x_0$? Give your answer intuitively (i.e. in simple words) together with the graph of two functions, one which is continuous at $x = x_0$ and one which is not.

(Solution:)

Continuous means that the function does not make a jump at $x = x_0$.

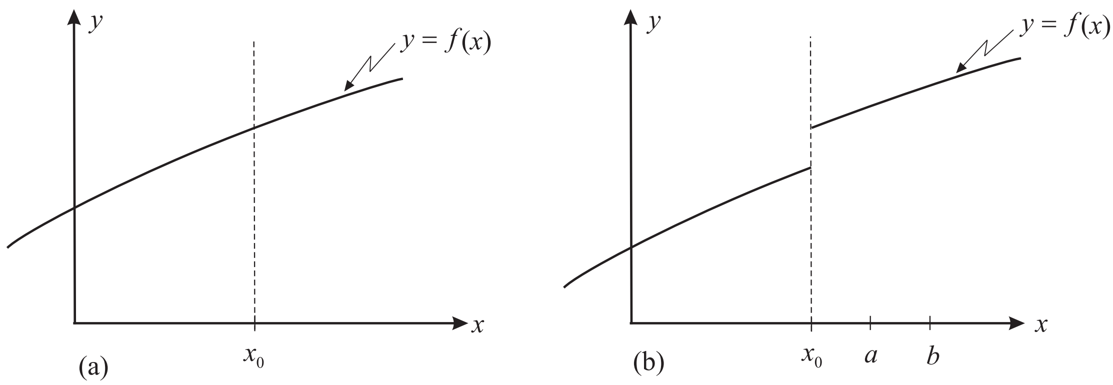


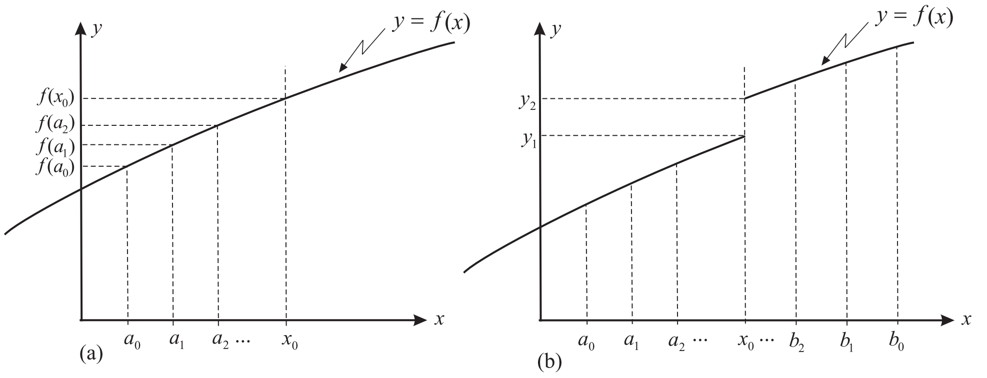
Fig9.1. **1**: Example of a continuous function (a) and of a discontinuous function (b) at $x = x_0$.

9.1. **b**) What does it mean that $f(x)$ is continuous, or is continuous in an interval (a, b)? Give examples in terms of fig 1.

(Solution:)

It means that $f(x)$ is continuous for all $x = x_0 \; \epsilon \; (a, b)$. The function $f(x)$ is 'continuous' means that it is continuous everywhere for all $x = x_0 \; \epsilon \; \mathcal{D}$, where $\mathcal{D}$ is the domain (i.e. the range of definition) of the function. $f(x)$ of fig. 1b is **discontinuous**[$\stackrel{\mathbf{G}}{=}$ `unstetig`] at $x = x_0$ but continuous at all other points, e.g. in the interval (a, b).

9.1. **c**) Give the definition of 'continuous at $x = x_0$' in a precise mathematical form and explain it with the help of fig. 2. Is it possible for (b), with a suitable re-definition of $f(x_0)$, to make $f(x)$ continuous at $x = x_0$?



`Fig`9.1. **2**: $f(x)$ in (a) is continuous at $x = x_o$, since each series $a_n \to x_0$ implies $f(a_n) \to f(x_0)$. $f(x)$ in (b) is discontinuous at $x = x_o$, since $a_n \to x_0$ and $b_n \to x_0$ but $\lim f(a_n) \neq \lim f(b_n)$.

(Solution:)

$$\boxed{\begin{array}{c}\text{The function } f(x) \text{ is continuous at } x = x_0 \textbf{ iff } \text{for each series} \\ a_n \to x_0 \quad \Rightarrow \quad f(a_n) \to f(x_0)\end{array}} \tag{1}$$

We have used the abbreviation **iff** = **if and only if**`[`$\stackrel{\mathbf{G}}{=}$ `dann und nur dann = genau dann]` also symbolized as $\Longleftrightarrow$.

(1) can be expressed in words like that:
In whatever way you approach to x_0 (i.e. by selecting a series $a_n \to x_0$) the corresponding function values (i.e. the $f(a_n)$) will approach the function value (i.e. $f(x_0)$) which was already defined there (i.e. $f(a_n) \to f(x_0)$).

(1) is obvious for fig. 2(a). From fig. 2(b) it is not clear what $f(x_0)$ is. However, since

$$\lim f(a_n) = y_1 \neq \lim f(b_n) = y_2 \tag{2}$$

(1) cannot be valid for whatever definition of the function $f(x)$ at $x = x_0$.

Rem: For the series c_n, mixing infinitely many members of a_n and infinitely many members of b_n, we have $c_n \to x_0$ but $\lim f(c_n)$ does not exist, so (1) cannot be valid.

9.1. **d**) What is the θ-function (**switching-function**`[`$\stackrel{\mathbf{G}}{=}$ `Einschaltfunktion]`). Give your answer in terms of a graph and formulas.

(Solution:)

$$\theta(x) = \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x > 0 \end{cases} \tag{3}$$

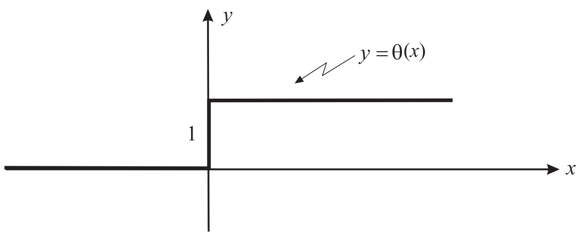


Fig9.1. **3**: The θ-function is discontinuous at $x = 0$. For $x = t =$ time it is the prototype of a switching-on process. It is continuous everywhere except at $x = 0$.

Rem 1: $\theta(x)$ is discontinuous for whatever definition we choose for $\theta(0)$. There are at least three versions for the definition of $\theta(x)$:

$$\theta(0) = 0, \quad \theta(0) = 1, \quad \theta(0) = \frac{1}{2} \tag{4}$$

Rem 2: However, we have a further option, namely to consider $\theta(0)$ undefined. Then the domain of definition of the θ-function is $\mathcal{D} = \mathbb{R} - \{0\}$. In this case a pure mathematician would say, the θ-function were continous everywhere, because 'everywhere' means 'everywhere in its domain of definition'. However, that way of speaking is counter-intuitive.

Rem 3: A similar situation holds for the tangent-function. In pure mathematical terminology, the tangent function is continous everywhere, because $\tan(\pi/2)$ is undefined, so the question of continuity does not arize there. A physicist, however, would say the tangent-function is discontinous at $x = \pi/2$, because whatever definition he adopts for $\tan(\pi/2)$ [finite, $+\infty$ or $-\infty$] the function is discontinous there.

Rem 4: Almost all discontinuous functions used in physics can be built with the help of $\theta(x)$ as the only discontinous function.

Rem 5: All discontinuous functions discussed so far are of a trivial type, called **piecewise continuous**[$\stackrel{G}{=}$ `st"uckweise stetig`], i.e. they are *discontinuous* only at a finite number of points but *continuous* in the intervals in between.

Rem 6: The following function is more seriously discontinuous, namely discontinuous *everywhere*:

$$f(x) = \begin{cases} 0 \text{ for } x = & \text{rational} \\ 1 \text{ for } x = & \text{irrational} \end{cases}$$

9.1. **e**) Classify your known functions according to continuity or discontinuity. Give only a rough answer.

(Solution:)

Almost all well-known functions given analytically, i.e. as formulas or as (convergent) power series, e.g. x^n, $\sin x$, $\cos x$, e^x, $\ln x$ and their composite functions e.g.

$$f(x) = e^x \sin x + x^3 \cos x \tag{5}$$

are continuous everywhere, except where a denominator becomes zero. To obtain other discontinuous functions one has to formulate an explicit **distinction** [$\stackrel{G}{=}$ `Fallunterscheidung`] as was done in (3).

9.**Q 2: Limit of a function**

9.2. **a**) What is the meaning of

$$\lim_{x \to x_0} f(x) \tag{1}$$

$$\lim_{x \to x_{0_+}} f(x) \qquad \text{(also called \textbf{limit from the right} [}\stackrel{G}{=} \texttt{rechtsseitiger Limes]}) \tag{2}$$

$$\lim_{x \to x_{0_-}} f(x) \qquad \text{(also called \textbf{limit from the left} [}\stackrel{G}{=} \texttt{linksseitiger Limes]}) \tag{3}$$

(Solution:)

1) (1) means that for each[1] sequence $a_n \to x_0$ $\quad \lim_{n\to\infty} f(a_n)$ exists and is the same, i.e. is independent of the particular choice of an a_n which goes to x_0.

E.g. if we have a different sequence $b_n \to x_0$ we will have also $\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} f(a_n)$.

$\lim_{x\to x_0} f(x)$ is this common limit (common for all possible $a_n \to x_0$).

2) (2) is the same but only for series $a_n \to x_0$ with the additional condition

$$a_n > x_0 \tag{4}$$

3) (3) is the same with the additional condition

$$a_n < x_0 \tag{5}$$

[1] $a_n \in \mathcal{D}$

9.Q 3: Continuity expressed by limits of functions

9.3. **a**) Re-express continuity with $\lim_{x\to x_0} f(x)$

(Solution:)

$$\boxed{f(x) \text{ continuous at } x = x_0 \Longleftrightarrow \lim_{x\to x_0} f(x) = f(x_0)} \tag{1}$$

In words: the function$f(x)$ is continuous at $x = x_0$ **iff** its limit for $x \to x_0$ exists and is equal to the functional value $f(x_0)$.

9.3. **b**) Re-express continuity with $\lim_{x\to x_{0_{\pm 0}}}$.

(Solution:)

$$\boxed{f(x) \text{ continuous at } x = x_0 \Longleftrightarrow \lim_{x\to x_{0+}} f(x) = \lim_{x\to x_{0-}} f(x) = f(x_0)} \tag{2}$$

In words: a function $f(x)$ is continuous **iff** the left side limit and the right side limit both exist and are equal to the functional value.

9.3. **c**) Calculate

$$\lim_{x\to 0_\pm} \theta(x) \tag{3}$$

(Solution:)

$$\lim_{x\to 0_+} \theta(x) = 1 \tag{4}$$

$$\lim_{x\to 0_-} \theta(x) = 0 \tag{5}$$

9.Ex 4: Limits of series built from continuous functions

Calculate the limits of the following series ($n \to \infty$).

9.4. **a**)

$$a_n = \sin\left(\frac{1}{n}\right) \tag{1}$$

HINT: sin is a continuous function.
RESULT: $a_n \to 0$

(Solution:)

$\frac{1}{n} \to 0$, since sin is continuous

$$\sin\left(\frac{1}{n}\right) \to \sin 0 = 0 \tag{2}$$

9.4. **b)**

$$a_n = e^{\sin(\frac{1}{n})} \tag{3}$$

HINT:

$$f(x) = e^{\sin(x)} \quad \text{is a continuous function} \tag{4}$$

RESULT:

$$a_n \to 1 \tag{5}$$

(Solution:)

$$a_n \to e^{\sin 0} = e^0 = 1 \tag{6}$$

9.4. **c)**

$$a_n = \ln \sum_{m=0}^{n} x^m \qquad \text{for} \quad |x| < 1 \tag{7}$$

HINT: ln is a continuous function. Its argument is a partial sum of the geometric series.
RESULT:

$$a_n \to \ln \frac{1}{1-x} \tag{8}$$

(Solution:)

$$\sum_{m=0}^{n} x^m \quad \xrightarrow{n\to\infty} \quad \frac{1}{1-x} \qquad \text{(sum of the geometric series)} \tag{9}$$

Since ln is continuous

$$\lim_{n\to\infty} \ln \sum_{m=0}^{n} x^m = \ln \sum_{m=0}^{\infty} x^m = \ln \frac{1}{1-x} \tag{10}$$

9.**Ex 5: Removable singularities**
Consider the function

$$y = f(x) = \frac{x}{x} \tag{1}$$

9.5. **a**) What is its domain $\mathcal{D}$?

(Solution:)

Domain $\mathcal{D}$ means its range of definition. Since division by zero is undefined, we have

$$\mathcal{D} = \mathbb{R} - \{0\} = \mathbb{R}^*, \quad \text{i.e. all } x \neq 0. \tag{2}$$

9.5. **b**) Is this function continuous at $x = 0$?
RESULT: no.

(Solution:)

Since the function is not defined at $x = x_0 = 0$ it cannot be continuous at $x = x_0$.

9.5. **c**) Extending the domain of f by the definition

$$f(0) = 5 \tag{3}$$

is f continuous now? Why or why not?

(Solution:)

For the series $a_n = \frac{1}{n} \neq 0,\ a_n \to 0$, we have

$$f(a_n) = \frac{a_n}{a_n} = 1 \to 1 \neq 5 \tag{4}$$

Therefore f is still discontinuous.

9.5. **d**) Redefine f at $x = 0$ so that f becomes continuous at $x = x_0 = 0$
RESULT:

$$f(0) = 1 \tag{5}$$

REM: We say that $x = 0$ was a **removable singularity**[$\stackrel{\mathbf{G}}{=}$ `hebbare Singularität`][2], of the function (1) at $x = 0$. The discontinuity could be **remedied**[$\stackrel{\mathbf{G}}{=}$ `geheilt`] by the additional definition (5).

9.**Ex 6: Limits of functions**
Calculate the following limits of functions.
9.6. **a**)

$$\lim_{x \to \frac{\pi}{2}} \sin x \tag{1}$$

[2] A discontinuity is an example of a **sigularity**[$\stackrel{\mathbf{G}}{=}$ `Singularität`].

HINT: $\sin x$ is continuous.
RESULT: $= 1$

(Solution:)

Because of continuity

$$\lim_{x\to\frac{\pi}{2}} \sin x = \sin\frac{\pi}{2} = 1 \tag{2}$$

9.6. **b)**

$$\lim_{x\to\frac{\pi}{2}+} \sin x \tag{3}$$

RESULT: $= 1$

(Solution:)

Since the limit exists, the left sided and right sided limits also exist and are equal.

9.6. **e)**

$$\lim_{x\to 0} \frac{x}{x} \tag{4}$$

RESULT: 1

(Solution:)

In the limit of a function $x \to x_0$ it is implied that $x \,\epsilon\, \mathcal{D}$, i.e. $x \neq 0$. We can then divide by x and obtain

$$\lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1 \tag{5}$$

9.6. **f)**

$$\lim_{x\to 0_+} \frac{1}{x} \tag{6}$$

RESULT: $= \infty$

9.6. **g)**

$$\lim_{x\to 0_-} \frac{1}{x} \tag{7}$$

RESULT: $= -\infty$

9.6. **h)**

$$\lim_{x\to 0} \frac{1}{x} \tag{8}$$

RESULT: The limit does not exist, i.e. (8) is a meaningless expression.

(Solution:)

Since the left and right side limits are not equal, the limit per se cannot exist.

9.6. **i)**

$$\lim_{x\to 0} \frac{x^2+x}{3x^2+2x} \tag{9}$$

HINT: **cancel**[$\stackrel{\mathbf{G}}{=}$ `Bruch kürzen`] the x's.
RESULT: $\frac{1}{2}$

(Solution:)

The domain $\mathcal{D}$ of $\frac{x^2+x}{3x^2+2x}$ contains all x different from the zeroes of the denominator $(x \neq 0, \quad x \neq -2/3)$.
Thus $x \to 0$ implies $x \neq 0$ and we can divide the function by x. Thus,

$$\lim_{x\to 0} \frac{x^2+x}{3x^2+2x} = \lim_{x\to 0} \frac{x+1}{3x+2} \stackrel{\clubsuit}{=} \frac{1}{2} \tag{10}$$

$\clubsuit$ Since $f(x) = \frac{x+1}{3x+2}$ is continuous at $x = 0$, the limit is $f(0) = \frac{1}{2}$.

9.6. **j)**

$$\lim_{\Delta x\to 0} \frac{(\Delta x)^2 + \triangle x}{3(\Delta x)^2 + 2 \triangle x} \tag{11}$$

HINT: $\triangle x$ is just another name for a variable, such as x, φ, α, etc. So this exercise is the same as (9).
RESULT: $\frac{1}{2}$

10 Differential and differentiation

10.Q 1: Tangent, derivative, differential

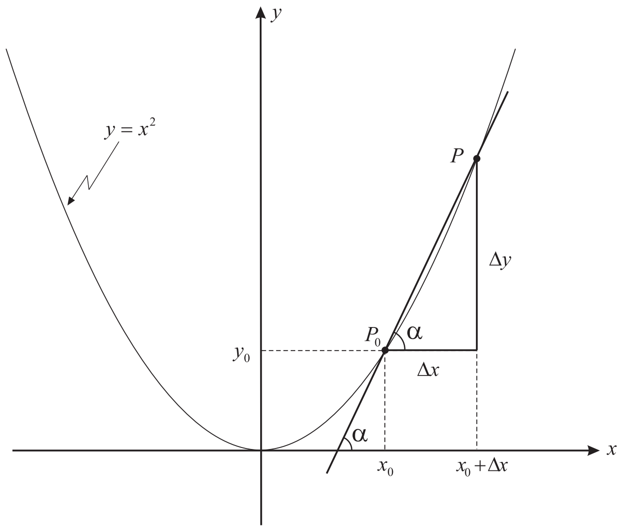


Fig10.1. **1**: In the case of the parabola ($y = x^2$) we see the increment Δy of the function value y while x increments from x_0 to $x_0 + \Delta x$

10.1. **a**) Give the coordinates of P_0. (In the following we consider P_0 as constant.)

(Solution:)

$$P_0 = (x_0, y_0) \qquad y_0 = x_0^2 \tag{1}$$

10.1. **b**) The variable point P is imagined to have arizen from P_0 by a **displacement**[$\stackrel{\mathbf{G}}{=}$ `Verschiebung, Verrückung`] along the parabola, whereby its coordinates have received the **increments**[$\stackrel{\mathbf{G}}{=}$ `Zuwächse`] $\Delta x, \Delta y$.

REM: It is usual to denote increments of a variable by prefixing it with Δ.

Give the coordinates of P expressed by $\Delta x, \Delta y$.

(Solution:)

$$P = (x_0 + \Delta x, y_0 + \Delta y) \tag{2}$$

10.1. **c**) We call Δx the **independent increment**, because x is the independent variable, and because both x and Δx can be chosen freely. Δy is then fixed, because the point P must move along the parabola. Therefore, we call Δy the **dependent increment.**

Calculate the dependent increment Δy, expressed by the independent increment Δx.

(Solution:)

$$y_0 + \Delta y = y = x^2 = (x_0 + \Delta x)^2 = x_0^2 + 2x_0(\Delta x) + (\Delta x)^2 \tag{3}$$

Because of $y_0 = x_0^2$

$$\Delta y = 2x_0(\Delta x) + (\Delta x)^2 \tag{4}$$

10.1. **d**) Calculate the **difference quotient** [$\stackrel{\mathbf{G}}{=}$ `Differenzenquotient`]

$$\frac{\Delta y}{\Delta x} \tag{5}$$

Rem: Here 'difference' is synonymous with 'increment'. Thus instead of 'difference quotient' the term 'increment quotient', though rarely used, would be more appropriate.

The **straigth line** [$\stackrel{\mathbf{G}}{=}$ `Gerade`] through P_0, P is called a **secant**. What is the geometrical meaning of the difference quotient for the secant?
Give a formula for α.

(Solution:)

$$\frac{\Delta y}{\Delta x} = 2x_0 + \Delta x = \tan\alpha \tag{6}$$

Hint: Note that both angles α in fig 1 are equal.

The difference quotient of a straight line (i.e. $\tan\alpha$) is called the **gradient** [$\stackrel{\mathbf{G}}{=}$ `Steigung`] or **slope** [$\stackrel{\mathbf{G}}{=}$ `Steigung`] of the straight line.

10.1. **e**) In the limit $\Delta x \to 0$ the secant becomes the tangent (lat. tangere = to touch) to the point P_0. Give the slope of the tangent.

Rem 1: The English word 'tangent' has two different meanings:

- **tangent** [$\stackrel{\mathbf{G}}{=}$ `Tangente`] = limit of a secant
- **tangent** [$\stackrel{\mathbf{G}}{=}$ `Tangens`] = tan = sin / cos

REM 2: Since the tangent is the limit of a secant, intersecting the curve in two points coming closer and closer together, one can express in a **picturesque** [$\stackrel{\mathbf{G}}{=}$ `bildlich`] way: 'the tangent intersects the curve in two infinitely neighbouring points'.
However, such a phrasing is mathematically incorrect, because there is not such a thing as 'two infinitely neighbouring points': Two points either coincide (i.e. are identical) or they have a finite distance.

(Solution:)

$$\Delta x \to 0 \quad \Rightarrow \frac{\Delta y}{\Delta x} \to 2x_0 = \tan\alpha \tag{7}$$

10.1. **f**) The slope of the tangent is called the **derivative** and is denoted by y'. Give the derivative of the function $y = x^2$.

(Solution:)

$x_0 \mapsto x$ (redenoting x_0 by x)

$$y' = 2x \tag{8}$$

10.1. **g**) The increment $\Delta y = 2x_0(\Delta x) + (\Delta x)^2$ is a sum of two terms (= summands). The first term is of first order in Δx because it contains Δx as a factor only once. The second term is of second order because it containes the factor Δx twice.

A **differential** (denoted by d instead of Δ) is an increment calculated approximately keeping only terms of lowest order. Calculate dx and dy and show that the derivative is the differential quotient

$$\boxed{y' = \frac{dy}{dx}} \qquad \boxed{\text{derivative} = \text{differential quotient}} \tag{9}$$

(Solution:)

$dx = \Delta x$, $dy = 2x_0\Delta x$ (neglecting $(\Delta x)^2$ in (4))
(redenoting x_0 by x)

$$\frac{dy}{dx} = 2x_0 = y' \tag{10}$$

10.1. **h**) What's the geometrical meaning of the differential dy?

(Solution:)

The differential dy is the dependent increment Δy, when the curve is replaced (approximated) by its tangent.

REM 1: The differential is the **tangential mapping** [$\stackrel{G}{=}$ `Tangentialabbildung`], i.e. the equation for the tangent. Short: The differential is the tangent (instead of the function).

REM 2: Though we have introduced the differential as an *approximation* of the increment, this should not lead to the erroneous conclusion that the differential itself is an inexact quantity or that differential calculus is an approximative method only. The differential is the *exact* equation for the tangent, but it is only an approximation for the secant.

10.**Ex 2: ☻ A second example**

Given the function

$$y = x^3 \quad \text{(Graph: cubic parabola)} \tag{1}$$

10.2. **a**) Starting from an arbitrary point (x, y) consider a displaced point $(x + \Delta x,\ \ y + \Delta y)$ on the curve. Calculate the increment Δy and the difference quotient $\Delta y/\Delta x$.

(Solution:)

$$\begin{aligned} y + \Delta y &= x^3 + \Delta y = \\ &= (x + \Delta x)^3 = (x + \Delta x)(x^2 + 2x(\Delta x) + (\Delta x)^2) = \\ &= x^3 + 2x^2\Delta x + x(\Delta x)^2 + x^2\Delta x + 2x(\Delta x)^2 + (\Delta x)^3 \end{aligned} \tag{2}$$

$$\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \tag{3}$$

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2 \tag{4}$$

10.2. **b**) Calculate the differential dx, dy, the differential quotient $\frac{dy}{dx}$ and the derivative y'.

(Solution:)

$$dx = \Delta x, \quad dy = 3x^2\Delta x = 3x^2 dx \tag{5}$$

$$y' = \frac{dy}{dx} = 3x^2 \tag{6}$$

10.2. **c**) Show that the limit of the difference quotient ($\Delta x \to 0$) is the differential quotient.

(Solution:)

$$(4) \quad \Rightarrow \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 3x^2 = y' \tag{7}$$

REM: When the independent variable x is time t and the dependent variable $y = f(t)$ is the position of a particle at time t, then the difference quotient

$$\frac{\Delta y}{\Delta t} = \overline{v} \tag{8}$$

is the average velocity $\overline{v}$ of the particle in the time interval $t \dots t + \Delta t$.
The differential quotient

$$\dot{y} = \frac{dy}{dt} = v(t) \tag{9}$$

is the **instantaneous velocity**[$\stackrel{\mathbf{G}}{=}$ `Momentangeschwindigkeit`] at time t.
Note that in physics derivatives with respect to time t are denoted by a dot ($\dot{}$) instead of a prime (’).

10.**Q 3: Derivatives of elementary functions**
Give the derivatives of the following functions

10.3. **a**) $y = a \quad (a = \text{const})$

(Solution:)

$$\boxed{a' = 0} \qquad \boxed{\text{The derivative of a constant is zero}} \tag{1}$$

10.3. **b**) $y = x^a \quad (a = \text{const})$

(Solution:)

$$\boxed{(x^a)' = a x^{a-1}} \quad (a = \text{const}) \quad (\textbf{power rule}[\stackrel{\mathbf{G}}{=} \texttt{Potenzregel}]) \tag{2}$$

10.3. **c**) $y = \sin x$

(Solution:)

$$\boxed{(\sin x)' = \cos x} \tag{3}$$

10.3. **d)** $y = \cos x$

(Solution:)

$$\boxed{(\cos x)' = -\sin x} \tag{4}$$

10.3. **e)** $y = e^x$

(Solution:)

$$\boxed{(e^x)' = e^x} \tag{5}$$

The (natural) exponential function is its own derivative

10.3. **f)** $y = \ln x$

(Solution:)

$$\boxed{(\ln x)' = \frac{1}{x}} \tag{6}$$

10.Q 4: Derivatives of composite functions

Given two functions $f(x), g(x)$ and the constant a. Give the derivative of

10.4. **a)** $y(x) = f(x) \pm g(x)$

(Solution:)

$$\boxed{\begin{array}{c}(f \pm g)' = f' \pm g' \\ \\ \textbf{The derivative of a sum is the sum of the derivatives}\end{array}} \tag{1}$$

10.4. **b)** $y(x) = af(x) \qquad (a = \text{const})$

(Solution:)

$$\boxed{(af)' = af' \qquad (a = \text{const})} \tag{2}$$

A constant a can be pulled before the derivative.

10.4. **c)** $y(x) = f(x)g(x)$

(Solution:)

$$\boxed{(fg)' = fg' + f'g} \quad (\textbf{Leibniz's product rule}) \tag{3}$$

10.4. **d**) $y(x) = \frac{f(x)}{g(x)}$

(Solution:)

$$\boxed{\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}} \quad (\textbf{quotient rule}) \tag{4}$$

10.4. **e**) $y(x) = f(g(x))$

(Solution:)

$$\boxed{y'(x) = f'(g(x))\, g'(x)} \quad (\textbf{chain rule}) \tag{5}$$

or short:

$$\boxed{y' = f'\, g'} \quad (\textbf{chain rule}) \tag{6}$$

The derivative of a composite function is the product of the derivatives of the composing functions

10.4. **f**) Apply e) for $f(x) = e^x$, $g(x) = ax, a =$const.

(Solution:)

The composite function is

$$y(x) = e^{ax} \tag{7}$$

$$y' = \frac{dy}{dx} = e^{ax} a \tag{8}$$

10.**Ex 5: Derivative for a very simple case**

Calculate the derivative $y'(x)$ of the function $y(x) = 3 + 2x$ using the rules for derivatives.

RESULT:

$$y'(x) = 2 \tag{1}$$

(Solution:)

We use the following rules:
derivative of a sum = sum of derivatives
derivative of a constant (3) is zero
a constant factor (2) can be pulled before the derivative.
Derivative of $x = x^1$ is

$$1x^{1-1} = x^0 = 1, \quad \text{i.e. } x' = 1 \tag{2}$$

Therefore:

$$y'(x) = (3 + 2x)' = 3' + (2x)' = 0 + 2x' = 2 \cdot 1 = 2 \tag{3}$$

10.**Ex 6: Differential quotient for a linear function**
Consider the function

$$y = -6 + 2x \tag{1}$$

10.6. **a**) Sketch it and describe its graph geometrically.

(Solution:)

See the following figure.
The graph is a straight line.

10.6. **b**) Why is y(x) called a linear function?

(Solution:)

In old fashioned terminology 'line' means straight line.

10.6. **c**) At $x = 7$ on the x-axis draw the increment $\Delta x = 2$, and on the y-axis the corresponding dependent increment Δy.

10.6. **d**) For an arbitrary independent increment (starting at x) and having length[3] Δx, calculate analytically[4] beginning with y and ending with $y + \Delta y$ and Δy itself of the corresponding dependent increment Δy.
Partial result:

$$\Delta y = 2\Delta x \tag{2}$$

(Solution:)

$$y = -6 + 2x \tag{3}$$

[3] In c) we had $x = 7, \Delta x = 2$
[4] i.e. with the formula $y = -6 + 2x$ and without the figure.

$$y + \Delta y = -6 + 2(x + \Delta x) = -6 + 2x + 2\Delta x = y + 2\Delta x \qquad (4)$$

$$\Delta y = 2\Delta x \qquad (5)$$

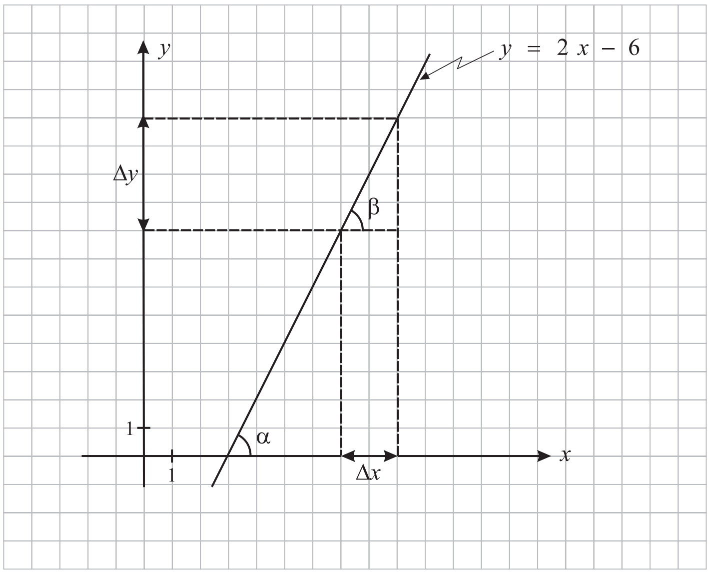


Fig10.6. **1**: Graphical representation of the linear function $y = -6 + 2x$ and its differential quotient $\Delta y/\Delta x$.

10.6. **e**) In fig.1 prove $\alpha = \beta$.
HINT: Use the following theorem of plane geometry: a straight line intersects two parallel lines at the same angle.

(Solution:)

The graph of $y = -6 + 2x$ is the straight line, the x-axis is one of the parallel lines.

10.6. **f**) Calculate the slope of the line.
HINT: The slope is $\tan\alpha$.
RESULT:

$$\Delta y/\Delta x = 2 \qquad (6)$$

(Solution:)

$$\tan\alpha = \tan\beta = \frac{\Delta y}{\Delta x} = 2 \qquad (7)$$

10.6. **g**) With a calculator calculate α (i.e. the angle between the x-axis and the line).
RESULT:

$$\alpha = 63.43^\circ \tag{8}$$

(Solution:)

$$\tan\alpha = 2, \quad \alpha = 63.43^\circ \tag{9}$$

10.6. **h**) The variable Δx is a small quantity of first order ($\Delta x \ll 1$), calculate the increments Δy and Δx in first order approximation. (For emphasis we have a formula in first order approximation, use dy and dx instead of Δy and Δx and call the increments differentials.)
RESULT:

$$dy = 2dx \tag{10}$$

(Solution:)

Both Δx and Δy have only first order contributions in Δx, therefore the first approximation is identical to the exact result (5). $\Delta x = dx$, $\Delta y = dy$. Thus (5) reads

$$dy = 2dx \tag{11}$$

10.6. **i**) Calculate the differential quotient $\frac{dy}{dx}$ and verify that it is equal to the slope and to the derivative of the function $y = -6 + 2x$.

(Solution:)

$$\frac{dy}{dx} = \frac{2dx}{dx} = 2 \tag{12}$$

10.**Ex 7: Derivative of the exponential function**

Consider

$$y(x) = e^x \qquad \text{(natural exponential function)} \tag{1}$$

10.7. **a**) Calculate the increment Δy.
HINT:

$$\Delta y = y(x + \Delta x) - y(x) \tag{2}$$

RESULT:

$$\Delta y = e^x(e^{\Delta x} - 1) \tag{3}$$

(Solution:)

$$\Delta y = e^{x+\Delta x} - e^x = e^x e^{\Delta x} - e^x = e^x(e^{\Delta x} - 1) \tag{4}$$

10.7. **b**) Calculate the corresponding differential.
HINT: dy is Δy in linear approximation in Δx. Use the power series for $e^{\Delta x}$.
RESULT:

$$dy = e^x dx \tag{5}$$

(Solution:)

$$\Delta y = e^x(1 + \Delta x + \frac{1}{2}(\Delta x)^2 + \cdots - 1) = e^x(\Delta x + \frac{1}{2}(\Delta x)^2 + \cdots) \tag{6}$$

In linear approximation ($\Delta x \equiv dx$)

$$dy = e^x dx \tag{7}$$

10.7. **c**) Calculate the differential quotient and verify that the derivative of the natural exponential function is identical to itself: $(e^x)' = e^x$.

(Solution:)

$$y' \stackrel{\text{def}}{=} \frac{dy}{dx} = e^x \qquad \text{q.e.d.} \tag{8}$$

10.7. **d**) Prove $(e^x)' = e^x$ again by using the power series of e^x and by assuming the derivative of the infinite sum is the infinite sum of the derivatives of the individual terms.

(Solution:)

$$\begin{aligned}
(e^x)' &= (1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots)' \\
&= x' + \tfrac{1}{2!}(x^2)' + \tfrac{1}{3!}(x^3)' + \tfrac{1}{4!}(x^4)' + \cdots \\
&= 1 + \tfrac{1}{2!}2x + \tfrac{1}{3!}3x^2 + \tfrac{1}{4!}4x^3 + \cdots \\
&= 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots \\
&= e^x
\end{aligned} \tag{9}$$

10.**Ex 8: The product rule**
Calculate the derivative of

$$y(x) = x^2 \sin x \tag{1}$$

RESULT:

$$y'(x) = x^2 \cos x + 2x \sin x \tag{2}$$

(Solution:)

$$y'(x) = x^2(\sin x)' + (x^2)' \sin x = x^2 \cos x + 2x \sin x \tag{3}$$

10.**Ex 9: The quotient rule**

10.9. **a**) Look up $(\tan x)'$ in a formulary.

10.9. **b**) Check the result with the help of the definition

$$\tan x = \frac{\sin x}{\cos x} \tag{1}$$

HINT: Use $\sin^2 x + \cos^2 x = 1$. (Remember: $\sin^2 x$ is an abbreviation for $(\sin x)^2$.)

(Solution:)

$$(\tan x)' = \frac{\cos x(\sin x)' - \sin x(\cos x)'}{(\cos x)^2} = \frac{\cos x \cos x + \sin x \sin x}{(\cos x)^2} = \frac{1}{\cos^2 x} \tag{2}$$

10.**T 10: Different notations for functions and their derivatives**

Consider a function

$$y = f(x) \tag{1}$$

Here f is the name of a function, x is the **independent variable** (also called the **argument** of the function), y is the **dependent variable**: To each x there corresponds a (unique) y given by (1), i.e. by the prescription f.

To save letters, sometimes the same letter is used for the function and the dependent variable, so (1) reads:

$$y = f(x) = y(x) \tag{2}$$

No confusion is possible: it is clear the first y is the dependent variable, the second y is the name of a function (e.g. $f = y = \sin$).

As a special case take $f = n$-th power, so (2) reads:

$$y = f(x) = y(x) = x^n \tag{3}$$

For the derivative then (at least) the following variety of notations exists:

$$y' = f' = \frac{dy}{dx} = \frac{d}{dx}\, y = \frac{dx^n}{dx} = \frac{d}{dx}\, x^n = y'\,(x) = (x^n)' \tag{4}$$

A composite function

$$y(x) = f(g(x)) \tag{5}$$

is written in pure mathematical texts as

$$y = f \circ g \tag{6}$$

which has the advantage that the irrelevant argument x does not appear.

In physics, on the other hand, very often one does not distinguish between the composite (y) and the outermost composing (f) function, since they often represent the same physical quantity only expressed in different coordinates, so (5) is written as

$$y = y(x) = y(g) = y(g(x)) \text{ with } g = g(x) \tag{7}$$

It is clear that the first y is the dependent variable, the second y is the composite function, and the third and fourth y is the outermost composing function.

As an example ($y \mapsto T, g \mapsto i$) let T be the temperature of a rod (measured in °C = °Celsius) at position i measured in inches, e.g.

$$T = T(i) = i^2 \tag{8}$$

(At position $i = 0$ the temperature is zero, two inches apart ($i = 2$) the temperature is 4 °C ($T = 4$)).

Now we want to express the temperature while position x is measured in meters (1 inch = 2.54 cm), so we have (approximately):

$$i = i(x) = 40x \tag{9}$$

meaning the following:
To x there corresponds i, e.g. to $x = 1$ (1m) there corresponds $i = 40$.
'To correspond' means that we consider an identical (the same) position.

Therefore we write:

$$\begin{aligned} T &= T(i) = T(i(x)) = T(40x) = T(x) \\ &= i^2 = [i(x)]^2 = 1600x^2 \end{aligned} \tag{10}$$

The first T is the dependent variable (measured temperature at a certain position), the second T is the temperature as a function of position expressed in inches. The same for the third and fourth T (outmost composing function). The last T is the composite function, namely the temperature as a function of position expressed in meters.
The second line of (10) inserts the special form (8) of our assumed temperature distribution.

With these notations the chain rule looks like that:

$$\text{For} \quad y(x) = y(z) \quad \text{with} \quad z = z(x)$$

$$\text{we have} \qquad\qquad (11)$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dz}\,\frac{dz}{dx}}$$

(**chain rule formulated with differentials**)

In the first differential quotient $(\frac{dy}{dx})$ the symbol y is considered as a function of x, i.e. y is the composite function.

In the second differential quotient $(\frac{dy}{dz})$ the symbol y is considered a function of z, i.e. is the left composing function.

10.Ex 11: The chain rule, 1. example

Calculate again the derivative of

$$y(x) = e^{ax} \qquad (1)$$

using the above chain rule formulated with differentials.

(Solution:)

We consider y as a composite function:

$$y(x) = y(z) = e^z \quad \text{with} \quad z = z(x) = ax \qquad (2)$$

According to the chain rule:

$$\frac{dy}{dx} = \underbrace{\frac{dy}{dz}}_{e^z}\,\underbrace{\frac{dz}{dx}}_{a} = ae^z = ae^{ax} \qquad (3)$$

10.Ex 12: The chain rule, 2. example

Calculate the derivatives of the following functions.

10.12. **a**)

$$y(x) = a\sin(kx), \quad a, k = \text{ constants} \qquad (1)$$

HINT: write the chain rule as

$$\frac{dy}{dx} = \frac{dy}{dz}\,\frac{dz}{dx} \quad \text{with } z = kx \qquad (2)$$

RESULT:

$$y'(x) = ak\cos(kx) \tag{3}$$

(Solution:)

With

$$y = a\sin z \tag{4}$$

we have

$$y'(x) = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = a\cos z \cdot k = ak\cos(kx) \tag{5}$$

10.12. **b)** $y = e^{e^x}$
RESULT:

$$y' = \exp(x + e^x) \tag{6}$$

(Solution:)

With $z = e^x, \quad y = e^z$ we have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^z e^x = e^{e^x} e^x = e^{x+e^x} = \exp(x + e^x) \tag{7}$$

10.**Ex 13: Velocity as the derivative with respect to time t.**
In a harmonic oscillator the mass-point is at position

$$x(t) = a\sin(\omega t + \chi) \tag{1}$$

$$\begin{array}{ll} a = \text{ amplitude}, & \omega = \text{ angular frequency}, \\ \chi = \text{ phase-shift}, & a, \omega, \chi = \text{ constants} \end{array} \tag{2}$$

Calculate the velocity

$$v = \dot{x}(t) = \frac{dx}{dt} \tag{3}$$

as a function of t ($v = v(t) =$ momentaneous velocity).
REM: when the independent variable is time t, it is usual in physics to use a dot ($\cdot$) instead of a prime ($'$) to denote the derivative.
RESULT:

$$v(t) = a\omega\cos(\omega t + \chi) \tag{4}$$

(Solution:)

With

$$z = \omega t + \chi \tag{5}$$

we have

$$v(t) = \frac{dx}{dt} = \frac{dx}{dz}\frac{dz}{dt} = a\cos z \cdot \omega = a\omega\ \cos(\omega t + \chi) \tag{6}$$

10.Ex 14: ☻☻ Velocity of a damped harmonic oscillator

In a **damped**[$\stackrel{\mathbf{G}}{=}$ `gedämpft`] harmonic oscillator the mass-point is at position

$$x(t) = ae^{-\sigma t}\sin(\omega t) \tag{1}$$

a, σ, $\omega = \text{const.}$
Calculate the velocity.

REM: It is a harmonic oscillator with an exponentially **decaying**[$\stackrel{\mathbf{G}}{=}$ `zerfallende, abnehmende`] amplitude $ae^{-\sigma t}$. So, strictly speaking, it is harmonic only approximately in a short time intervall in which the amplitude can be considered constant.

(Solution:)

$$v = \dot{x}(t) = a[e^{-\sigma t}\omega(\cos\omega t) - \sigma e^{-\sigma t}\sin(\omega t)] \tag{2}$$

10.Ex 15: Derivative of x^x

Calculate the derivative of

$$y(x) = x^x \tag{1}$$

HINT: first prove

$$y(x) = e^{x\ln x} \tag{2}$$

RESULT:

$$y' = x^x(1 + \ln x) \tag{3}$$

(Solution:)

$$y(x) = x^x = e^{\ln x^x} = e^{x\ln x} \tag{4}$$

With

$$z = x\ln x, \quad y = e^z \tag{5}$$

the chain rule yields

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^z(x\ln x)' \stackrel{\clubsuit}{=} e^z(x \cdot (\ln x)' + 1\ln x) = \tag{6}$$

$$= e^z(x \cdot \frac{1}{x} + \ln x) = x^x(1 + \ln x) \tag{7}$$

$\boxed{\clubsuit \text{ product rule}}$

11 Applications of differential calculus

11.Q 1: Minimax problems

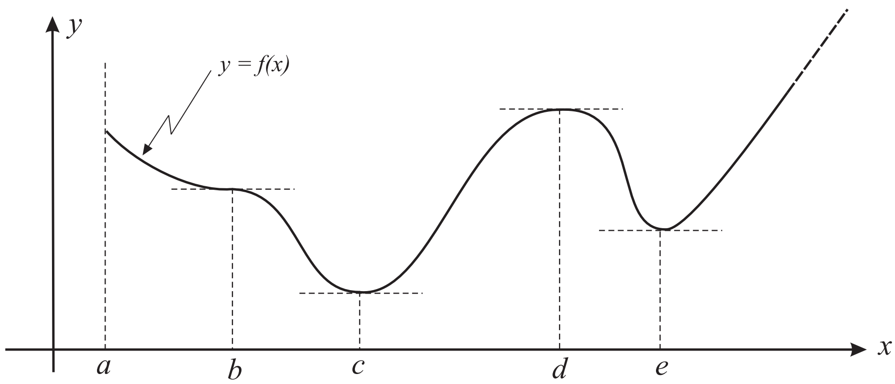


Fig11.1. **1**: A function defined in the interval $[a, \infty)$ with stationary points at $x = b$, c, d, e; local extremas at $x = c$, d, e and saddle point at $x = b$.

Fig. 1 shows the graph of a function $y = f(x)$ defined in the interval $[a, \infty)$.
11.1. **a**) How do we calculate the minimum of $y = f(x)$?

(Solution:)

We determine all points x for which

$$\boxed{f'(x) = 0 \quad \text{(stationary points)}} \tag{1}$$

holds. In our case this yields[5]

$$x = b,\ c,\ d,\ e. \tag{2}$$

Now we calculate f at these stationary points *and* also at the boundaries of the domain (at $x = a$ and $x = \infty$ in our case):

$$f(a),\ f(b),\ f(c),\ f(d),\ f(e),\ f(\infty) = \infty \tag{3}$$

We choose $f(c)$ since this is the lowest value. Thus:
The function has a *minimum at* $x = c$ and the minimum is (i.e. *has the value*) $f(c)$.

11.1. **b**) Does the function have a maximum? (Give both a precise and a sloppy answer.)

(Solution:)

[5] e is not Euler's number e.

Precise answer: No, since for larger and larger x's ($x > e$) we obtain even larger values for $y = f(x)$.
Sloppy answer: The function's maximum is at $x = \infty$ and is (has the value) $f(\infty) = \infty$.

11.1. **c**) What is at $x = e$ and what is a precise definition for that term?

(Solution:)

At $x = e$ the function has a **local minimum**, i.e. when the domain is restricted to a sufficiently small interval

$$[e - \varepsilon,\ e + \varepsilon] \quad (\varepsilon > 0) \tag{5}$$

around e, the function has an (**absolute**, also called a **global**) **minimum** at $x = e$.

REM: $x = a$ and $x = d$ are local maxima.

11.1. **d**) Why is (1) called a **stationary point**?

(Solution:)

$f'(x) = 0$ can also be written as

$$\boxed{dy = 0 \qquad \text{stationary point}} \tag{1'}$$

or in full

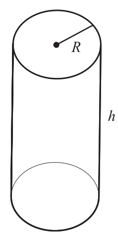
$$dy = f'(x)\ dx = 0 \tag{1''}$$

i.e. the tangent is horizontal. So (1′) says that the function does not change, i.e. in an old fashioned language, it is **stationary** in *linear approximation*.
REM: The exact increment Δy is not zero, but in linear approximation in

$$dx \equiv \Delta x = x - c \text{ it is } \Delta y \approx dy = 0 \tag{6}$$

11.**Ex 2: Shape of maximum volume with given surface**

We would like to construct a **cup**[$\overset{\mathbf{G}}{=}$ `Becher`] out of gold in the **shape**[$\overset{\mathbf{G}}{=}$ `Form`] of a cylinder with radius R and height h (see fig.1) containing maximum volume V.



`Fig`11.2. **1**: Calculation of cylindrical cup (height h, radius R) with given surface having maximum volume.

Since the **available**[$\stackrel{\mathbf{G}}{=}$ `zur Verfügung stehend`] **amount**[$\stackrel{\mathbf{G}}{=}$ `Menge`] of gold is limited, the **area**[$\stackrel{\mathbf{G}}{=}$ `Fläche`] of the cup is given as A_0 (the top of the cup is open).

11.2. **a**) Calculate V and area A_0 as a function of R and h.
RESULT:

$$V = h\pi R^2, \quad A_0 = 2\pi Rh + \pi R^2 = \text{ fixed} \tag{1}$$

11.2. **b**) Eliminate h and express $V = V(R)$ for the given A_0.
RESULT:

$$V = V(R) = \frac{1}{2}A_0 R - \frac{1}{2}\pi R^3 \tag{2}$$

(Solution:)

$$h = \frac{A_0 - \pi R^2}{2\pi R}, \quad V = \frac{1}{2}R(A_0 - \pi R^2) \tag{3}$$

11.2. **c**) Calculate R for the optimal cup.
RESULT:

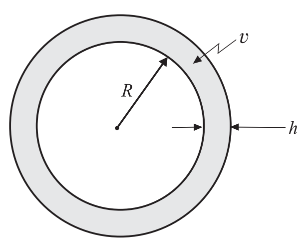
$$R = \sqrt{\frac{A_0}{3\pi}} \tag{4}$$

(Solution:)

$$\text{Extremum: } 0 \stackrel{!}{=} \frac{dV}{dR} = \frac{1}{2}A_0 - \frac{3}{2}\pi R^2 \tag{5}$$

11.**Ex 3: Gold necessary for a gold ball**

We would like to construct a ball with inner radius R and wall thickness h. Calculate the amount necessary (i.e. volume v).



`Fig`11.3. **1**: Volume v of the rind of a ball with inner radius R and thickness h.

11.3. **a**) In a formulary look up the volume $V = V(r)$ of a sphere of radius r.
RESULT:

$$\boxed{V = \frac{4}{3}\pi r^3} \qquad (V = \text{ volume of sphere with radius } r) \tag{6}$$

11.3. **b**) The answer to our problem is therefore

$$v = \frac{4}{3}\pi(R+h)^3 - \frac{4}{3}\pi R^3 \tag{7}$$

However, we want the answer only in linear approximation in the small quantity h ($h \ll R$), and to **save**[$\overset{\mathbf{G}}{=}$ `sparen`] computation we apply differential calculus:

$$v = dV, \quad h = dr \tag{8}$$

(indeed: v is the increment of $V(r)$ while incrementing r from R to $r = R + dr$.) Calculate v by differentiating (6).
RESULT:

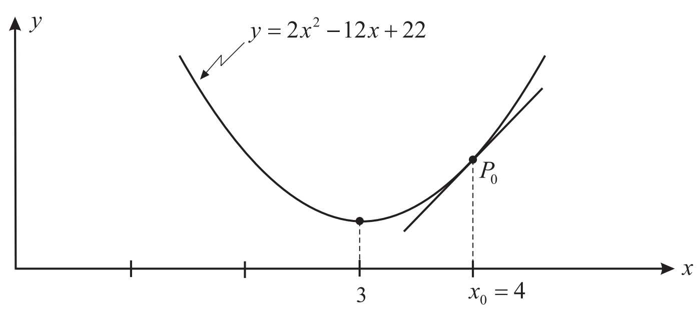
$$v = 4\pi r^2 h \tag{9}$$

(Solution:)

$$V'(r) = \frac{dV}{dr} = \frac{4}{3}\pi 3r^2 = 4\pi r^2 \tag{10}$$

$$v = dV = 4\pi r^2 dr = 4\pi r^2 h \tag{11}$$

11.Ex 4: ☻☻ The differential as the equation for the tangent



`Fig`11.4. **1**: Graph of a quadratic function. The differential is the equation for the tangent (at any point P_0).

Consider the quadratic function

$$y = f(x) = 2x^2 - 12x + 22 \tag{1}$$

11.4. **a**) Show that it has an extremum at $x = 3$.

(Solution:)

$$y' = 4x - 12 = 0 \;\Rightarrow\; x = 3 \quad \text{q.e.d.} \tag{2}$$

11.4. **b**) Show that the extremum is a minimum.

HINT: As will be discussed more fully in the next chapter, the extremum is a minimum if the second derivative (i.e. the derivative of the derivative $= (y')' = y''$) is positive.

(Solution:)

$$y'' = (4x - 12)' = 4 > 0, \quad \text{i.e. minimum} \tag{3}$$

11.4. **c**) Find the equation of the tangent at the point $P_0(x_0, y_0)$ for $y_0 = f(x_0),\ x_0 = 4$.
HINT: use the fact that the differential

$$dy = f'(x_0)dx \tag{4}$$

is the tangential mapping (the tangential or linear approximation to the function) i.e. the equation of the tangent. Write dx and dy in (4) in terms of $(x, y) =$ running point of the tangent and $(x_0, y_0) = P_0$.
RESULT:

$$y = y_0 + f'(x_0)(x - x_0) \tag{5}$$

(Solution:)

$$dx = x - x_0, \quad dy = y - y_0 \tag{6}$$

Thus (4) reads

$$y - y_0 = f'(x_0)(x - x_0) \tag{7}$$

11.4. **d**) Verify that (5) is the tangent by checking that it goes through P_0 and that at P_0 it has the same slope as the curve.

(Solution:)

1) For $x = x_0,\ y = y_0$ (5) is valid, i.e. the straight line (5) passes through P_0.
2) The slope of the straight line (5) is $y' = f'(x_0)$ i.e. identical to the slope of the curve.

11.**Ex 5: Average as the best guess for a measured quantity**
(**average**[$\stackrel{\mathbf{G}}{=}$ `Durchschnitt`], **guess**[$\stackrel{\mathbf{G}}{=}$ `Voraussage`])

A student measured the length l of a **rod**[$\stackrel{\mathbf{G}}{=}$ `Stab`] several (n) times, obtaining the results

$$l_i, \quad i = 1, 2, \cdots n \tag{1}$$

What should he report to his professor as the "true" value l for the length of the rod? We assume that the true length of the rod was constant while it was being measured and that the discrepancies in (1) are due to errors in the measurements. According to **Gauss** , a single measuring error

$$\Delta l_i = l_i - l \tag{2}$$

should get a **penalty**[$\stackrel{\mathbf{G}}{=}$ `Strafe`] proportional to the **square** of Δl_i (**principle of least squares**[$\stackrel{\mathbf{G}}{=}$ `Prinzip der kleinsten Fehlerquadrate`])[6] i.e the quantity l should be chosen so that the quantity (P = penalty = sum of error squares)

$$P = \sum_{i=1}^{n} (\Delta l_i)^2 \tag{3}$$

becomes minimal. Show that l is the average of l_i:

$$l = \bar{l} = \frac{1}{n} \sum_{i=1}^{n} l_i \qquad \text{(average)} \tag{4}$$

HINT 1: write down $P(l)$ and differentiate it with respect to l with l_i = constant. The derivative of a sum is the sum of the derivatives.
INTERMEDIATE RESULT:

$$\sum_{i=1}^{n} (l - l_i) = 0 \tag{5}$$

HINT 2: Separate it into two sums. l is constant here, i.e. it can be pulled before the sum.

(Solution:)

$$P = P(l) = \sum_{i=1}^{n} (l_i - l)^2 \tag{6}$$

[6]More exactly: principle of minimum sums of error squares.

$$P' = \frac{dP}{dl} = \sum_{i=1}^{n} \frac{d}{dl}(l_i - l)^2 \tag{7}$$

with $z = l_i - l$, $\frac{dz}{dl} = -1$, the chain rule yields

$$P' = \sum_{i=1}^{n} 2z \cdot (-1) = -2\sum_{i=1}^{n}(l_i - l) \stackrel{!}{=} 0 \tag{8}$$

According to hint 2 this reads

$$\sum_{i=1}^{n} l_i = \sum_{i=1}^{n} l = l\sum_{i=1}^{n} 1 = nl \tag{9}$$

i.e. we have obtained (4). q.e.d.
Rem 1: As usual we have chosen a vertical bar in (4) to denote the average of a quantity.
Rem 2: According to Gauss (3), a large error (i.e. $\Delta l = 10$ mm) is punished very severely $[(\Delta l)^2 = 100 \text{ mm}^2]$, whereas a small error (e.g. $\Delta l = 1$ mm) gives only a mild penalty $((\Delta l)^2 = 1 \text{ mm}^2)$.
Rem 3: The principle of least squares is only valid for **random errors**[$\stackrel{\mathbf{G}}{=}$ `zufällige Fehler`] and when the error is composed of a *large number* of random contributions with **both signs**[$\stackrel{\mathbf{G}}{=}$ `beiderlei Vorzeichen`]. A typical example is the observation of the position of a star. Light travelling through the atmosphere suffers small derivations in all directions.[7]
The principle (4) is not valid for **systematic errors**, e.g. when the measuring rod was calibrated incorrectly, or when the student only concentrated during the first measurement $i = 1$.

11.**Ex 6: Error propagation**
(**Error propagation**[$\stackrel{\mathbf{G}}{=}$ `Fehlerfortpflanzung`])
In the laboratory a student has the **task**[$\stackrel{\mathbf{G}}{=}$ `Aufgabe`] of determinig the outer radius R of a gold ball.

[7] Temperature gradients due to turbulence lead to a variable **diffraction index**[$\stackrel{\mathbf{G}}{=}$ `Brechungsindex`]) partially cancelling each other out.

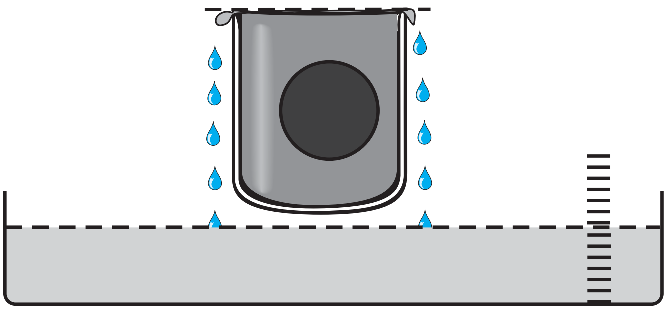


Fig11.6. **1**: Archimedes was the first to determine the volume of a complicated figure (a king's crown in his case, a ball with radius R in our case) by the amount of overflowing water.

The ball is **immersed**[$\stackrel{G}{=}$ `eintauchen`] into a full bottle and the student measures the volume V of the overflown water, which is identical to the volume

$$V = \frac{4}{3}\pi R^3 \tag{1}$$

of the gold ball. From that the student calculates

$$R = \left(\frac{3}{4\pi}V\right)^{\frac{1}{3}} \tag{2}$$

We assume the measurement of V has a relative error ε_V (e.g. $\varepsilon_V = 0.1\% = 0.001$). We would like to **estimate**[$\stackrel{G}{=}$ `abschätzen`] the relative error ε_R of R calculated by (2).

11.6. **a**) We treat the relative errors $\varepsilon_V, \varepsilon_R$ and the corresponding absolute errors as differentials. Identify these differentials.

Hint: The absolute error of the volume is $\Delta V = V_m - V$, where V_m is what the student has measured and V is the exact (unknown) value of the volume. The absolute error of the radius is $\Delta R = R_m - R$, where R is the exact (unknown) value of the radius and R_m is what the student will calculate (report) based upon his unexact measurement V_m. The relative errors are $\epsilon_V = \frac{\Delta V}{V}, \epsilon_R = \frac{\Delta R}{R}$.

Result: Absolute error in the measurement of V is $dV = \varepsilon_V V$, absolute error in the determination of R is

$$dR = \varepsilon_R R \tag{3}$$

11.6. **b**) Calculate the relative error ε_R of the R that the student should report.
Hint: Calculate the relationship between the differentials dR and dV by differentiating (1) with respect to R.

RESULT:

$$\varepsilon_R = \tfrac{1}{3}\varepsilon_V \tag{4}$$

(Solution:)

$$\frac{dV}{dR} = 4\pi R^2 \tag{5}$$

$$dV = 4\pi R^2\ dR \tag{6}$$

$$\frac{dV}{V} = \frac{4\pi R^2\ dR}{\frac{4}{3}\pi R^3} = 3\frac{dR}{R} \tag{7}$$

$$\varepsilon_V = 3\varepsilon_R \tag{8}$$

12 Higher derivatives, Taylor's formula

12.**Q 1: Higher derivatives**
The second derivative is the derivative of the derivative

12.1. **a**) For $y(x) = \sin x$ calculate
$y', y'', y''', y^{(4)}$ and the n-th derivative $y^{(n)}$.

(Solution:)

$$\begin{aligned} y &= \quad \sin x \\ y' &= \quad \cos x \\ y'' &= -\sin x \\ y''' &= -\cos x = y^{(3)} \\ y'''' &= \quad \sin x \ = y^{(4)} \end{aligned} \tag{1}$$

$$y^{(n)} = \begin{cases} (-1)^k \sin x & \text{for} \quad n = \text{even}: \quad n = 2k \\ (-1)^k \cos x & \text{for} \quad n = \text{odd}: \quad n = 2k+1 \end{cases} \tag{2}$$

with $k \,\epsilon\, \mathbb{N}_o$.

REM: The function itself is sometimes called the zeroth derivative.

12.1. **b**) For $y(x) = e^x$ calculate $y^{(n)}$.

(Solution:)

$$y^{(n)} = e^x \tag{3}$$

12.1. **c**) For $y(x) = x^5$ calculate $y^{(n)}$.

(Solution:)

$$\begin{aligned} y' &= 5x^4 \\ y'' &= 20x^3 \\ y''' &= 60x^2 \\ y'''' &= 120x \\ y^{(5)} &= 120 \\ y^{(n)} &= 0 \quad \text{for } n \geq 6 \end{aligned} \tag{4}$$

12.**Q 2: Taylor's formula**

12.2. **a**) Develop a function $y = f(x)$ about the point $x = 0$.

(Solution:)

Taylor's formula:

$$\boxed{f(x) = f(0) + f'(0)\,x + \frac{1}{2}f''(0)\,x^2 + \frac{1}{3!}f'''(0)\,x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\,x^n} \tag{1}$$

REM: When x is time, and 0 is now, Taylor's formula can be used to forecast weather: Truncate the formula including the first three terms only.
$f(0)$ is the weather (e.g. temperature) now. $f'(0)$ is the change of weather now, and $f''(0)$ is the change of change (acceleration) of weather now. So (1) can be used to forecast weather at time x, e.g. tomorrow.

12.2. **b**) Generalize to the development about an arbitrary point x_o.

(Solution:)

$$\boxed{f(x) = f(x_o + h) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) h^n \qquad \text{with } x = x_o + h} \tag{2}$$

REM: A slightly different notation for (2) is

$$\boxed{\Delta y = y'(x)\;\Delta x + \frac{1}{2!}y''(x)(\Delta x)^2 + \frac{1}{3!}y'''(x)(\Delta x)^3 + \cdots} \tag{3}$$

Here we have written, see fig. 1, $\Delta y = f(x_o + h) - f(x_o)$, $\quad \Delta x = h$, $\quad y' = f'(x_o)$, etc.

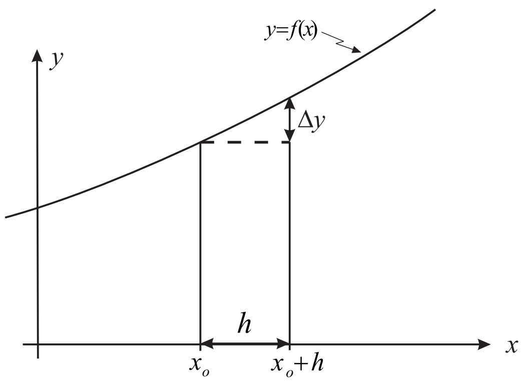


Fig12.2. **1**: While x increments from x_o to $x_o + h$, the function value y increments from $f(x_o)$ to $f(x_o) + \Delta y$. Δy is given by Taylor's formula in terms of the higher derivatives of $y = f(x)$ at $x = x_o$.

REM: The **gist** [$\stackrel{G}{=}$ `Knackpunkt`] of Taylor's formula is it gives the whole function $f(x)$ if we know all its higher derivatives at a single point x_0. Of course, Taylor's

formula is valid only if, among other assumptions not formulated here, the function $f(x)$ is differentiable an infinite number of times.
If we know only the first few higher derivatives at x_0, we can still use Taylor's formula, **truncated**[$\stackrel{G}{=}$ abgeschnitten] after the first few terms, since it yields an approximative value for $f(x_0+h)$ for small values of h.

12.Q 3: Distinguishing local minima from local maxima

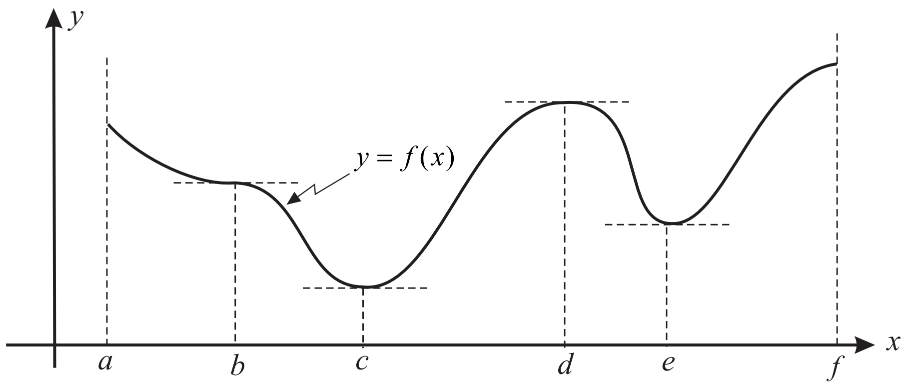


Fig12.3. **1**: Local minima at $x = c, e$, local maximum at d, and saddle point at b can be distinguished with the help of higher derivatives.

In the function $y = f(x)$ shown in fig. 1, defined in the intervall $[a, f]$, we see stationary points at $x = b, c, d, e$ (i.e. $f'(b) = 0$, etc.)
12.3. **a**) How can we decide what is a minimum and what is a maximum?

(Solution:)

$$\boxed{f'(x_0) = 0, \quad f''(x_0) < 0} \quad (\textbf{local (or relative) maximum} \text{ at } x = x_0) \tag{1}$$

$$\text{proof:} \quad \begin{array}{ll} \text{left of } d : f'(x) & > 0 \\ \text{right of } d : f'(x) & < 0 \end{array} \tag{2}$$

i.e. $f'(x)$ is decreasing: $f''(x_0) < 0$. Similarly:

$$\boxed{f'(x_0) = 0, \quad f''(x_0) > 0} \quad (\textbf{local (or relative) minimum} \text{ at } x = x_0) \tag{3}$$

REM: c is the absolute minimum, but of course, it is also a local (or relative) minimum.

12.3. **b**) How can we recognize that $x = b$ is a saddle point?

(Solution:)

$$\boxed{f'(x_0) = f''(x_0) = 0, \quad f'''(x_0) \neq 0} \qquad (\textbf{saddle point} \text{ at } x = x_0) \tag{4}$$

REM: The last condition in (4) is necessary. As a concrete example for this situation consider

$$y = x^4 \tag{5}$$

at $x = x_0 = 0$
We have

$$f'(x_0) = f''(x_0) = f'''(x_0) = 0, \quad f''''(x_0) > 0 \quad \text{(minimum)} \tag{6}$$

In these rare cases the reader should consult a formulary.

12.**Ex 4: Taylor's formula to construct power series**
Use Taylor's formula to derive the power series for e^x and $\sin x$.
HINT 1: develop around $x_0 = 0$.
HINT 2 for $\sin x$: use $n = 2k+1,\ k = 0, 1, \cdots \infty$ to select only odd n in the sum.
RESULT:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \tag{1}$$

(Solution:)

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n \tag{2}$$

1) For

$$f(x) = e^x, \quad f^{(n)}(x) = e^x, \quad f^{(n)}(0) = 1 \tag{3}$$

$$\text{thus,} \quad e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{4}$$

2) For $f(x) = \sin x$ we have $f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, \cdots$ which can be summarized as

$$f^{(n)}(x) = \begin{cases} (-1)^k \sin x \text{ for } n = 2k \\ (-1)^k \cos x \text{ for } n = 2k+1 \end{cases} \tag{5}$$

$$f^{(n)}(0) = \begin{cases} 0 \text{ for } n = 2k \\ (-1)^k \text{ for } n = 2k+1 \end{cases} \tag{6}$$

$$\text{thus,} \qquad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \tag{7}$$

12.Ex 5: Taylor's formula in linear approximations

Truncate[$\overset{\mathbf{G}}{=}$ `abschneiden`] Taylor's formula to linear approximation in h, identify differentials and show that we obtain

$$\boxed{dy = f'(x)\ dx} \tag{1}$$

(Solution:)

For $y = f(x)$ the increments are $\Delta y = f(x_0 + h) - f(x_0)$, $\Delta x = h$. In linear approximation in h Taylor's formula reads

$$f(x_0 + h) = f(x_0) + f'(x_0)h \tag{2}$$

i.e.

$$\Delta y = f'(x_0)\Delta x \tag{3}$$

To **distinguish**[$\overset{\mathbf{G}}{=}$ `kennzeichen`] it as a formula for linear approximation we write d instead of of Δ.

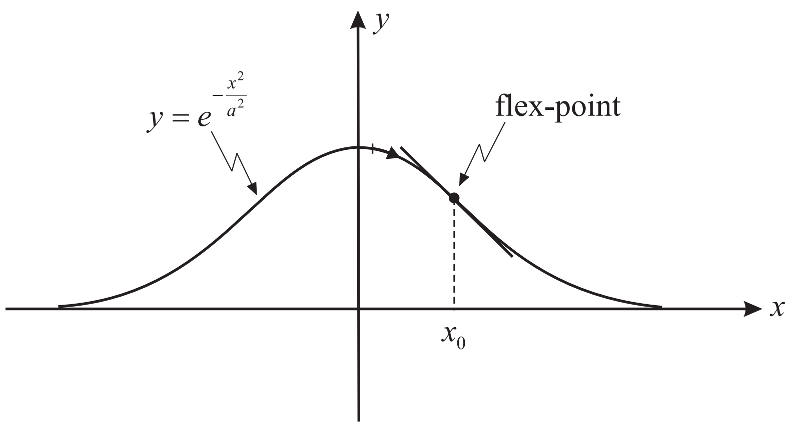
$$dy = f'(x_0)dx \tag{4}$$

x is as good as x_0 to denote an arbitrary point.

12.Ex 6: ☻☻ Qualitative analysis of the Gaussian bell-shaped curve

The **gaussian bell-shaped curve**[$\overset{\mathbf{G}}{=}$ `Gausssche Glockenkurve`] is given by

$$y = e^{-\frac{x^2}{a^2}}, \quad a = \text{ const.}, \quad a \neq 0 \tag{1}$$



`Fig`12.6. **1**: The Gaussian has maximum at $x = 0$ and flex-points at $x = \pm x_0$ where a driver (small arrow) has to change the sign of his direction: $y''(x_0) = 0$.

12.6. **a**) From the graph of e^x show that the Gaussian (1) is always positive and $y(\pm\infty) = 0$.

(Solution:)

The exponential function is positive everywhere (positive definite).

For $x \to \pm\infty$

$$x^2 \to +\infty, \quad \frac{x^2}{a^2} \to \infty, \quad -\frac{x^2}{a^2} \to -\infty \tag{2}$$

According to the graph of e^x we have $e^{-\infty} = 0$.

12.6. **b**) Show that the Gaussian is an **even function**[$\stackrel{\mathbf{G}}{=}$ `gerade Funktion`], i.e. that the graph is mirror-symmetric with respect to the y-axis.

(Solution:)

$$y(-x) = e^{-\frac{x^2}{a^2}} = y(x) \qquad \text{q.e.d.} \tag{3}$$

12.6. **c**) Show that the only extremum is at $x = 0$.

(Solution:)

With

$$z = -\frac{x^2}{a^2}, \quad \frac{dz}{dx} = -\frac{2x}{a^2}, \quad y = e^z \tag{4}$$

the chain rule yields

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = -\frac{2x}{a^2}e^z \stackrel{!}{=} 0 \tag{5}$$

Since $-\frac{2}{a^2} \neq 0, \quad e^z \neq 0, \quad$ we find $x = 0 \quad$ q.e.d.

12.6. **d**) Show that the extremum is a maximum.

(Solution:)

$$y'' \stackrel{\clubsuit}{=} -\frac{2}{a^2}e^z - \frac{2x}{a^2}y' \stackrel{(5)}{=} -\frac{2}{a^2}e^z + \left(\frac{2x}{a^2}\right)^2 e^z \tag{6}$$

$\boxed{\clubsuit \text{ product rule applied to (5)}}$

$$y''(0) = -\frac{2}{a^2} < 0 \quad \Rightarrow \quad \text{maximum} \tag{7}$$

12.6. **e**) At $x = x_0$ (and at $x = -x_0$) the Gaussian has a **flex-point** [$\stackrel{G}{=}$ `Wendepunkt`]. Look up the conditions for a flex-point in a formulary.
RESULT:

$$\boxed{\begin{array}{l} y''(x_0) = 0 \\ y'''(x_0) \neq 0 \end{array}} \qquad \textbf{(flex point)} \tag{8}$$

REM: When you "drive" along the graph in the direction of the small arrow in fig. 1, you must turn the **steering wheel** [$\stackrel{G}{=}$ `Steuerrad`] to the right before the flex-point, i.e. y' is decreasing, i.e. $y'' < 0$. After the flex-point you must turn the steering wheel to the left, i.e. y' is increasing, i.e. $y'' > 0$. Therefore, at the flex-point x_0 you have $y''(x_0) = 0$ i.e. you drive straight ahead. The second condition in (8) is necessary to ensure the curve really turns to the opposite side.

12.6. **f**) From (8) calculate the flex-point x_0 for the Gaussian.
RESULT:

$$x_0 = \pm \frac{a}{\sqrt{2}} \tag{9}$$

(Solution:)

According to (6) $y'' = 0$ yields

$$\frac{2}{a^2} = \frac{4}{a^4} x^2, \qquad 1 = \frac{2}{a^2} x^2, \qquad x^2 = \frac{a^2}{2}, \quad x = \pm \frac{a}{\sqrt{2}} \tag{10}$$

12.**Ex 7: Extrapolation with Taylor's formula**

An **economist** [$\stackrel{G}{=}$ `Wirtschaftswissenschaftler`] would like to predict the GNP (= **gross national product** [$\stackrel{G}{=}$ `Bruttosozialprodukt`][8])

$$G(t) = GNP \tag{1}$$

for the year 2008 (i.e. for $t = 2008$ years) using the following known data:

$$G(1998) = 10^{12} \text{ euros} \tag{2}$$

$$G(1999) = 1.001 \cdot 10^{12} \text{ euros} \tag{3}$$

$$G(2000) = 1.004 \cdot 10^{12} \text{ euros} \tag{4}$$

12.7. **a**) He uses Taylor's formula in linear approximation and (2) and (3) to determine

[8] The price of all goods produced by a nation in one year

$\dot{G}(1998)$.
HINT: $\dot{}$ is the derivative with respect to t.
RESULT:

$$\dot{G}(1998) = 10^9 \frac{\text{euros}}{\text{year}} \tag{5}$$

(Solution:)

Taylor's formula in linear approximation is

$$\Delta G = \dot{G}(1998)\Delta t \tag{6}$$

For $\Delta t = 1$ year

$$\Delta G = 0.001 \cdot 10^{12} \text{ euros } = 10^9 \text{ euros} \tag{7}$$

12.7. **b**) Based on Taylor's formula in second order approximation (relying on (4) and (5)) he calculates $\ddot{G}(1998)$.
HINT: apply Taylor's formula for the time interval 1998 $\cdots$ 2000.
RESULT:

$$\ddot{G}(1998) = 10^9 \text{ euros (year)}^{-2} \tag{8}$$

(Solution:)

Taylor's formula in second order is

$$\Delta G = \dot{G}(1998)\Delta t + \frac{1}{2}\ddot{G}(1998)(\Delta t)^2 \tag{9}$$

$$\Delta t = 2 \text{ years}, \quad \Delta G = G(2000) - G(1998) = 4 \cdot 10^9 \text{ euros} \tag{10}$$

$$4 \cdot 10^9 \text{ euros} - 10^9 \frac{\text{euros}}{\text{year}} \text{ 2 years} = \frac{1}{2}\ddot{G}(1998)\ 4(\text{year})^2 \tag{11}$$

$$2 \cdot 10^9 \text{ euros} = 2\ddot{G}(1998)(\text{year})^2 \tag{12}$$

$$\ddot{G}(1998) = 10^9 \frac{\text{euros}}{(\text{year})^2} \tag{13}$$

12.7. **c**) Now he applies Taylor's formula again in second order to calculate $G(2008)$.
RESULT:

$$1.06 \cdot 10^{12} \text{ euros} \tag{14}$$

(Solution:)

With $\Delta t = 10$ years we have

$$\begin{aligned} G(2008) &= G(1998) + \Delta t\ \dot{G}(1998) + \tfrac{1}{2}(\Delta t)^2\ \ddot{G}(1998) \\ &= 10^{12} \text{ euros } + 10 \text{ years } \cdot 10^9 \tfrac{\text{euros}}{(\text{year})} + \tfrac{1}{2}100\ (\text{years})^2 10^9 \text{ euros year}^{-2} \\ &= (10^{12} + 10^{10} + 50 \cdot 10^9) \text{ euros} \\ &= 1.06 \cdot 10^{12} \text{ euros} \end{aligned} \tag{15}$$

Rem: There are better **extrapolations** than the one just given, e.g. by choosing $t_0 = 2000$ as the base point for the development and using $G(1999)$ for the calculation of the first derivative, etc. Alternatively, we could simply draw a quadratic function (i.e. a parabola) in t through the three given points (2) (3) and (4).

13 Integrals

13.Q 1: The integral as an area

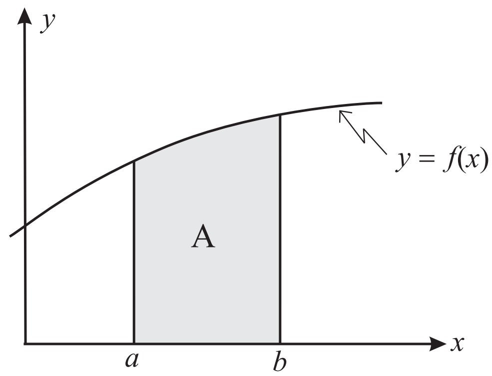


Fig13.1. **1**: The area A bounded by the graph of the function, the x-axis and two verticals at a and b is the integral of the function from a to b

13.1. **a**) Give the mathematical notation for the value A of the shaded area,

(Solution:)

$$A = \int_a^b f(x)\, dx \tag{1}$$

13.1. **b**) and give the phrasing (i.e. in words) for that symbol.

(Solution:)

'A is the integral of $f(x)$ from a to b'.

13.1. **c**) What is, in this connection, the name of a, of b, and of $f(x)$?

(Solution:)

a is the **lower boundary**[$\stackrel{\mathbf{G}}{=}$ `untere Grenze`], b is the upper boundary, $f(x)$ is the **integrand**.

13.1. **d**) Give an intuitive explanation for the name 'integral', for the symbolism and for the integral sign.

(Solution:)

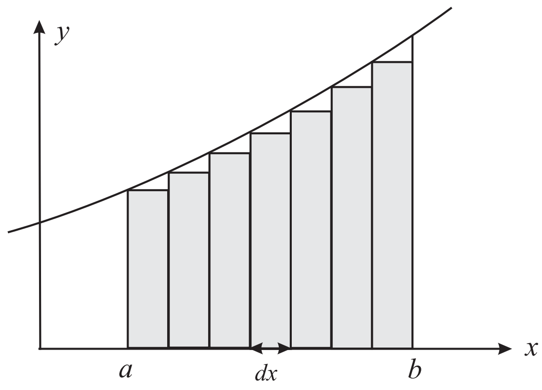


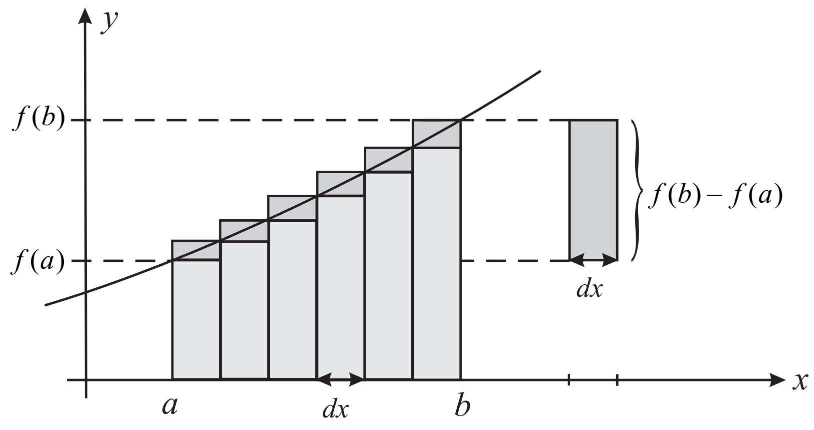
Fig13.1. **2**: The integration interval $[a, b]$ is divided into n subintervals of length $dx = \frac{b-a}{n}$ ($n = 7$ in fig.2). So the integral (area A under the graph) can be approximated as the sum of the shaded small rectangles.

A is approximately the sum (Leibniz has introduced the integral sign as a stylized S from S = sum) of the shaded rectangle above. The whole area is integral, i.e. all its pieces (rectangles) together. Each rectangle has the area

$$f(x) \cdot dx \tag{2}$$

where dx is the breadth of a rectangle and $f(x)$ is its height, whereas x is the left lower corner of the rectangle. dx is the increment of x during one rectangle.

Rem: The error of this approximation can be made as small as we like, if dx is made sufficiently small, or the number n of subintervals is made sufficiently large, see fig.3.



`Fig`13.1. **3**: The approximation in this figure is slightly too big, thus it is called an **upper sum**[$\stackrel{\mathbf{G}}{=}$ `Obersumme`], since in each interval we have included the darkly shaded small rectangles. The lightly shaded rectangles give the **lower sum**[$\stackrel{\mathbf{G}}{=}$ `Untersumme`] only. The error of either approximation is less than the difference between the upper sum and the lower sum, given by the dark rectangle on the right, i.e. less than $dx\ (f(b) - f(a))$ which can be made arbitrarily small, when dx is made sufficiently small.

13.Q 2: Indefinite integrals

13.2. **a**) What is an indefinite integral (in words) and in a precise and a **sloppy**[$\stackrel{\mathbf{G}}{=}$ `schluderig`] formula

(Solution:)

The integral (area) is considered as a function of the upper boundary

$$I(x) = \int_a^x f(\xi) d\xi \tag{1}$$

REM: Since x is used for the upper boundary, a new name ξ has been used for the **integration variable**. Very often x' is used instead of ξ.
I = integral, instead of A = area.

The sloppy form is

$$I(x) = \int_a^x f(x)\, dx \tag{2}$$

REM: This form is ambiguous if the integral itself depends on x as a parameter:

$$f(\xi) = f(x, \xi) \tag{3}$$

13.2. **b**) Prove

$$\boxed{\int_a^a f(x)\,dx = 0} \qquad (4)$$

(Solution:)

the area is zero, or: each $dx = 0$.

13.Q 3: Integration as the inverse of differentiation

13.3. **a**) Give the main theorem of **calculus**[$\stackrel{\mathbf{G}}{=}$ `Infinitesimalrechnung`][9] in words and in 2 formulae.

(Solution:)

Integration is the inverse of differentiation

$$\boxed{I(x) = \int_a^x f(\xi)d\xi \quad \Rightarrow \quad I'(x) = f(x)} \qquad (1)$$

$$\boxed{I(x) = \int_a^x f'(\xi)d\xi = f(x) - f(a)} \qquad (2)$$

Rem 1: These formulae are valid only if the function $f(\xi)$ does not depend upon the parameter x, e.g. not in the case $f(\xi) = \sin(x\xi)$.

Rem 2: Because of the importance of these formulae (1) and (2) we give them again in a more concise and symbolic form:
Introduce the **operator** $\mathcal{D}$ of differentiation (= **differential operator**) acting on differentiable functions $f(x)$ as the **operands**:

$$\mathcal{D}f(x) = f'(x) \qquad \text{(a)}$$

(Note, that it is usual to write operators to the left of the operands (e.g. $f(x)$), i.e. the operators are acting to the right.)
and introduce an operator of integration (= **integral operator**)

$$\mathcal{J}f(x) = I(x) \qquad \text{(b)}$$

where $I(x)$ is defined in (1),
and introduce the **identical operator** = **trivial operator** id

$$\text{id } f(x) = f(x) \qquad \text{(c)}$$

and, finally, introduce the symbol $\circ$ for **composition**[$\stackrel{\mathbf{G}}{=}$ `Hintereinanderausführen`] of operators (= **operator product**), then the main theorem of calculus about the interchangeability of integration and differentiation can symbolically be written as:

$$\boxed{\mathcal{D} \circ \mathcal{J} = \mathcal{J} \circ \mathcal{D} = \text{ id}} \qquad \text{(1')}$$

[9]'calculus' is a common word including both differential and integral calculus.

i.e. first integrating ($\mathcal{J}$) and then ($\circ$) differentiating ($\mathcal{D}$) is the same ($=$) as doing nothing (id). (Since operators are acting to the right, operator products, ($\circ$) must be read from right to left, so $\mathcal{D} \circ \mathcal{J}$ means:
apply first $\mathcal{J}$ to a possible operand, giving an intermediate result, and then apply the operator $\mathcal{D}$ to that intermediate result.
In the second part of (1') we have ignored the constant of integration ($-f(a)$) occuring in (2).
So, (1') is only valid if we collect (classify) the functions into **equivalence classes**, where two functions are called **equivalent** ($\sim$) if they differ only by a constant:

$$f(x) \sim g(x) \qquad \Longleftrightarrow \qquad f(x) = g(x) + \text{ const.} \tag{d}$$

The **operator relation** (= **operator equality**) (1') is valid when acting on such equivalence classes as the operands.

13.3. **b**) Prove it intuitively with the integral as an area.

(Solution:)

Proof of (1):

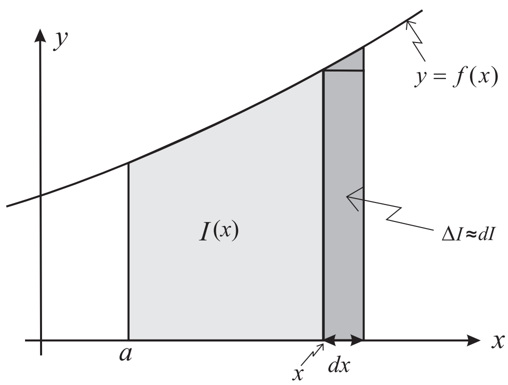


Fig13.3. **1**: $I(x)$ is the area from a to x. Its increment ΔI (darkly shaded) is (in linear approximation) what is under the integral sign. Thus the derivative of the integral with respect to its upper boundary is the integrand.

We write the derivative as a differential quotient:

$$I'(x) = \frac{dI}{dx} \tag{3}$$

ΔI is the dark-shadowed area. dI is the corresponding differential, i.e. ΔI in lowest order, i.e we can neglect the upper triangle (which is of order $(dx)^2$), i.e. dI is the darkly shadowed rectangle

$$dI = f(x)dx \tag{4}$$

which proves (1).

Proof of (2):

$$I(x) = \int_a^x \frac{df}{d\xi}\, d\xi = \int_a^x df \tag{5}$$

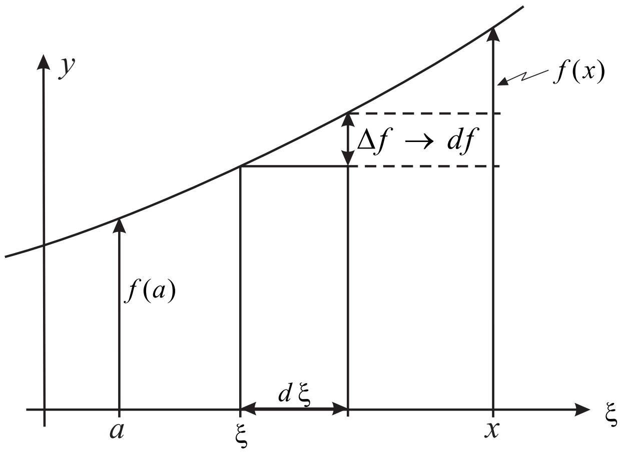


Fig13.3. **2**: The integral of all increments df (corresponding to the interval $\xi \ldots \xi + d\xi$) is just $f(x) - f(a)$

which is the sum of all increments df, which just[10] gives $f(x) - f(a)$.

13.3. **c**) What is an **antiderivative**[$\overset{\mathbf{G}}{=}$ `Stammfunktion, Aufleitung`] of $f(x)$ and to what extent is it **ambiguous**[$\overset{\mathbf{G}}{=}$ `vieldeutig`]? What is an integration constant?

(Solution:)

The antiderivative is a function $I(x)$ whose derivative is the given function $f(x)$:

$$I'(x) = f(x) \tag{6}$$

The antiderivative is ambiguous for a constant C, i.e. when $I(x)$ is a antiderivative any other antiderivative of $f(x)$ has the form

$$I(x) + C \tag{7}$$

C is called the **integration constant**. (It is constant with respect to x.)

[10] $f(x) - f(a)$ is the sum of all exact increments Δf. For sufficiently large n (sufficiently small $d\xi$) this is also the sum of all df.

REM: In the English literature the word ‘antiderivative’ is rarely used, instead it is simply called an integral.

13.3. **d**) Describe in words what is the main method to calculate indefinite integrals and definite integrals.

(Solution:)

We try to guess a function $I(x)$, called an antiderivative, whose derivative is the given integrand $f(x)$. According to the theorem c) we must have

$$\boxed{\int_a^x f(\xi)\, d\xi = I(x) + C = I(x) - I(a) =: [I(\xi)]_a^x} \tag{8}$$

REM 1: A bracket with attached lower and upper boundaries (as in (8)) means the difference of the bracketed expression at both boundaries.

REM 2: In an older notation, but still widely in use, instead of the brackets, i.e. instead of $[I(\xi)]_a^x$, only a right bar at the end is written, i.e $I(\xi)\ |_a^x$. However, in some cases that could be ambiguous, e.g. in case of $5 + \sin\xi\ |_a^x$ one does not know if the 5 is included or not.

REM 3: Even the notation $[I(\xi)]_a^x$ is a shorthand only and can be ambiguous. A completely corrrect notation reads: $[I(\xi)]_{\xi=a}^{\xi=x}$.

REM 4: That $C = -I(a)$ can be checked by setting $x = a$, where the integral vanishes.

For a definite integral simply take x definite, e.g. $x = b$.

13.3. **e**) In a formulary in the chapter ‘Indefinite integrals’ you can find an entry like

$$\int \sin x\ dx = -\cos x \tag{9}$$

What’s the meaning of that information. (What are the antiderivatives? How, do you calculate definite integrals from them? What are the boundaries in(9)?)

(Solution:)

(9) says that $-\cos x$ is an antiderivative of $\sin x$
(i.e. the derivative of the right-hand side of (9) is the integrand: $(-\cos x)' = \sin x$)
The general antiderivative is then obtained by adding a constant (namely an integration constant). So (9) could also be written as

$$\int \sin x\ dx = -\cos x + C \tag{9'}$$

A definite integral is obtained by taking the antiderivative at the boundaries and then subtracting:

$$\int_a^b \sin x\ dx = [-\cos x]_a^b = (-\cos b) - (-\cos a) \tag{10}$$

$$\boxed{\begin{array}{c}\textbf{For the calculation of definite integrals}\\ \textbf{the integration constant drops out}\end{array}} \tag{11}$$

With boundaries (9) reads

$$\int_a^x \sin\xi \; d\xi = -\cos x \tag{9''}$$

for **suitable**[$\stackrel{\mathbf{G}}{=}$ `geeignet`] a.[11]

$$\boxed{\begin{array}{c}\textbf{antiderivative} \equiv \textbf{indefinite integral} \equiv\\ \equiv \textbf{integral as a function of its upper boundary}\end{array}} \tag{12}$$

13.**Qx 4: Linear combination of integrals**

Express in formulae and prove the following statements:

13.4. **a**) A constant can be pulled before the integral.

HINT: Use indefinite integrals and prove by differentiation.

(Solution:)

$$\int cf(x)\; dx = c\int f(x)\; dx \tag{1}$$

$\boxed{\textbf{A constant can be pulled before the integral}}$

In a non-sloppy notation that reads

$$\int_a^x cf(\xi)\; d\xi \stackrel{?}{=} c\int_a^x f(\xi)\; d\xi \tag{1'}$$

PROOF OF (1′):
Differentiation of (1′) with respect to x yields

$$cf(x) \stackrel{?}{=} \frac{d}{dx}\, c\int_a^x f(\xi)\; d\xi \stackrel{\spadesuit}{=} c\,\frac{d}{dx}\int_a^x f(\xi)\; d\xi = cf(x) \tag{2}$$

$\boxed{\spadesuit \text{ A constant can be pulled before the derivative}}$

which is true, i.e. both sides of (1′) are antiderivatives of $cf(x)$. Thus both sides of (1′) can differ by an (integration) constant only. That constant is zero since (1′) is true for $x = a$, when both sides are zero. q.e.d.

13.4. **b**) An integral of a sum is the sum of the integrals

(Solution:)

[11] a must be chosen so that for $x = a$ the right hand side of (9″) vanishes: $\cos a = 0$, as can be seen from (10). ((9″) is just (10) with x written instead of b and changing the name of the integration variable from x to ξ.)

$$\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx \tag{3}$$

$\boxed{\textbf{An integral of a sum is the sum of the integrals}}$

Again, differentiation with respect to the upper boundary yields (The derivative of a sum is the sum of the derivatives):

$$f(x) + g(x) = f(x) + g(x) \tag{4}$$

which is true. Thus, both sides of (3) can differ by an (integration) constant only, which must be zero, because (3) is valid if upper boundary = lower boundary.

13.4. **c**) An integral of a linear combination is the linear combination of the integrals.

(Solution:)

$$\int \lambda f(x) + \mu g(x)\ dx = \lambda \int f(x)\ dx + \mu \int g(x)\ dx \qquad (\lambda, \mu = \text{const}) \tag{5}$$

Proof by combining (1)(3)

13.Q 5: Additivity of the integral in the integration range

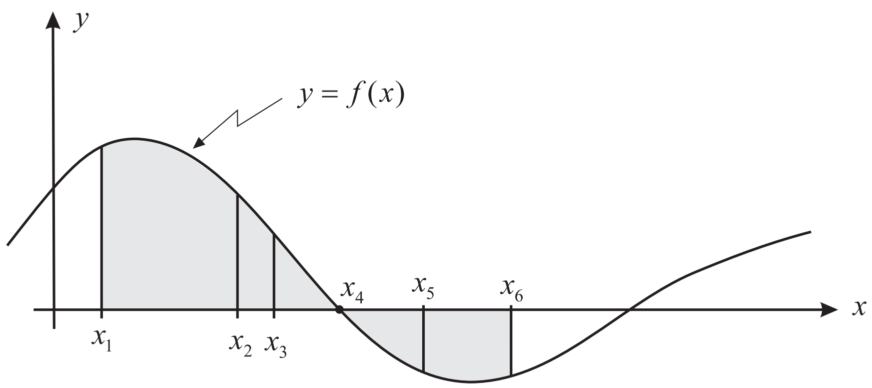


Fig13.5. **1**: Since an integral gives the negative value of the area when it lies below the x-axis, the shaded area must be calculated by the difference of two integrals: one up to x_4 and one from x_4 upwards.

13.5. **a**) Using the meaning of integrals as area, prove:

(Only for $f(x) \geq 0$, for $x_1 < x < x_3$)

$$\boxed{\int_{x_1}^{x_3} f(x)\, dx = \int_{x_1}^{x_2} f(x)\, dx + \int_{x_2}^{x_3} f(x)\, dx} \tag{1}$$

(additively of integrals with respect to the integration interval)

(Solution:)

it is the sum of two partial areas

13.5. **b**) What ist the meaning of

$$\int_{x_2}^{x_1} f(x)\, dx \tag{2}$$

where the upper boundary is lower than the lower boundary

(Solution:)

$$\boxed{\int_{x_2}^{x_1} f(x)\, dx = -\int_{x_1}^{x_2} f(x)\, dx} \tag{3}$$

i.e. minus the area since by going from x_2 to x_1 the increments dx must be negative.

13.5. **c**) What is the geometric meaning of

$$\int_{x_5}^{x_6} f(x)\, dx \tag{4}$$

(Solution:)

Since each $f(x)$ is negative, the integral gives $-A$, where A is the area.

REM: Sometimes one says that the integral gives the oriented area, which can be negative.

13.5. **d**) Give an expression for the shaded area A.

(Solution:)

$$A = \int_{x_1}^{x_4} f(x)\, dx - \int_{x_4}^{x_6} f(x)\, dx \tag{5}$$

where x_4 is the zero of $f(x)$ i.e. $f(x_4) = 0$.

13.**Ex 6: Antiderivatives**

Find (all) antiderivatives of the following functions (C = integration constant). In each case test by differentiation.

13.6. **a**)

$y'(x) = x$ RESULT: $y(x) = \frac{1}{2}x^2 + C$

13.6. **b)**

$y'(x) = x^2$ RESULT: $y(x) = \frac{1}{3}x^3 + C$

13.6. **c)**

$y'(x) = 1$ RESULT: $y(x) = x + C$

13.6. **d)**

$y'(x) = x^n, \qquad (n \neq -1)$ RESULT: $y(x) = \frac{1}{n+1}x^{n+1} + C$

13.6. **e)**

$y'(x) = \frac{1}{x}$ RESULT: $y(x) = \ln|x| + C$

TEST: For the region $x \geq 0$, the proposed result is $y(x) = \ln x + C$, which yields $y' = 1/x$.
For the region $x < 0$, the proposed result is $y(x) = \ln(-x) + C$. The chain rule with $z = -x$ yields

$$y'(x) = \frac{1}{z}(-1) = \frac{1}{-x}(-1) = \frac{1}{x}$$

13.6. **f)** Free fall on the earth

$\dot{x}(t) = v_0 + gt \qquad (v_o,\ g = \text{const.})$ RESULT: $x(t) = v_0 t + \frac{1}{2}gt^2 + x_0$
$(x_0 = C = \text{integration constant})$

REM: This example corresponds to the free fall on earth with the (constant) **gravitational acceleration on earth**[$\stackrel{\mathbf{G}}{=}$ `Erdbeschleunigung`] g.
$\dot{x}(t) = v(t)$ is the instantaneous velocity (in the downward direction). v_0 is the initial velocity (at the initial time $t_0 = 0$).
$v_0 \neq 0$ in case the body was given an initial push.

13.6. **g)**

$y'(\varphi) = \cos\varphi$ RESULT: $y(\varphi) = \sin\varphi + C$

13.6. **h)**

$\dot{y}(t) = \cos(\omega t) \qquad (\omega = \text{const.})$ RESULT: $y(t) = \frac{1}{\omega}\sin(\omega t) + C$

(Solution:)

Test:

$$\left(\frac{1}{\omega}\sin(\omega t) + c\right)' = \frac{1}{\omega}\cdot\omega\cos(\omega t) \quad \text{q.e.d.} \tag{1}$$

where we have applied the chain rule with

$$z = \omega t, \quad \frac{dz}{dt} = \omega \tag{2}$$

13.6. **i)** $y'(x) = e^{\alpha x}$ RESULT: $y(x) = \frac{1}{\alpha}e^{\alpha x} + C$

13.6. **j)** $y'(x) = \frac{1}{\sqrt{x}}$

HINT: write as a power. Use d).

(Solution:)

$$y'(x) = x^{-\frac{1}{2}}, \qquad n = -\frac{1}{2} \qquad \text{in d)} \tag{3}$$

$$y(x) = \frac{1}{-\frac{1}{2}+1}\,x^{-\frac{1}{2}+1} + C = 2x^{\frac{1}{2}} + C = 2\sqrt{x} + C \tag{4}$$

Test:

$$(2\sqrt{x})' = 2(x^{\frac{1}{2}})' = 2\cdot\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}} \tag{5}$$

13.**Ex 7: Definite integrals**

Calculate the following definite integrals.

HINT: use the antiderivatives from the previous exercise.

13.7. **a)**

$$\int_0^a x^2 dx \tag{1}$$

RESULT: $= \frac{1}{3}a^3$

(Solution:)

$$\int_0^a x^2 dx \stackrel{\spadesuit}{=} \left[\frac{1}{3}x^3\right]_0^a = \frac{1}{3}a^3 \tag{2}$$

$\boxed{\spadesuit \text{ an antiderivative of } x^2 \text{ is } \frac{1}{3}x^3}$

13.7. **b)**

$$\int_0^{2\pi} \cos\varphi \, d\varphi \tag{3}$$

RESULT: $= 0$

(Solution:)

$$\int_0^{2\pi} \cos\varphi \, d\varphi = [\sin\varphi]_0^{2\pi} = \underbrace{\sin(2\pi)}_{0} - \underbrace{\sin 0}_{0} \tag{4}$$

13.7. **c)**

$$\int_a^b dx \tag{5}$$

HINT: the integrand is 1.
RESULT: $= b - a$

(Solution:)

$$\int_a^b dx = \int_a^b 1 \, dx \stackrel{\spadesuit}{=} [x]_a^b = b - a \tag{7}$$

$\boxed{\spadesuit \ x \text{ is an antiderivative of } 1}$

13.7. **d)**

$$\int_0^{t_0} a\cos(\omega t) + b\, e^{\alpha t} \, dt \tag{8}$$

RESULT:

$$= \frac{a}{\omega}\sin(\omega t_0) + \frac{b}{\alpha}(e^{\alpha t_0} - 1) \tag{9}$$

(Solution:)

$$\int_0^{t_0} a\cos(\omega t) + be^{\alpha t} \, dt \stackrel{\spadesuit}{=} a\int_0^{t_0} \cos(\omega t) dt + b\int_0^{t_0} e^{\alpha t} dt = \tag{10}$$

$$= a\left[\frac{1}{\omega}\sin(\omega t)\right]_0^{t_0} + b\left[\frac{1}{\alpha}e^{\alpha t}\right]_0^{t_0} = \frac{a}{\omega}\sin(\omega t_0) + \frac{b}{\alpha}(e^{\alpha t_0} - 1) \tag{11}$$

♠ Integral of a sum = sum of integrals
Constants like a and b can be pulled in front of the integral

---

13.**Ex 8: Indefinite integrals**
Calculate the following indefinite integrals.
HINT: this is the same type of exercise as the last one, differing only in notation:

antiderivative ≡ indefinite integral

13.8. **a)**

$$\int x\,dx \tag{1}$$

RESULT:

$$\int x\,dx = \frac{1}{2}x^2 \tag{2}$$

(Solution:)

Test:

$$\left(\frac{1}{2}x^2\right)' = \frac{1}{2}\cdot 2x = x$$

i.e. the intergrand of the integral (1) is obtained.

13.8. **b)**

$$\int x^2\,dx \tag{3}$$

RESULT:

$$\int x^2\,dx = \frac{1}{3}x^3 \tag{4}$$

13.8. **c)**

$$\int dx \tag{5}$$

RESULT:

$$\int dx = x \tag{6}$$

13.8. **d)**

$$\int \frac{1}{x}\,dx \tag{7}$$

RESULT:

$$\boxed{\int \frac{1}{x}\,dx = \ln|x|} \tag{8}$$

(The integral of 1/x is the absolute value of the natural logarithm.)

REM: The logarithm is defined only for positive arguments.
If x is positive the absolute sign in (8) is irrelevant and can be omitted: $|x| = x$.
If x is negative, we have $|x| = -x$. Using the chain rule with the substitution $z = -x$ we can check (8) like this:

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \left[\frac{d}{dz}\ln z\right]\ \frac{dz}{dx} = -\frac{1}{z} = \frac{1}{x} \tag{1}$$

q.e.d.

13.8. **e)**

$$\int \cos\varphi\ d\varphi \tag{9}$$

RESULT:

$$\int \cos\varphi\ d\varphi = \sin\varphi \tag{10}$$

13.8. **f)**

$$\int \cos(\omega t)\,dt \tag{11}$$

RESULT:

$$\int \cos(\omega t)\,dt = \frac{1}{\omega}\sin\omega t \tag{12}$$

13.8. **g)** ☻☻ Write (2) in a mathematically correct form.
RESULT:

$$\int_0^x \xi\,d\xi = \frac{1}{2}x^2 \tag{13}$$

(Solution:)

$$\int_0^x \xi \, d\xi \stackrel{\spadesuit}{=} \left[\frac{1}{2}\xi^2\right]_0^x = \frac{1}{2}x^2 \tag{14}$$

♠	The antiderivative of ξ is $\frac{1}{2}\xi^2$ Test: $\frac{d}{d\xi}\frac{1}{2}\xi^2 = \frac{1}{2}\cdot 2\cdot \xi$ The definite integral is the antiderivative taken at the boundaries followed by taking their difference.

13.8. **h**) Why (2) is not a mathematically correct notation?

(Solution:)

boundaries are not specified. It is implied that the upper boundary is x, but then it is not correct to use the same symbol for the upper boundary as for the integration variable.

13.8. **i**)

$$\int \sin(kx)\, dx \tag{15}$$

Result:

$$\int \sin(kx)\, dx = -\frac{1}{k}\cos(kx) \tag{16}$$

(Solution:)

test:

$$\left(-\frac{1}{k}\cos(kx)\right)' = -\frac{1}{k}(-\sin(kx))\, k = \sin(kx) \tag{17}$$

13.8. **j**) Write (16) in a mathematically correct form, i.e. as a definite integral.
Result:

$$\int_{\frac{\pi}{2k}}^x \sin(kx)\, dx = -\frac{1}{k}\cos(kx) \tag{18}$$

(Solution:)

$$\int_a^x \sin(k\xi)\, d\xi = \left[-\frac{1}{k}\cos(k\xi)\right]_a^x = -\frac{1}{k}\cos(kx) + \frac{1}{k}\cos(ka) \tag{19}$$

We must have $\cos(ka) = 0$. Take e.g. $ka = \frac{\pi}{2} \Rightarrow a = \frac{\pi}{2k}$.

13.8. **k**)

$$\int_0^x x\xi \, d\xi \tag{20}$$

HINT: x is a constant.
RESULT:

$$= \int_0^x x\xi \, d\xi = \frac{1}{2}x^3 \tag{21}$$

(Solution:)

$$\int_0^x x\xi \, d\xi = x \int_0^x \xi \, d\xi = \left[x\frac{1}{2}\xi^2 \right]_0^x = \frac{1}{2}xx^2 = \frac{1}{2}x^3 \tag{22}$$

13.8. **l**) Write (21) in sloppy notation. Why is that obviously wrong?

(Solution:)

In sloppy notation (21) would be

$$\int_0^x x^2 dx = \frac{1}{3}x^3 \tag{21}$$

Here the distinction between the integration variable ξ and the constant x (occuring in(21) in the integrand and in the upper boundary) is lost. So, we see that sloppy notation can be dangerous. However, it is safe in most cases.

13.Ex 9: Integration as the inverse of differentiation

13.9. **a**) In a formulary look up the integral

$$\int x\sqrt{x^2 - a^2} \, dx, \quad a = \text{ const.}$$

RESULT: $\frac{1}{3}\sqrt{(x^2 - a^2)^3}$

13.9. **b**) Check that result by differentiating.
HINT: first unify $\sqrt{}$ and ^{3} to a unique exponent.

(Solution:)

$$\left(\frac{1}{3}\sqrt{(x^2 - a^2)^3}\right)' = \frac{1}{3}\left((x^2 - a^2)^{\frac{3}{2}}\right)' \stackrel{\clubsuit}{=} \frac{1}{3} \cdot \frac{3}{2}(x^2 - a^2)^{\frac{3}{2}-1}\, 2x = x\sqrt{x^2 - a^2}$$

$\boxed{\clubsuit \text{ chain rule with } z = x^2 - a^2,\ \frac{d}{dz} z^{\frac{3}{2}} = \frac{3}{2} z^{\frac{3}{2}-1},\ \frac{dz}{dx} = 2x}$

13.Ex 10: Area under the sine curve

13.10. **a**) Calculate the shaded area A under one bosom of the sine curve.

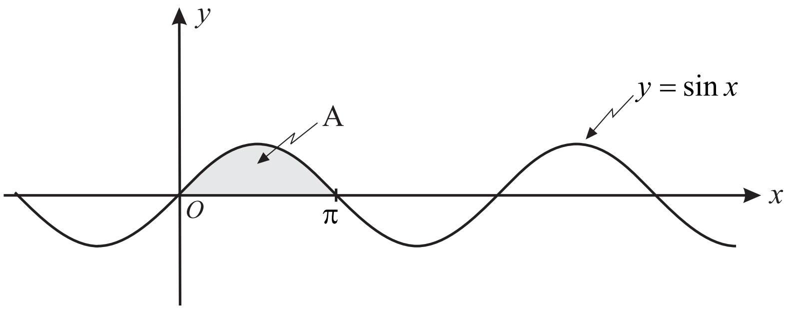


Fig13.10. **1**: Area A under one half period of a sine curve is calculated.

HINT: do not use a formulary, use the differentiation rules for sin and cos instead.
RESULT:

$$A = 2 \tag{1}$$

(Solution:)

$$A = \int_0^\pi \sin x \, dx \overset{\spadesuit}{=} [-\cos x]_0^\pi = -\underbrace{\cos \pi}_{-1} + \underbrace{\cos 0}_{1} = 2 \tag{2}$$

$\boxed{\spadesuit \text{ the antiderivative of } \sin x \text{ is } -\cos x}$

13.10. **b**) Calculate the shaded area A between the sine curve and the x-axis in the interval $[x_1, x_2]$ where

$$-\pi < x_1 \leq 0 \leq x_2 < \pi. \tag{3}$$

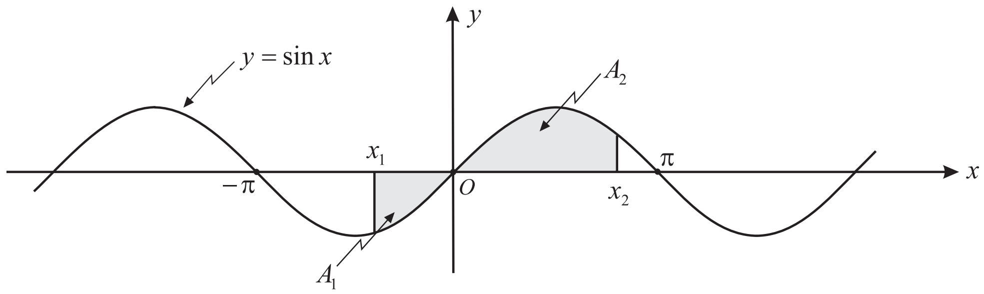


Fig13.10. **2**: To calculate an area (which by definition is always positive) we have to split the integration interval into $[x_1, 0]$ and $[0, x_2]$.

HINT: The area of the integral is only positive in the interval $[0, \pi]$, in the interval $[-\pi, 0]$ the integral is minus the area. Thus you have to calculate the partial areas A_1 and A_2 separately.
RESULT:

$$A = 2 - \cos x_1 - \cos x_2 \tag{4}$$

(Solution:)

$$A_2 = \int_0^{x_2} \sin x \, dx = [-\cos x]_0^{x_2} = -\cos x_2 + \underbrace{\cos 0}_{1} = 1 - \cos x_2 \tag{5}$$

$$A_1 = -\int_{x_1}^{0} \sin x \, dx = [\cos x]_{x_1}^{0} = \underbrace{\cos 0}_{1} - \cos x_1 = 1 - \cos x_1 \tag{6}$$

$$A = A_1 + A_2 = 2 - \cos x_1 - \cos x_2 \tag{7}$$

13.10. **c**) What is the geometrical meaning of the integral

$$\int_{x_1}^{x_2} \sin x \, dx \tag{8}$$

RESULT: It is the difference of two areas

$$A_2 - A_1. \tag{9}$$

13.10. **d**) **Justify** [$\stackrel{\mathbf{G}}{=}$ `begründe`] geometrically the following equations.

$$\int_0^{2\pi} \sin \varphi \, d\varphi = 0 \tag{4}$$

$$\int_x^{x+2\pi} \sin\varphi \, d\varphi = 0 \tag{5}$$

$$\int_x^{x+2\pi} \cos\varphi \, d\varphi = 0 \tag{6}$$

(Solution:)

In each full period interval $[x, x+2\pi]$ the sine (and also the cosine) has the same amount of area counted negatively as area counted positively, thus **canceling**[$\stackrel{\mathbf{G}}{=}$ `sich aufheben`] each other out to zero.

13.**Ex 11: Area of a triangle calculated by an integral**

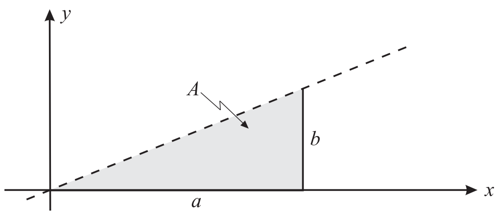


Fig13.11. **1**: The area of the right triangle is half the area of a rectangle with side lengths a and b.

13.11. **a**) Calculate the area A of the shaded triangle using the fact that it is half of a rectangle.
RESULT:

$$A = \frac{1}{2}ab \tag{1}$$

13.11. **b**) Give the equation of the dotted straight line.
HINT: Use the fact that y is proportional to x. Determine the constant of proportionality at $x = a$.
RESULT:

$$y = \frac{b}{a}x \tag{2}$$

(Solution:)

$$y = \alpha x, \quad \alpha = \text{ constant of proportionality} \tag{3}$$

(3) is true for $x = 0$. To make it true for $x = a$ we must have $b = \alpha a \;\Rightarrow\; \alpha = \frac{b}{a}$.

13.11. **c**) Express the area A of the triangle by an integral.
RESULT:

$$A = \int_0^a \frac{b}{a} x \, dx \tag{4}$$

13.11. **d**) A constant can be pulled in front of the integral. Why is $\frac{b}{a}$ a constant?
RESULT:

$$A = \frac{b}{a} \int_0^a x \, dx \tag{5}$$

(Solution:)

A 'constant' means a constant with respect to the integration variable x, i.e. a and b are independent of x. For a fixed triangle a and b do not change, while x ranges from 0 to a.

13.11. **e**) What is the antiderivative of x?
RESULT:

$$\frac{1}{2}x^2 \tag{6}$$

(Solution:)

Test:

$$\left(\frac{1}{2}x^2\right)' = \frac{1}{2}(x^2)' = \frac{1}{2} \cdot 2x = x \qquad \text{q.e.d.} \tag{7}$$

13.11. **f**) Calculate the integral (5).
HINT: To calculate an integral take the antiderivative at the upper and lower boundaries of the integral and form its difference.
RESULT: See (1)

(Solution:)

$$A = \frac{b}{a} \cdot \left[\frac{1}{2}x^2\right]_0^a = \frac{b}{a}\left(\frac{1}{2}a^2 - \frac{1}{2}0^2\right) = \frac{1}{2}ab$$

13.**Ex 12: ☻☻ Average of $\sin^2$ and $\cos^2$ is $\frac{1}{2}$**

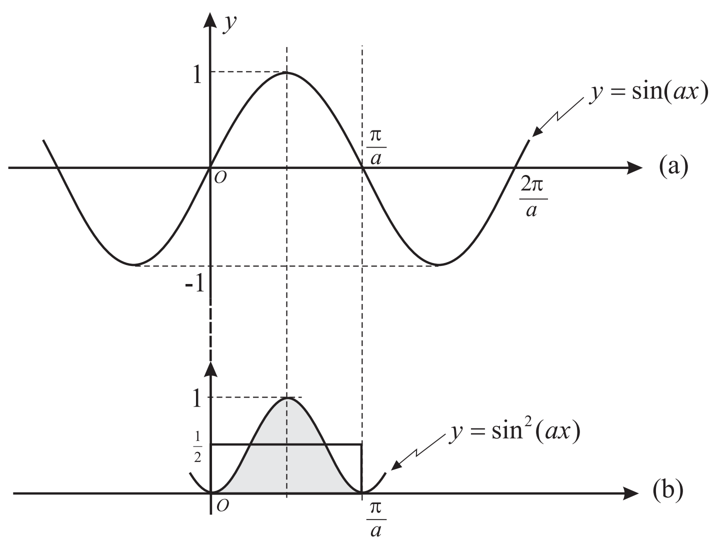


Fig13.12. **1**: Sine squared curve (b) is obtained from the sine curve by squaring the sine curve (a). Since $(-1)^2 = 1$, (b) is always positive. The average $h = \frac{1}{2}$ is defined by the condition that the bold area $h \cdot \frac{\pi}{a}$ is equal to the shaded area under one period of sine squared.

REM 1: Because of squaring, the period of $\sin^2(ax)$ is half the period of $\sin(ax)$.

REM 2: Because of

$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x)$$

the curves (a) and (b) in fig.1 are essentially the same.

In fig.1 you can see the graphs of $y = f(x) = \sin(ax)$ and $y = f^2(x) = \sin^2(ax)$. From the graphs check the following statements a) - f).

13.12. **a**) f and f^2 have the same zeros.

13.12. **b**) f^2 is non-negative [i.e. $f^2 \geq 0$]

13.12. **c**) The **range**[$\stackrel{\mathbf{G}}{=}$ `Wertebereich`] of f^2 is the interval $[0, 1]$.

13.12. **d**) The maxima and minima of f become maxima of f^2.

13.12. **e**) The zeros of f are **flex-points**[$\stackrel{\mathbf{G}}{=}$ `Wendepunkte`] of f which become minima for f^2.

13.12. **f**) The period of f is $\frac{2\pi}{a}$, while f^2 has half that.

13.12. **g**) Calculate the shaded area under one half period of f^2.
HINT: Use a formulary for the antiderivative of f^2.

RESULT:

$$A = \frac{\pi}{2a} \tag{1}$$

(Solution:)

$$A = \int_0^{\frac{\pi}{a}} \sin^2(ax)\, dx = \left[\frac{x}{2} - \frac{\sin(2ax)}{4a}\right]_0^{\frac{\pi}{a}} = \frac{\pi}{2a} - \frac{\sin(2a\frac{\pi}{a})}{4a} = \frac{\pi}{2a} \tag{2}$$

13.12. **h**) Imagine that the graph of f^2 is a **mountain range**[$\stackrel{\mathbf{G}}{=}$ `Gebirge`]. What is its **average height**[$\stackrel{\mathbf{G}}{=}$ `durchschnittliche Höhe`] h?

HINT: The average height h is defined so that the area of the **solid rectangle**[$\stackrel{\mathbf{G}}{=}$ `Rechteck mit fetten Umrissen`] is equal to A.

RESULT:

$$h = \frac{1}{2} \tag{3}$$

(Solution:)

$$A = \frac{\pi}{2a} \stackrel{!}{=} h \cdot \frac{\pi}{a} \;\Rightarrow\; h = \frac{1}{2} \tag{4}$$

13.12. **i**) The average of $\sin^2$ and $\cos^2$ occur very often in physics, so because of its importance we restate our result as follows:

$$\boxed{\overline{\sin^2} = \overline{\cos^2} = \frac{1}{2}} \tag{5}$$

In words:

The average of sine squared (and cosine squared) [**when taken over a full period**] **is one half.**

REM: In (5) we have, as usual, used an **upper bar**[$\stackrel{\mathbf{G}}{=}$ `Balken`] to denote the **average**[$\stackrel{\mathbf{G}}{=}$ `Durchschnitt`].

Why does our result (5) also hold for $\cos^2$?

(Solution:)

cos and sin differ only by a translation (= shift along the x-axis). Thus the same is valid for $\cos^2$ and $\sin^2$. Areas are invariant under translations.

13.**Ex 13: Derivative of an integral with respect to its lower boundary**

Calculate the derivative with respect to the lower boundary of the integral, i.e. calculate[12]

$$\frac{d}{dx}\int_x^a f(\xi)\, d\xi \tag{1}$$

[12]It is assumed that both f and a do not depend upon x.

HINT: What happens when you interchange boundaries?
RESULT: $-f(x)$

(Solution:)

$$\frac{d}{dx}\int_x^a f(\xi)\,d\xi = \frac{d}{dx}\left[-\int_a^x f(\xi)\,d\xi\right] = -\frac{d}{dx}\int_a^x f(\xi)\,d\xi = -f(x) \tag{2}$$

14 Application of integrals to geometry

14.Ex 1: An amulet out of gold

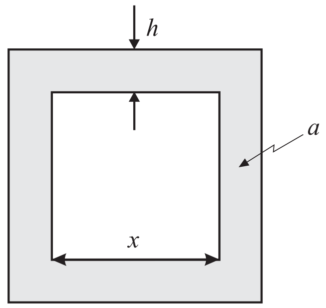


Fig14.1. **1**: The shaded area element a (inner side length x, width h) is calculated in linear approximation, i.e. is treated as a differential: $a = dA$.

Calculate the **shaded**[$\stackrel{G}{=}$ `schraffierte`] area a of an amulet of quadratic shape, see fig.1. x = inner side length, h = width of **frame**[$\stackrel{G}{=}$ `Rahmen`]. Calculate in linear approximation of the small quantity h ($h \ll x$), i.e. treat a and h as differentials.
14.1. **a**) Calculate a by using its integral, i.e. the area of a square

$$A = A(x) = x^2 \tag{1}$$

HINT: the outer square has side length $x + 2h$.
RESULT:

$$a = 4x\,h \tag{2}$$

(Solution:)

$$a = A(x + 2h) - A(x) = dA \tag{3}$$

$$\frac{dA}{dx} = 2x, \quad dx = 2h \tag{4}$$

$$a = 2x\,dx = 4x\,h \tag{5}$$

REM: A more elementary deduction is:

$$dA = (x + 2h)^2 - x^2 = x^2 + 4xh + 4h^2 - x^2 = 4xh \tag{3'}$$

since the quadratic term $4h^2$ can be negelected in a differential. However, this method is computationally more complicated (and can be significantly more so in other examples) since in ($3'$) we do not take into account early enough that dA and h should be treated as differentials.

14.1. **b**) Normally however, in integral calculus we do not know the integral (e.g. $A = x^2$), but, on the contrary, we are in the process of calculating it.

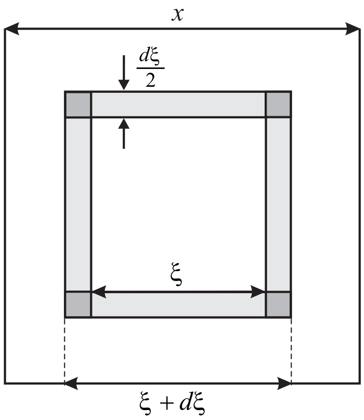


Fig14.1. **2**: The same area element dA is calculated directly. The darkly shaded area elements are of second order and can be neglected. A new variable ξ is introduced going from $\xi = 0$ to $\xi = x$.

It is the **tremendous**[$\stackrel{\mathbf{G}}{=}$ `gewaltig`] power of integral calculus to first determine (e.g. intuitively, geometrically, etc) the differential and then by equation manipulation to get the integral. Thus instead of using the integral (1), calculate geometrically its differential dA, i.e. the shaded area in the above figure between the square with side lengths ξ and the square with side lengths $\xi + d\xi$.
HINT: because dA is a differential, calculate it only in linear approximation in $d\xi$.
RESULT:

$$dA = 2\xi \, d\xi \tag{6}$$

(Solution:)

The width of a **beam**[$\stackrel{\mathbf{G}}{=}$ `Balken`] of the **frame**[$\stackrel{\mathbf{G}}{=}$ `Rahmen`] is $\frac{1}{2}d\xi$. Thus the shaded area is[13]

$$dA = 4 \cdot \frac{d\xi}{2} \cdot \xi \tag{7}$$

---

[13]We need to have the formula for the area of a rectangle $A = ab$, of which (1) is a special case. So our deduction is circular. However, this first, very simple example shows most clearly how we get from the differential to the integral and what we can neglect in calculating the differential.

Here we have neglected the four darkly shaded squares at the corners. However, that area is of second order in $d\xi$, namely $4\left(\frac{d\xi}{2}\right)^2$, and can therefore be neglected in the differential (7).

14.1. **c**) Calculate the area of the square (with side lengths x) by integrating (6).
RESULT:

$$A = x^2 \tag{8}$$

(Solution:)

$$A = \int_0^x dA = \int_0^x 2\xi \, d\xi = \left[2 \cdot \frac{1}{2}\xi^2\right]_0^x = x^2 \tag{1}$$

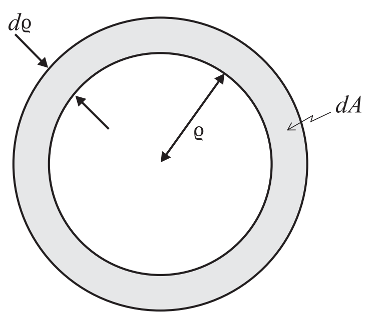
14.**Ex 2: Area of a circle**
Similarly, calculate the shaded area dA of the gold ring $d\varrho$ (without using the formula

$$A = \pi r^2 \tag{1}$$

for the area of a circle, however, you may use the formula

$$c = 2\pi r \tag{2}$$

for the circumference (perimeter) of a circle.)



`Fig`14.2. **1**: To calculate the area of a circle we first calculate the shaded area element dA as a rectangle with side lengths $2\pi\varrho$ and $d\varrho$.

HINT: the notation dA implies that it is a differential, i.e. is calculated in linear approximation in $d\varrho$.
14.2. **a**) Calculate the shaded area dA as if it were a rectangle with side lengths $d\varrho$

and c, where $c =$ circumference (perimeter) of a circle.
RESULT:

$$dA = 2\pi\varrho\, d\varrho \tag{3}$$

14.2. **b**) Integrate dA to get the formula for the area of a circle with radius r.
RESULT:

$$A = A(r) = \pi r^2 \tag{4}$$

(Solution:)

$$A = \int_0^r dA = \int_0^r 2\pi\varrho\, d\varrho = \left[2\pi\frac{1}{2}\varrho^2\right]_0^r = \pi r^2 \tag{5}$$

REM: It is not so easy, as it was in our previous rectangular example, to prove that (3) is correct, i.e. we have only made second order errors in $d\varrho$ while replacing the shaded area by a rectangle of side lengths $d\varrho$ and $2\pi\varrho$.

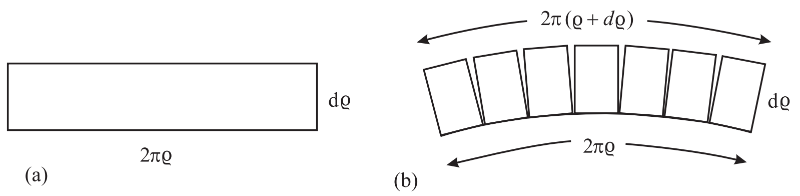


Fig14.2. **2**: Our result (3), based on the rectangle (a), is slightly too small since, intuitively, streching the rectangle (a) to the ring (b) would tear out one side of the rectangle. However, that error is of second order in $d\varrho$.

In **non-rigorous**[$\stackrel{\mathbf{G}}{=}$ `nicht-strenge`] mathematics, which most physicicsts use, one develops an intuitive feeling for the correctness e.g. of (3). In the following we give some additional intuitive arguments to **corrobarate**[$\stackrel{\mathbf{G}}{=}$ `bekräftigen`] that feeling. First we have the feeling that dA is slightly (hopefully only of second order in $d\varrho$) smaller than the shaded area: when you take the rectangle (a) in fig.2 and try to **bend**[$\stackrel{\mathbf{G}}{=}$ `biegen`] it onto a circle (b) the outer periphery will **tear out**[$\stackrel{\mathbf{G}}{=}$ `ausreißen`] because it must be stretched from the length $2\pi\varrho$ to the larger length $2\pi(\varrho + d\varrho)$.
On the other hand, bending a larger rectangle

$$dA = d\varrho \cdot 2\pi(\varrho + d\varrho) \tag{6}$$

would result in a compression of the inner periphery. Thus (3) is too small and (6) is too large. Since (3) and (6) differ only by the second order quantity $2\pi(d\varrho)^2$, both are equivalent in linear approximation, i.e. (3) is correct.

14.Ex 3: Area of a circle calculated in polar coordinates

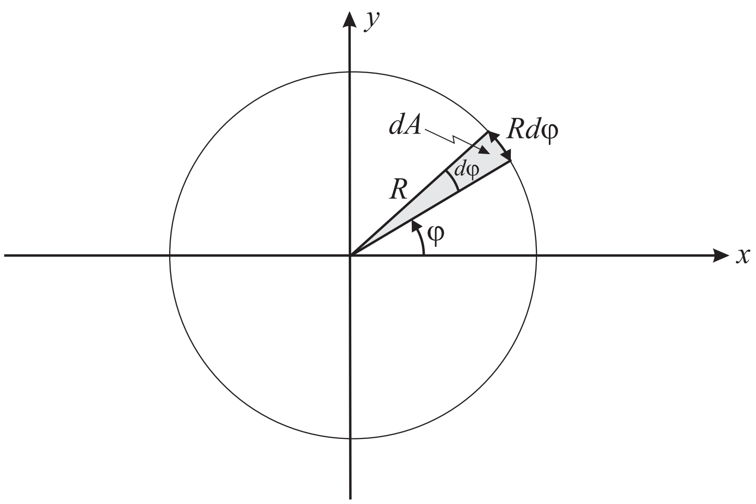


Fig14.3. **1**: The area of a circle (radius R) is calculated again using the shaded triangles as area elements.

We calculate the area of a circle again using the shaded differential dA.

14.3. **a**) Calculate dA.

HINT: calculate dA in first order approximation as a rectangular triangle with base R and the arc length of the angle $d\varphi$ as its perpendicular.

RESULT:

$$dA = \frac{1}{2} R^2 \, d\varphi \tag{1}$$

(Solution:)

The area of a rectangular triangle is

$$dA = \frac{1}{2} ab \tag{2}$$

where a = base = R, b = perpendicular = arc length = $R \cdot d\varphi$.

14.3. **b**) Integrate (1) to obtain the area of a circle.

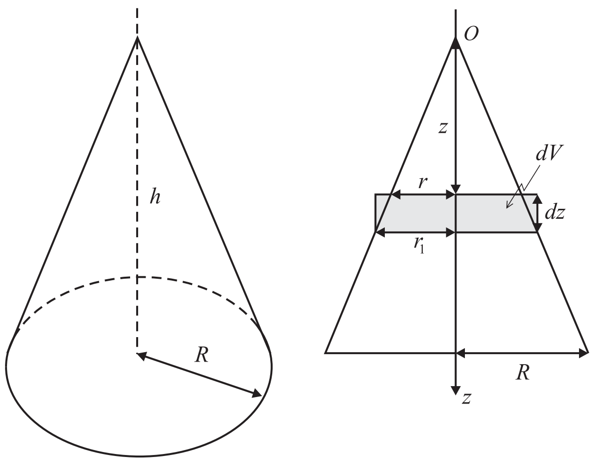
(Solution:)

$$A = \int dA = \frac{1}{2} R^2 \int_0^{2\pi} d\varphi \tag{3}$$

The integrand here is 1, its antiderivative is x (or φ since the integration variable is φ). Thus,

$$A = \frac{1}{2}R^2\,[\varphi]_0^{2\pi} = \frac{1}{2}R^2(2\pi - 0) = \pi R^2 \tag{4}$$

14.Ex 4: Volume of a cone
(**cone**[$\stackrel{\mathbf{G}}{=}$ `Kegel`])



`Fig`14.4. **1**: The volume of a cone (height h, base is a circle of radius R) is calculated by integrating the shaded volume element dV. dV is estimated by taking r and r_1 as the radius. Thickness is dz, where the axis of the cylinder is the z-axis.

We would like to calculate the volume of a cone with a circle of radius R as the base and with height h. We choose a z-axis in the axis of the cone, $z = 0$ being the top of the cone and $z = h$ being the base of the cone.

14.4. **a**) Calculate the radius r of the sphere which is a **cross-section**[$\stackrel{\mathbf{G}}{=}$ `Querschnitt`] of the cone at height z.
HINT: r is proportional to z. Determine the constant of proportionality for $z = h$.
RESULT:

$$r = \frac{R}{h}z \tag{1}$$

(Solution:)

$r = \alpha z,\ \alpha =$ constant of proportionality. For the top in particular, we have $z = r = 0$. For $z = h$ we must have

$$r = R = \alpha h \quad \Rightarrow \quad \alpha = \frac{R}{h} \quad \Rightarrow \quad (1) \tag{2}$$

14.4. **b**) Approximate the shaded volume element dV by a disk (= cylinder) of radius r and thickness dz.
RESULT:

$$dV = \pi \frac{R^2}{h^2} z^2 \, dz \tag{3}$$

(Solution:)

dV = volume of the cylinder = height × area of a circle = $dz \cdot \pi r^2 \stackrel{(1)}{=} \pi \frac{R^2}{h^2} z^2 \, dz$

14.4. **c**) (3) will be slightly too small, so choose it too large by taking r_1 instead of r, and show that (3) is correct in linear approximation.

(Solution:)

The larger volume is (in (3) replace $z \to z + dz$)

$$dV' = \pi \frac{R^2}{h^2} (z + dz)^2 dz \tag{3'}$$

Since (3) and (3′) differ only by second order quantities ($\pi \frac{R^2}{h^2} 2z(dz)^2$ or higher) and the correct value of the shaded volume element lies in-between, (3) is correct for the differentials.

14.4. **d**) Integrate (3) to obtain the volume of the cone.
RESULT:

$$V = \frac{1}{3} \pi h R^2 \tag{4}$$

(Solution:)

$$V = \int_0^h dV = \int_0^h \frac{\pi R^2}{h^2} z^2 \, dz = \left[\frac{\pi R^2}{h^2} \frac{1}{3} z^3 \right]_0^h = \frac{1}{3} \frac{\pi R^2}{h^2} h^3 = \frac{1}{3} \pi h R^2 \tag{5}$$

14.**Ex 5: Surface of a sphere**

We will calculate the area A of the surface of a sphere with radius R, see fig.1. First we will calculate the shaded surface element (≡ surface differential) dA. It is bounded by two circles of radii r and r_1.

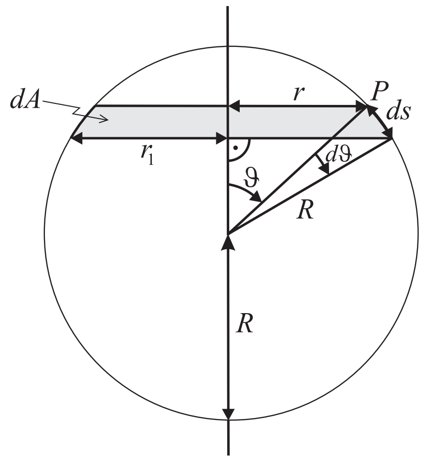


Fig14.5. **1**: Surface of a sphere (radius R) is calculated by integrating the shaded surface elements dA (defined by $\vartheta \ldots \vartheta + d\vartheta$).
dA is estimated as a rectangle with side lengths ds and the circumference (perimeter) of a circle with radius r (or r_1).

14.5. **a**) Calculate r, r_1 and the periphery element ds.
HINT: for $\sin(\vartheta + d\vartheta)$ use Taylor's formula.
RESULTS:

$$r = R \sin\vartheta \tag{1}$$

$$r_1 = R \sin\vartheta + R \cos\vartheta \, d\vartheta \tag{2}$$

$$ds = R \, d\vartheta \tag{3}$$

(Solution:)

1) Both **sides**[$\overset{\mathbf{G}}{=}$ `Schenkel`] of the angle $d\vartheta$ have length R. Because of the right angle, we have $r = R \sin\vartheta$ (side projection).
2) Similarly

$$r_1 = R \sin(\vartheta + d\vartheta) \overset{\clubsuit}{=} R(\sin\vartheta + \underbrace{(\sin\vartheta)'}_{\cos\vartheta} d\vartheta) = R \sin\vartheta + R \cos\vartheta \, d\vartheta \tag{4}$$

$\clubsuit$ Taylor's formula in first order (= linear) approximation as suitable for differentials

3)

$$ds = R \, d\vartheta \quad \text{(length of arc with centri-angle } d\vartheta) \tag{5}$$

14.5. **b**) Calculate dA (and make plausible that your expression is correct in linear approximation) by **cutting off**[$\stackrel{\mathbf{G}}{=}$ `aufschneiden`] our surface element at the periphery element ds and **unbending**[$\stackrel{\mathbf{G}}{=}$ `abwickeln`] it into a plane, which is not possible without **tearing**[$\stackrel{\mathbf{G}}{=}$ `zerreißen`] (see fig. 2a) or **squashing**[$\stackrel{\mathbf{G}}{=}$ `zerquetschen`] it (see fig. 2b).

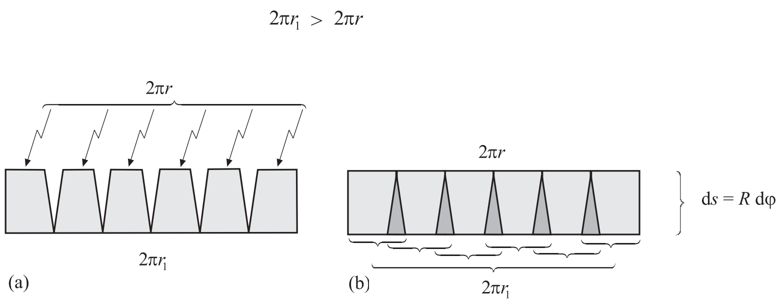


Fig14.5. **2**: (6) is too small since bending the rectangle (b) onto a sphere would tear one side open, while (6′) is too large (see fig a) because then one side ($2\pi r_1$) would be compressed to $2\pi r$.

REM: Observe that the periphery element ds is **perpendicular**[$\stackrel{\mathbf{G}}{=}$ `senkrecht`] to the tangent at P on the circle r. Therefore, ds becomes the height of the resulting rectangle.

RESULT:

$$dA = 2\pi r\, ds \tag{6}$$

(Solution:)

(6) corresponds to the rectangle in fig.2b. The true dA is slightly larger, since the **dark-shaded**[$\stackrel{\mathbf{G}}{=}$ `dunkel-schraffiert`], overlapping, small triangles are lacking in (6). Calculating the rectangle (fig.2a)

$$dA_1 = 2\pi r_1 ds \stackrel{(2)}{=} 2\pi\ R \sin\vartheta\, ds + 2\pi R \cos\vartheta\, d\vartheta\, ds \tag{6'}$$

dA_1 is slightly larger than the true dA since dA_1 also contains the small white open triangles in fig.2a. Since (6) and (6′) differ only by a second order term $2\pi R \cos\vartheta\ d\vartheta\ ds$, (6) is correct in linear approximation, i.e. for a differential. We prefer (6) instead of (6′) because it is simpler.

14.5. **c**) Integrate (6) with (1) (3) from $\vartheta = 0$ to $\vartheta = \pi$ to obtain the surface of the sphere.

RESULT:

$$\boxed{A = 4\pi R^2} \quad (A = \text{ surface of a sphere with radius } R) \tag{7}$$

(Solution:)

$$\begin{aligned} A &= \int dA = \int_0^\pi 2\pi R \sin\vartheta R\, d\vartheta \\ &= 2\pi R^2 \int_0^\pi \sin\vartheta\, d\vartheta = 2\pi R^2 \left[-\cos\vartheta\right]_0^\pi \\ &= 2\pi R^2 (\underbrace{-\cos\pi}_{1} + \underbrace{\cos 0}_{1}) = 4\pi R^2 \end{aligned} \tag{8}$$

14.**Ex 6: Volume of a sphere**

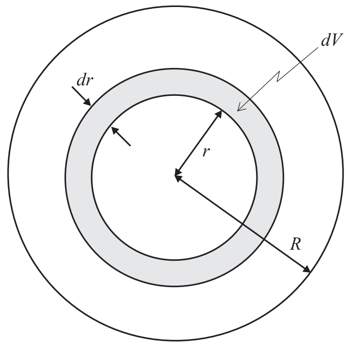


Fig14.6. **1**: The volume V of a sphere with radius R is the integral of the shaded volume elements dV.

14.6. **a**) We treat the shaded volume element dV as a plate (**cuboid**[$\stackrel{\mathbf{G}}{=}$ `Quader`]) with ground area as the surface of a sphere with radius r and height[14] dr. Calculate dV.

RESULT:[15]

$$dV = 4\pi r^2\, dr \tag{1}$$

[14] Note that the radius element dr (see fig.1) is perpendicular to the tangential plane at a point P on the sphere r, so after cutting, tearing and squashing to form a plane cuboid, dr remains perpendicular to the ground plate of the cuboid.

[15] If we had taken the surface of the sphere $(r+dr)$ we would have obtained $dV_1 = 4\pi(r+dr)^2 dr = 4\pi r^2 dr + 4\pi 2r(dr)^2 + 4\pi(dr)^3$ which is slightly too large, whereas (1) was slightly too small. Since dV and dV_1 are identical in linear approximation, (1) is correct as a differential.

14.6. **b**) Calculate the volume of a sphere by integrating dV.

RESULT:

$$\boxed{V = \frac{4}{3}\pi R^3} \quad (V = \text{ volume of a sphere with radius } R) \tag{2}$$

(Solution:)

$$V = \int dV = \int_0^R 4\pi r^2 dr = \left[4\pi \frac{1}{3} r^3\right]_0^R = \frac{4}{3}\pi R^3 \tag{3}$$

15 Substitution method and partial integration

15.Q 1: Substitution method

15.1. **a**) What is the **substitution method** for calculating integrals?

(Solution:)

When we have to calculate an integral:

$$\int_a^b f(x)\,dx \tag{1}$$

we choose a (suitable) new variable y connected to the old one (x) by, let's say,

$$y = g(x) \tag{2}$$

and express everything in (1), i.e. the integrand $f(x)$, the differential dx and the boundaries a and b in the new variable (coordinate) y.

REM 1: In (2) we **substitute**[$\stackrel{\mathbf{G}}{=}$ `ersetzen`] the integration variable x by the new integration variable y. Therefore the name 'substitution method'.

REM 2: Intuitively this procedure is **obvious**[$\stackrel{\mathbf{G}}{=}$ `offensichtlich`]: the integral is just a sum. We express every summand in new variables without changing their values. We make sure that we have corresponding (the same) boundaries.

15.1. **b**)☻ Perform this procedure explicitely in the general case (2).

HINT: Use the inverse function $h = g^{-1}$, i.e.

$$x = h(y), \quad y = h^{-1}(x) = g(x) \tag{3}$$

(Solution:)

Transformation of the integrand:

$$f(x) = f(h(y)) \tag{4}$$

REM 1: As an example $T = f(x)$ could be the temperature at position x expressed in meters. $x = h(y) = \frac{1}{40}y$, $\quad y =$ position in inches. Then (4) reads: $T = f(\frac{1}{40}y)$. The temperature T at a definite physical point is the same, irrespective if position is expressed in meters (x) or in inches (y); $\quad x = \frac{1}{40}y$.

Transformation of the differential, using (3):

$$\frac{dx}{dy} = h'(y) \qquad \Rightarrow \qquad dx = h'(y)\,dy \tag{5}$$

Transformation of the boundaries:
$x = a, x = b$ by (3) correspond to

$$y = h^{-1}(a), \quad y = h^{-1}(b) \tag{6}$$

Thus we arrive at:

$$\boxed{\int_a^b f(x)\,dx = \int_{h^{-1}(a)}^{h^{-1}(b)} f(h(y))\,h'(y)\,dy} \quad \textbf{(substitution method)} \tag{7}$$

REM 2: By a suitable choice of h, the right hand side of (7) might be simpler than its left hand side.

15.1. **c**) Perform the procedure of substitution of integrals to calculate

$$I = \int_{x=0}^{x=1} e^{ax}\,dx, \quad a = \text{const}, \quad a > 0 \tag{8}$$

Unfortunately, we only know the indefinite integral

$$\int^x e^y dy = e^x + \text{const.} \tag{9}$$

Choose a suitable **substitution of variables**[$\stackrel{\mathbf{G}}{=}$ `Variablensubstitution`], so that the integral (8) acquires the form (9).

REM: In (8) we have given the boundaries as $x = 0$ and $x = 1$. This '$x =$ ' is **superfluous**[$\stackrel{\mathbf{G}}{=}$ `überflüssig`], since the differential dx indicates what is the integration variable (namely x). Since in the substitution method the integration variable will be changed, we make this additional designation to make clear that 0 and 1 are x-boundaries (i.e. bounderies expressed in the variable x).

HINT: Choose

$$y = g(x) = ax \tag{10}$$

(Solution:)

Transformation of the integral:

$$e^{ax} = e^y \tag{11}$$

Transformation of the differential:

$$\frac{dy}{dx} = y'(x) \stackrel{(10)}{=} a \qquad \Rightarrow \qquad dx = \tfrac{1}{a}\,dy \tag{12}$$

Transformation of the boundaries:
$x = 0, x = 1$ corresponds to

$$y = 0, \quad y = a \tag{13}$$

Thus we arrive at the final result

$$I = \int_{y=0}^{y=a} e^y \tfrac{1}{a}\, dy = [\tfrac{1}{a} e^y]_0^a = \tfrac{1}{a}(e^a - 1) \tag{14}$$

15.**Q 2: ☻ Partial integration**

15.2. **a**) Give the formula for partial integration.

(Solution:)

$$\boxed{\int_a^x u(t)v'(t)\; dt = [u(t)v(t)]_a^x - \int_a^x u'(t)v(t)\; dt} \quad (\textbf{Partial integration}) \tag{1}$$

15.2. **b**) Express it in words.

(Solution:)

If the integrand is a product of a function and of the derivative of another function, the integral is minus a similar integral where the differentiation is shifted from the one factor to the other one, with the addition of a **boundary term**[$\stackrel{\mathbf{G}}{=}$ `Randterm`] which is the difference of the product of both functions at the boundaries.

Rem: Short: The derivative can be shifted over from one factor to the other if we accept the penalty of a minus sign and of a **boundary term**[$\stackrel{\mathbf{G}}{=}$ `Randterm`] $[\cdots]_a^x$.

15.2. **c**) Prove it by differentiation.

(Solution:)

Differentiating both sides of (1) leads to

$$u(x)v'(x) = u(x)v'(x) + u'(x)v(x) - u'(x)v(x) \tag{2}$$

[For the differentiation of the integrals we have used the main theorem of calculus. For the product uv we have used the product rule.]
(2) is true. So, since the derivative of both sides of (1) are equal, the sides itself can only differ by a constant. But this constant must be zero, since (1) is true for $x = a$. q.e.d.

15.**Ex 3: The substitution method**

15.3. **a**)

$$\int_a^b \frac{x^3}{2-3x^4} dx \tag{1}$$

Hint: substitute

$$z = 2 - 3x^4 \qquad \Longrightarrow \qquad \frac{1}{2-3x^4} = \frac{1}{z} \tag{2}$$

RESULT:

$$= -\tfrac{1}{12}\ln\left|\frac{2-3b^4}{2-3a^4}\right| \tag{3}$$

(Solution:)

$$\frac{dz}{dx} \overset{(2)}{=} -12x^3, \quad dz = -12x^3\,dx \quad \Rightarrow \quad x^3\,dx = -\tfrac{1}{12}\,dz \tag{4}$$

Thus

$$\int_a^b \frac{x^3}{2-3x^4}dx \overset{(2)(4)}{=} -\tfrac{1}{12}\int \frac{dz}{z} \overset{\spadesuit}{=} [-\tfrac{1}{12}\ln|z|]_{z=2-3a^4}^{z=2-3b^4} \tag{5}$$

$$= -\tfrac{1}{12}\left(\ln\left|2-3b^4\right| - \ln\left|2-3a^4\right|\right) \overset{\clubsuit}{=} -\tfrac{1}{12}\ln\left|\frac{2-3b^4}{2-3a^4}\right| \tag{6}$$

$\spadesuit$ The antiderivative of $\frac{1}{x}$ is $\ln|x|$.
$\clubsuit$ $\ln x - \ln y = \ln\frac{x}{y}$

15.3. **b)**☻ The last example is quite general if the integrand is a fraction with the **numerator**[$\overset{\mathbf{G}}{=}$ `Zähler`] being the derivative of the **denominator**[$\overset{\mathbf{G}}{=}$ `Nenner`]. Thus calculate

$$\int_a^b \frac{f'(x)}{f(x)}dx \tag{7}$$

with the substitution

$$z = f(x) \tag{8}$$

RESULT:

$$\boxed{\int_a^b \frac{f'(x)}{f(x)}dx = \ln\left|\frac{f(b)}{f(a)}\right|} \qquad (\textbf{logarithmic integration}) \tag{9}$$

(Solution:)

$$dz \overset{(8)}{=} f'(x)\,dx \tag{10}$$

$$\int_a^b \frac{f'(x)}{f(x)}dx \overset{(18)}{=} \int_{f(a)}^{f(b)} \frac{dz}{z} = [\ln|z|]_{f(a)}^{f(b)} = \ln|f(b)| - \ln|f(a)| = \ln\left|\frac{f(b)}{f(a)}\right| \tag{11}$$

15.Ex 4: Calculation of arc lengths (rectifications)

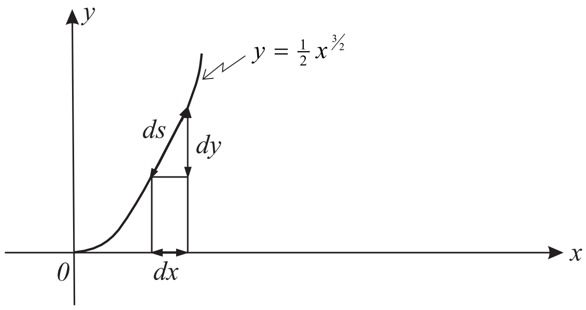


Fig15.4. 1: The line element ds of the curve is calculated as the hypotenuse of a right triangle with dx as the base and dy as the perpendicular.

We will calculate the length s of the curve

$$y = \tfrac{1}{2}x^{3/2} \tag{1}$$

in the interval $0 \leq x \leq 1$ (solid line in above figure).
15.4. **a**) Calculate the line element ds (expressed by dx).
HINT 1: calculate dy by differentiating (1).
HINT 2: approximate ds as the hypotenuse of a right triangle with base dx and perpendicular dy; use Pythagoras.
RESULT:

$$ds = \sqrt{1 + \tfrac{9}{16}x}\ dx \tag{2}$$

(Solution:)

$$y' = \frac{dy}{dx} = (\tfrac{1}{2}x^{3/2})' = \tfrac{1}{2} \cdot \tfrac{3}{2}x^{1/2}, \tag{3}$$

$$dy = \tfrac{3}{4}x^{1/2}\ dx \tag{4}$$

Pythagoras

$$ds^2 = dx^2 + dy^2 \tag{5}$$

CONVENTION:

$$\boxed{\begin{array}{c} ds^2 \;\text{ means }\; (ds)^2, \;\text{ not }\; d(s^2); \\ dx^2 \;\text{ means }\; (dx)^2, \;\text{ not }\; d(x^2)\;, \text{ etc.} \end{array}} \tag{6}$$

$$ds = \sqrt{dx^2 + \tfrac{9}{16}x\, dx^2} = \sqrt{1 + \tfrac{9}{16}x}\; dx \tag{7}$$

15.4. **b**) Express the length s as an integral.
RESULT:

$$s = \int ds = \int_0^1 \sqrt{1 + \tfrac{9}{16}x}\; dx \tag{8}$$

15.4. **c**) Evaluate integral (8) with the help of the **substitution**

$$u = 1 + \tfrac{9}{16}\, x \tag{9}$$

RESULT:

$$s = \frac{61}{54} \tag{10}$$

(Solution:)

$$du \overset{(9)}{=} \tfrac{9}{16}\, dx \tag{11}$$

Thus by (8):

$$\begin{aligned} s &= \tfrac{16}{9} \int_1^{25/16} \sqrt{u}\; du = \tfrac{16}{9} \cdot \left[\tfrac{2}{3} u^{\frac{3}{2}}\right]_1^{25/16} = \tfrac{32}{27} \left[u^{\frac{3}{2}}\right]_1^{\frac{25}{16}} = \\ &= \frac{32}{27}\left(\left(\frac{25}{16}\right)^{3/2} - 1\right) = \frac{32}{27}\left(\frac{5^3}{4^3} - 1\right) = \frac{32}{27}\left(\frac{125}{64} - 1\right) = \\ &= \frac{32}{27} \cdot \frac{125 - 64}{64} = \frac{32}{27} \cdot \frac{61}{64} = \frac{61}{54} \end{aligned} \tag{12}$$

(61 is a prime number.)

15.**Ex 5: ☻ Any quantity is the integral of its differentials**
Transform the general integral

$$I(x) = \int_a^x f(\xi)\, d\xi \tag{1}$$

with the substitution

$$I = I(\xi). \tag{2}$$

HINT: observe that differentiation is the inverse of integration, i.e.

$$I' = f \tag{3}$$

RESULT:

$$I = \int dI \tag{4}$$

REM: in words:

$$\boxed{\begin{array}{c}\text{any quantity is the integral of its differentials}\\ \text{(or in other words: of its elements).}\end{array}} \tag{5}$$

This is an exact justification (by the substitution method) of our procedure of first calculating the differential dI of a desired quantity I and then integrating (i.e. looking for the antiderivative).

(Solution:)

$$dI = I'(\xi)\, d\xi \overset{(3)}{=} f(\xi)\, d\xi \tag{6}$$

The substitution method applied to (1) yields

$$I(x) = \int_{\xi=a}^{\xi=x} f(\xi)\, d\xi \overset{(6)}{=} \int_{I=I(a)}^{I=I(x)} dI = [I]_{I=I(a)}^{I=I(x)} = I(x) - I(a) \overset{(1)}{=} I(x) \tag{7}$$

which, in shorter notation, is (4).

16 ☻ Improper integrals

16.Q 1: Improper integrals

$$y = x^{-n}, \quad (n = 1, 2, 3, \cdots) \tag{1}$$

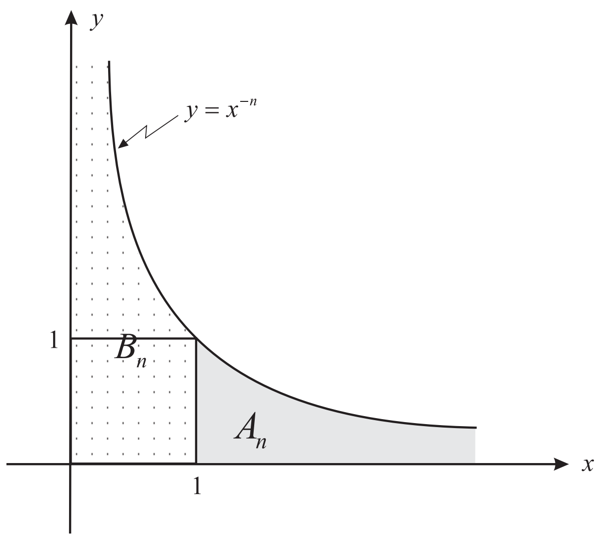


Fig16.1. **1**: We calculate the dotted (B_n) and the shaded (A_n) areas for several n's. Since a function value or the range of the integrals is infinite, they are improper integrals, which are limits of ordinary integrals.

16.1. **a**) For $n > 1$ calculate the shaded area A_n.

(Solution:)

$$A_n = \int_1^\infty x^{-n}\, dx = \left[\frac{1}{1-n}\, x^{-n+1}\right]_1^\infty = \frac{1}{1-n}\left(\frac{1}{\infty}\right)^{n-1} - \frac{1}{1-n} \tag{2}$$

Since $n - 1 > 0$ the first term involving ∞ vanishes, thus we get

$$A_n = \frac{1}{n-1} \tag{3}$$

16.1. **b**) The same for $n = 1$. (The curve is then called a **hyperbola**[$\stackrel{\mathbf{G}}{=}$ `Hyperbel`].)

(Solution:)

$$A_1 = \int_1^\infty \frac{1}{x}\, dx = [\ln|x|]_1^\infty = [\ln x]_1^\infty = \ln\infty = \infty \tag{4}$$

i.e. the shaded area is infinite.

The area under the hyperbola is infinite

16.1. **c**) The **dotted**[$\stackrel{G}{=}$ `punktierte`] area B_n for $n < 1$.

(Solution:)

Using a)

$$B_n = \int_0^1 x^{-n}\, dx = \left[\frac{1}{1-n}\, x^{-n+1}\right]_0^1 = \frac{1}{1-n} - \frac{1}{1-n}\, 0^{1-n} \tag{5}$$

We have $1-n > 0$, so the second term vanishes and we find

$$B_n = \frac{1}{1-n} \tag{6}$$

16.1. **d**) The same for $n = 1$.

(Solution:)

Using b)

$$B_1 = \int_0^1 \frac{1}{x}\, dx = [\ln|x|]_0^1 = \ln 1 - \ln 0 = 0 - (-\infty) = \infty \tag{7}$$

i.e. the area is infinite.

16.1. **e**) Why are the above **improper integrals**[$\stackrel{G}{=}$ `uneigentliche Integrale`]? Reformulate the above results using limits of proper integrals

(Solution:)

∞ occurs either as a boundary, or at a boundary the integrand is ∞. Since ∞ is not a **fully-fledged**[$\stackrel{G}{=}$ `vollwertig`] number, something is meaningless. Hence, the above integrals are called improper.
The above improper integrals can be written as limits of proper integrals:

$$A_n = \lim_{x\to\infty} \int_1^x \xi^{-n}\, d\xi = \frac{1}{n-1} \qquad (n > 1) \tag{8}$$

$$A_1 = \lim_{x\to\infty} \int_1^x \frac{1}{\xi}\, d\xi = \infty \tag{9}$$

$$B_n = \lim_{\epsilon\to 0_+} \int_\epsilon^1 \xi^{-n}\, d\xi = \frac{1}{1-n} \qquad (n < 1) \tag{10}$$

$$B_1 = \lim_{\epsilon \to 0_+} \int_{\epsilon}^{1} \frac{1}{\xi} \, d\xi = \infty \tag{11}$$

REM: $\epsilon \to 0_+$ means $\epsilon \to 0$ whereby only limiting processes with $\epsilon > 0$ are considered.
Also $x \to \infty$ implies $x \neq 0$, otherwise the integrand is not defined.

17 ☻ Partial derivatives and total differential. Implicit functions

17.**Q 1: Partial derivatives**
Consider a function $z = z(x, y)$ of *two* independent variables x, y, e.g. conceived as the surface of a mountains.

REM: To save letters we write $z = z(x, y)$ instead of $z = f(x, y)$ to denote an arbitrary function of two independent variables x and y. Thus z is both the dependent variable and the name of a function. No confusion is possible.

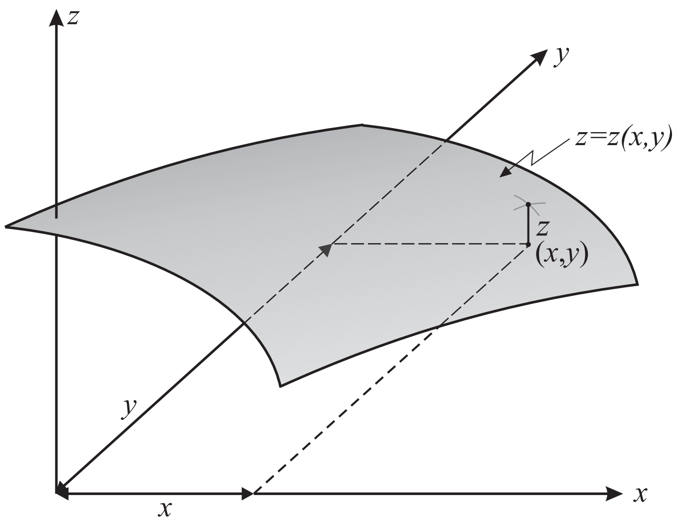


Fig17.1. **1**: A function $z = z(x, y)$ can be viewed as the height z of a mountain above a base point (x, y).

17.1. **a**) What is (in words) the **partial derivative**[$\stackrel{\mathbf{G}}{=}$ `partielle Ableitung`]

$$\frac{\partial z}{\partial x} \tag{1}$$

(Solution:)

$\frac{\partial z}{\partial x}$ is the derivative of z with respect to x, when the other variables, in our case y only, are held constant (i.e. while differentiating, they are treated *as if* they where constants).

17.1. **b**) In case of

$$z = \sin(xy) + y \tag{2}$$

calculate all first order and second order partial derivatives.

(Solution:)

$$\frac{\partial z}{\partial x} = y\cos(xy) \tag{3}$$

$$\frac{\partial z}{\partial y} = x\cos(xy) + 1 \tag{4}$$

$$\frac{\partial^2 z}{\partial x^2} = -y^2\sin(xy) \tag{5}$$

$$\frac{\partial^2 z}{\partial y^2} = -x^2\sin(xy) \tag{6}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \cos(xy) - xy\sin(xy) \tag{7}$$

17.1. **c**) Give alternative notations for (higher) partial derivatives.

(Solution:)

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\, z = \partial_x\, z = z_{,x} = z_{|x} \tag{8}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\, z = \partial_x \partial_y z = z_{,xy} = z_{|xy} \tag{9}$$

17.1. **d**) Give relations between higher partial derivatives.

(Solution:)

The order of the partial derivatives is irrelevant. One example is (7), a further one is:

$$z_{|yyx} = z_{|xyy} = z_{|yxy} \tag{10}$$

17.**Q 2: Taylor's formula in 2 variables**

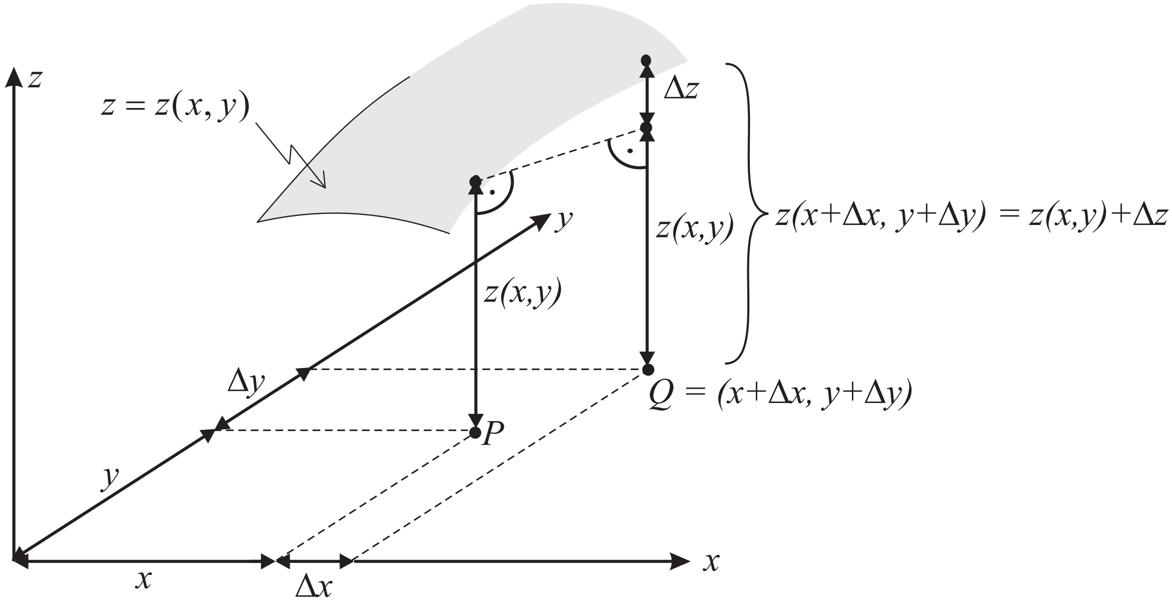


Fig17.2. **1**: The increment Δz of the functional value z depends upon two independent increments $dx = \Delta x$ and $dy = \Delta y$.

Starting from a point $P = (x, y)$ with height $z = z(x, y)$ we go to a displaced point $Q = (x + \Delta x, y + \Delta y)$ by two independent increments Δx and Δy.

17.2. **a**) Give the Taylor formula for the corresponding dependent increment

$$\Delta z = z(x + \Delta x, y + \Delta y) - z(x, y) \tag{1}$$

up to the second order (inclusive) and give an example of a (neglected) third order term.

(Solution:)

$$\boxed{\begin{aligned} \Delta z = \; & \tfrac{\partial z}{\partial x}\Delta x + \tfrac{\partial z}{\partial y}\Delta y + \\ & + \; \tfrac{1}{2}\, \tfrac{\partial^2 z}{\partial x^2}(\Delta x)^2 + \tfrac{\partial^2 z}{\partial x \partial y}(\Delta x)(\Delta y) + \tfrac{1}{2}\, \tfrac{\partial^2 z}{\partial y^2}(\Delta y)^2 + O(3) \end{aligned}} \tag{2}$$

(Taylor's formula in 2 variables in second order approximation)
where $O(3)$ includes all terms of third order or higher, including e.g. the term

$$\frac{1}{3!}\frac{\partial^3 z}{(\partial x)^3}\,(\Delta x)^3 \quad \text{or} \quad \frac{\partial^3 z}{(\partial x)^2 \partial y}\,(\Delta x)^2\,(\Delta y) \tag{3}$$

REM: Again the **gist**[$\overset{\mathbf{G}}{=}$ `Knackpunkt`] of Taylor's formula (2) is that we know the value of the function $z(x + \Delta x, y + \Delta y) = z(x, y) + \Delta z$ at the neighbouring point $Q = (x + \Delta x, y + \Delta y)$ if we know all its higher partial derivatives

$z(x,y), \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y} \cdots$ at the undisplaced point $P = (x, y)$. Or, if we know only the first few of them, we know the function value at least approximatively for small values of Δx and Δy.

17.2. **b**) For the above function

$$z = \sin(xy) + y \tag{4}$$

calculate z in the neighbourhood of $P_0 = (0,0)$ in second order approximation.

(Solution:)

In our trivial example ($x = y = 0$) all partial derivatives up to the second order are zero, except

$$\frac{\partial z}{\partial y} = 1 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x \partial y} = 1 \tag{5}$$

Thus

$$\Delta z = \Delta y + \Delta x \Delta y \tag{6}$$

Because of (1) we have

$$z(\Delta x, \Delta y) = z(0,0) + \Delta z = \Delta z \tag{7}$$

Replacing $\Delta x \mapsto x, \quad \Delta y \mapsto y$ we obtain:

$$z = z(x,y) = y + xy \tag{8}$$

REM: The same result (8) could be obtained from (4) by developing sin in linear approximation of its argument: $\sin \varepsilon \approx \varepsilon$.

17.2. **c**) Starting from the above Taylor formula derive the formula for the **total differential** dz.

REM: Instead of 'total differential' the synonymous term '**complete differential**' is also used.

(Solution:)

A differential (denoted by d) is an increment (denoted by Δ) in the lowest order of approximation. For the independent increments there is no difference between increment and differential:

$$\Delta x = dx, \quad \Delta y = dy \tag{9}$$

and (2) reads in lowest (=first) order approximation:

$$\boxed{dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy} \qquad \textbf{total differential} \tag{10}$$

REM: Here, and in similar cases, it is implicitly assumed that $\Delta x, \Delta y$ are both of the same (i.e. first) order.

17.2. **d**) Explain the qualifier *'total'*.

(Solution:)

Putting $dy = 0$ in (10) we obtain a partial differential

$$dz_x = \frac{\partial z}{\partial x}\, dx \tag{11}$$

It is a special case of dz when all other independent increments, besides dx, are zero. (10) says that the total differential (i.e. when all independent differential are present) is simply the sum of all partial differentials. This must be so, because differentials are always calculated in the lowest (here: linear) approximation.

17.2. **e**) Give (in words) the geometric meaning of dz using the above figure.

(Solution:)

dz is the increment Δz when the real surface $z(x, y)$ is replaced by its **tangential plane**[$\stackrel{\mathbf{G}}{=}$ `Tangentialebene`] above P.

17.2. **f**) Generalize to the formula for the total differential for a function

$$y = y(x_1, x_2, \ldots, x_n) \tag{12}$$

of n independent variables $x_1, \ldots, x_n$.

(Solution:)

$$\boxed{dy = \sum_{i=1}^{n} \frac{\partial y}{\partial x_i}\, dx_i} \qquad \textbf{total differential} \text{ in } n \text{ variables} \tag{13}$$

REM: From this important formula, we can immediately deduce that an extremal point is given by

$$\boxed{\frac{\partial y}{\partial x_1} = \frac{\partial y}{\partial x_2} = \cdots = \frac{\partial y}{\partial x_n} = 0 \quad \textbf{extremum}} \tag{14}$$

i.e. **all partial derivatives are zero**, since at an extremum (e.g. minimum or maximum) $dy = 0$ (in first order approximation) for any values of the independent increments dx_i.
(In more detail: choose e.g. $dx_1 \neq 0$ but $dx_i = 0$ for the remaining increments, to deduce $\frac{\partial y}{\partial x_1} = 0$.)

17.Q 3: Implicit functions

17.3. **a**) Explain why by

$$f(x, y) = 0, \tag{1}$$

with a given function $f(x, y)$, we define a function $y = y(x)$ (called an implicit function, more correctly: an implicitly defined function)

(Solution:)

For each x (considered as a paramter) $f(x, y) = 0$ is an equation for y. The solution of that equation $y = y(x)$ is the implicit function.

REM: When there are several solutions for a fixed x, then $y(x)$ is a multi-valued function.

17.3. **b**) Use the example

$$f(x, y) \equiv x^2 + y^2 - 1 \tag{2}$$

and give in that case the function $y(x)$ in explicit form. Give the geometric meaning of that example.

(Solution:)

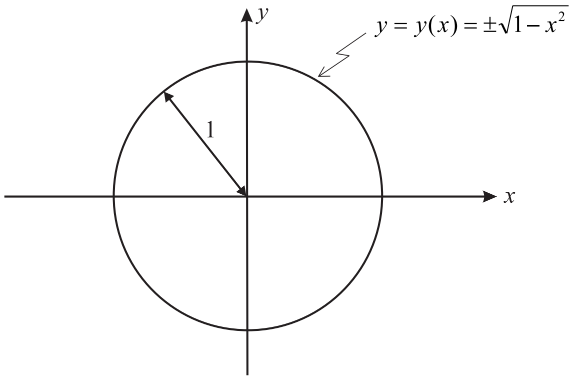


Fig17.3. **1**: The unit circle can be viewed as the graph of the double valued function $y = y(x) = \pm\sqrt{1 - x^2}$. $y(x)$ is the implicit function given by the equation of the circle: $x^2 + y^2 = 1$

(2) reads

$$x^2 + y^2 = 1 \tag{3}$$

which, geometrically, is the **unit circle**[$\overset{\mathbf{G}}{=}$ `Einheitskreis`]. Solving that equation for y (with x as a parameter) we obtain the implicit function

$$y = y(x) = \pm\sqrt{1 - x^2} \tag{4}$$

REM 1: Since, in our example, (3) does not have a unique solution, the implicit function $y = y(x)$ is not uniquely defined, but is a double-valued function ($\pm$).

Rem 2: The function (4) (as a real valued function) is only defined in the interval [-1, 1].

Rem 3: 'Explicit' or 'implicit' is not an attribute of the function, but refers only to a chosen way of defining it.

17.3. **c**) Is it possible to give the function $y(x)$ implicitly defined by

$$y^5x + y^4(x^2 - 2x) + yx + 3 = 0 \tag{5}$$

in explicit form?

(Solution:)

For given x, (5) is an algebraic equation of the fifth order, which cannot be solved for y in the general case. Therefore $y(x)$ cannot be given in explicit form.

Rem 1: The ordinary citizen can only solve linear equations and quadratic equations, e.g. (in y)

$$xy^2 + (x^2 - 1)y + (x^7 + 2) = 0 \tag{6}$$

Mathematicians can also solve third and fourth order algebraic equations. Equations of order higher than 4 cannot be solved using root symbols only, except in special cases.

Rem 2: The equations (3)(6) are called 'algebraic', where the word 'algabra' is used in an old fashioned meaning involving addition and multiplication (including natural exponents) only.

17.3. **d**)

$$e^y = x \tag{7}$$

defines the function $y(x)$ implicitly. Bring that implicit definition into the form (1) and give $y(x)$ in explicit form.

(Solution:)

$$f(x, y) \equiv e^y - x = 0 \tag{8}$$

$$y = \ln x \tag{9}$$

Obviously, the implicitly defined function $y = y(x)$ is just the **inverse function**[$\stackrel{\mathbf{G}}{=}$ `Umkehrfunktion, inverse Funktion`] of the function on the left hand side of (7).

17.3. **e**) The same for

$$\sin y = x \tag{10}$$

Sketch the graph of that function.

(Solution:)

(10) is the sine-function, except for the unusual choice of variables:

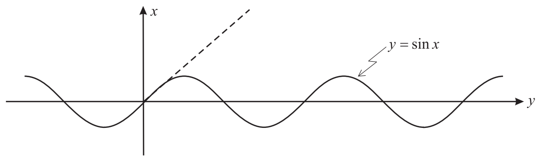


Fig17.3. **2**: Function ($y = \sin x$) and inverse function ($y =$ arc sin$x$) have the same graph, but $x \Longleftrightarrow y$ is interchanged.

By interchanging $x \Longleftrightarrow y$ (reflexion of the graph at the dotted **half angle line**[$\stackrel{\mathbf{G}}{=}$ `Winkelhalbierende`]) we get the following graph:

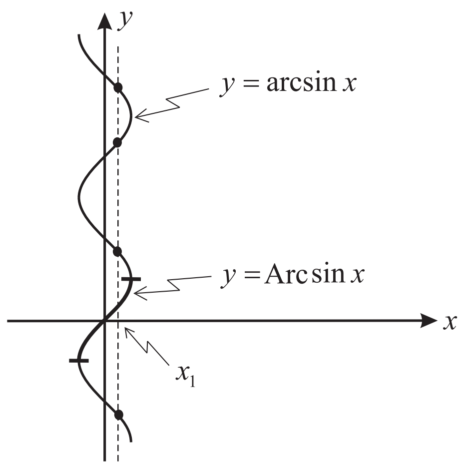


Fig17.3. **3**: Here, the labels at the axes (x, y) are as usual. So the graph of the inverse function is obtained by a mirror symmetry at the angle line between these axes.

Again, the implicitly defined function $y = y(x)$ is just the inverse function of sin and thus is denoted by

$$y = \arcsin x \tag{11}$$

Rem 1: Since y has the geometrical meaning of an angle (older terminology: an arc), y is the arc (lat: arcus) whose sin is x, that's why, the inverse function is called arcsin.

REM 2: $y = \arcsin x$ is an infinite valued function: To a definite value of x, the corresponding function values y are depicted by small circles in the above graph.

REM 3: To obtain a unique function, denoted by $y = \text{Arcsin}\ x$, i.e. with a capital A, one takes arbitrarily the branch depicted by a **bold line**[$\overset{G}{=}$ `fette Kurve`] in the above graph. It is called the **principal branch**[$\overset{G}{=}$ `Hauptast`] of the graph (or of the function). Every restriction of the graph, so that the function becomes unique is called a **branch**[$\overset{G}{=}$ `Ast, Zweig`] of the function. In an infinite-valued function, the function has an infinite number of branches. The function value y defined by $y = \text{Arcsin}\ x$ is called the **principal value**[$\overset{G}{=}$ `Hauptwert`] of the multiple-valued function $y = \arcsin x$.

REM 4: 'ln' and 'arcsin' are just newly defined mathematical symbols introduced for the solution of the equations (7) and (10) of the implicit definitions. Therefore it is a matter of taste if we say that an explicit form of the function is possible or not.

REM 5: Since arcsin is the inverse function of sin, we have the equations

$$\boxed{\arcsin(\sin x) = x} \tag{12}$$

(Since arcsin is a multiple valued function, (12) is true for a suitably chosen branch only

$$\boxed{\sin(\arcsin x) = x} \tag{13}$$

(The multi-valuedness of arcsin does not matter here because sin cancels it)

17.Q 4: Implicit differentiation

17.4. **a**) Using the total differential of

$$z = f(x, y) \tag{1}$$

derive the formula

$$\boxed{y'(x) = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}} \quad \text{(implicit differentiation)} \tag{2}$$

for the derivative $y'(x)$ of the function $y(x)$ implicitly defined by

$$f(x, y) = 0. \tag{3}$$

(Solution:)

$$dz = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0 \tag{4}$$

REM Geometric interpretation:
We choose the dependent increments dx, dy so that always $f(x, y) = 0$, i.e. we follow

a **contour line**[$\stackrel{\mathbf{G}}{=}$ `Höhenlinie`], i.e. a path without slope, remaining always on the altitude $f = 0$. The x, y-values thus followed are connected by $y = y(x)$ and dx, dy are corresponding differentials of $y = y(x)$.

Solving (4) for dy/dx leads to (2)

17.4. **b**) For the example

$$x^2 + y^2 = 1 \tag{5}$$

calculate $y'(x)$ by implicit differentiation.

(Solution:)

$$f(x, y) = x^2 + y^2 - 1 = 0 \tag{6}$$

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y \tag{7}$$

leading to the result

$$y' = -\frac{x}{y} \tag{8}$$

17.4. **c**) And, alternatively, by first calculating $y(x)$ explicitly.

(Solution:)

$$y(x) = \sqrt{1 - x^2} \tag{9}$$

$$y'(x) = \frac{1}{2\sqrt{1 - x^2}} \, (-2x) = -\frac{x}{\sqrt{1 - x^2}} \tag{10}$$

using (9) we see that this result is equivalent to (8)

REM: So in general, the result of implicit differentiation does not give y' in explicit form, when $y(x)$ is not known explicitly. However, as in this example, implicit differentiation may be easier than differentiation of the explicit function. And in other special examples (8) may have such a form that y drops out, etc.

17.Ex 5: Error propagation of multiple error sources

A student measures the side lengths a and b of a rectangle and calculates its area using

$$\boxed{A = ab} \tag{1}$$

What is the relative error ε_A if the relative errors of the measured side lengths are assumed to be ε_a and ε_b.

Hint 1: The area A is a function of two variables: $A = A(a, b) = ab$.

Hint 2: Treat ε_a, ε_b, ε_A and the corresponding absolute errors as differentials; use the complete differential of A.

Hint 3: If the *measured* value of a side is a_m and the exact (unknown) value is a, then the *absolute* error is $\Delta a = a_m - a$ and the *relative* error is $\epsilon_a = \Delta a/a$.
$A_m = a_m b_m$ is the proposed value for the area, whereas the exact (unknown) value for the area is $A = ab$. The absolute error of the area is $\Delta A = a_m b_m - ab$, the relative error is $\epsilon_A = \Delta A/A$. Treating as differentials means $\Delta = dA$, i.e. ΔA is calculated in *linear* approximation in da and db only.

Result:

$$\boxed{\varepsilon_A = \varepsilon_a + \varepsilon_b \qquad \textbf{(for factors, relative errors are additive)}} \tag{2}$$

(Solution:)

$$da = a\,\varepsilon_a, \qquad db = b\,\varepsilon_b, \qquad dA = \varepsilon_A\,A \tag{3}$$

Complete differential of A:

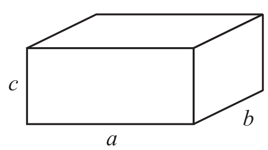
$$dA = \frac{\partial A}{\partial a}da + \frac{\partial A}{\partial b}db = b\,da + a\,db \tag{4}$$

$$\varepsilon_A = \frac{dA}{A} = \frac{b}{A}da + \frac{a}{A}db = \frac{da}{a} + \frac{db}{b} = \varepsilon_a + \varepsilon_b \tag{5}$$

Rem: Think about solving the same problem when $\epsilon_a, \epsilon_b, \epsilon_A$ are not treated as differentials but as exact quantities. Regain (5) by a linear approximation.

17.**Ex 6: Container with maximum volume**

We would like to construct a container of maximum volume V in the form of a **cuboid**[$\stackrel{\mathbf{G}}{=}$ `Quader`] with side lengths a, b, c, under the **auxiliary condition**[$\stackrel{\mathbf{G}}{=}$ `Nebenbedingung`] that the **surface area**[$\stackrel{\mathbf{G}}{=}$ `Oberfläche`] is given (= fixed) as A_0. Calculate a, b, c.



`Fig`17.6. **1**: What are the side lengths of a, b, c of a cuboid with maximum volume but given surface area?

17.6. **a**) Express $V = V(a, b, c)$ and the surface area $A = A(a, b, c)$. Eliminate c with the help of $A = A_0 =$ given, to calculate $V = V(a, b)$.
RESULT:

$$V = V(a,b) = \frac{ab}{a+b}\left(\frac{1}{2}A_0 - ab\right) \tag{1}$$

(Solution:)

$$V = abc, \qquad A = 2(ab + ac + bc) = A_0 \tag{2}$$

$$ab + c(a+b) = \frac{1}{2}A_0, \qquad c = \frac{\frac{1}{2}A_0 - ab}{a+b} \tag{3}$$

$$V = V(a,b) = \frac{ab}{a+b}\left(\frac{1}{2}A_0 - ab\right) \tag{4}$$

17.6. **b**) What is the condition for maximum $V = V(a, b)$?
RESULT:

$$\frac{\partial V}{\partial a} = \frac{\partial V}{\partial b} = 0 \tag{5}$$

17.6. **c**) As a **preliminary** [$\stackrel{\mathbf{G}}{=}$ `Vorbereitung`] for applying the product rule to $V(a, b)$, calculate

$$\frac{\partial}{\partial a}\frac{ab}{a+b} \quad \text{and} \quad \frac{\partial}{\partial a}\left(\frac{1}{2}A_0 - ab\right) \tag{6}$$

RESULTS:

$$\frac{b^2}{(a+b)^2} \quad \text{and} \quad -b \tag{7}$$

(Solution:)

The quotient rule yields ($A_0, b =$ const.)

$$\frac{\partial}{\partial a}\frac{ab}{a+b} = \frac{(a+b)b - ab}{(a+b)^2} = \frac{b^2}{(a+b)^2} \tag{8}$$

$$\frac{\partial}{\partial a}\left(\frac{1}{2}A_0 - ab\right) = -b \tag{9}$$

17.6. **d) Evaluate**[$\stackrel{\mathbf{G}}{=}$ `auswerten, vereinfachen`] the condition

$$\frac{\partial V}{\partial a} = 0 \tag{10}$$

HINT 1: use the product rule.
HINT 2: since we are looking for a maximum, we have $a > 0, \quad b > 0, \quad a + b > 0$, so we can divide the resulting equation by $\frac{b^2}{(a+b)^2}$ (which is not equal to zero).
RESULT:

$$-a(a+b) + \left(\frac{1}{2}A_0 - ab\right) = 0 \tag{11}$$

(Solution:)

The product rule yields

$$\frac{\partial V}{\partial a} = \frac{ab}{a+b}(-b) + \frac{b^2}{(a+b)^2}\left(\frac{1}{2}A_0 - ab\right) = 0. \tag{12}$$

Dividing by $\frac{b^2}{(a+b)^2}$ yields

$$-a(a+b) + \left(\frac{1}{2}A_0 - ab\right) = 0 \tag{13}$$

17.6. **e**) Similarly, evaluate $\frac{\partial V}{\partial b} = 0$.
HINT: $V = V(a, b)$ has a **formal symmetry** in a, b, i.e. $V(a, b)$ goes into itself by the interchange $a \Longleftrightarrow b$. We can therefore apply this interchange directly to (11).
RESULT:

$$-b(a+b) + \left(\frac{1}{2}A_0 - ab\right) = 0 \tag{14}$$

17.6. **f**) Subtract (11) - (14) to deduce $a = b$.
HINT: A product can be zero only if at least one of its factors is zero.

(Solution:)

Subtraction yields

$$(b-a)(a+b) = 0. \tag{15}$$

$a + b > 0$ thus $b - a = 0$.

17.6. **g**) Calculate a and b from (11) and c from (3) and finally V.
RESULT:

$$a = b = c = \sqrt{\frac{A_0}{6}}, \qquad V = \left(\frac{A_0}{6}\right)^{\frac{3}{2}} \tag{16}$$

(Solution:)

Since $a = b$, (11) yields

$$-2a^2 + \tfrac{1}{2}A_0 - a^2 = 0 \qquad \text{i.e.} \tag{17}$$

$$\tfrac{1}{2}A_0 = 3a^2 \tag{18}$$

$$a^2 = \frac{1}{6}A_0, \quad a = b = \sqrt{\frac{A_0}{6}} \tag{19}$$

By (3)

$$c = \frac{\frac{1}{2}A_0 - \frac{A_0}{6}}{2\sqrt{\frac{A_0}{6}}} = \frac{\frac{1}{3}A_0}{\frac{2\sqrt{A_0}}{\sqrt{6}}} = \frac{1}{6}\sqrt{A_0}\sqrt{6} = \frac{\sqrt{A_0}}{\sqrt{6}} = \sqrt{\frac{A_0}{6}}$$

$$V = abc = \left[\left(\frac{A_0}{6}\right)^{\frac{1}{2}}\right]^3 = \left(\frac{A_0}{6}\right)^{\frac{3}{2}} \tag{20}$$

Rem: The following is a celebrated and important theorem:

> The **shape**[$\stackrel{\mathbf{G}}{=}$ `form`] with given surface area $A = A_0$ and with maximum volume V is a sphere.
> And conversely: the shape with given volume $V = V_0$ and with minimum surface area A is again a sphere. (21)

It is much more difficult to prove that theorem, requiring differential calculus with an infinite number of variables, the so-called **calculus of variations**[$\stackrel{\mathbf{G}}{=}$ `Variations-rechnung`] (being a subbranch of **functional analysis**[$\stackrel{\mathbf{G}}{=}$ `Funktionalanalysis`]).

Our result was much more modest:
among all cuboids the **cube**[$\stackrel{\mathbf{G}}{=}$ `Würfel`] has the largest volume, when surface area is given
(or smallest surface area, when volume is given).

17.Ex 7: Complete differential as the tangential plane

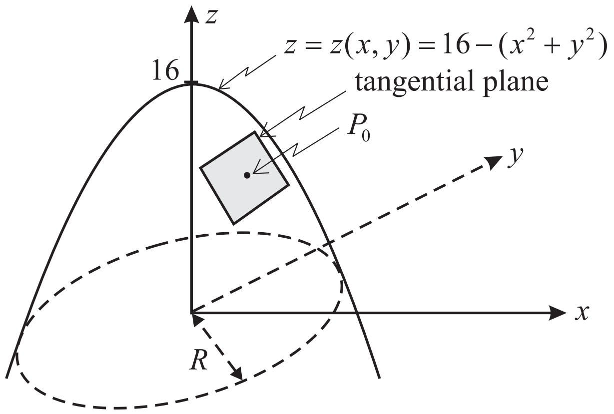


Fig17.7. **1**: Rotation paraboloid $z = 16 - (x^2 + y^2)$ intersects x-y-plane in a circle of radius R. We will calculate the equation of the tangential plane (= shaded rectangle) at P_0.

$$z = z(x, y) = 16 - (x^2 + y^2) \tag{1}$$

is the equation of a rotation paraboloid (as will become more evident in the following exercise b).

17.7. **a**) Calculate its extremum and height.

RESULT:

$$x = y = 0, \qquad z = 16 \tag{2}$$

(Solution:)

$$\frac{\partial z}{\partial x} = -2x \overset{!}{=} 0 \quad \Rightarrow \quad x = 0 \tag{3}$$

$$\frac{\partial z}{\partial y} = -2y \overset{!}{=} 0 \quad \Rightarrow \quad y = 0 \tag{4}$$

$$z = z(0, 0) = 16 \tag{5}$$

17.7. **b**) Show that the intersection with the x-y-plane is a circle with radius $R = 4$.

HINT: intersection with the x-y-plane means that $z = 0$; use Pythagoras to recognize the equation of a circle.

(Solution:)

$$z = 0 \quad \Longleftrightarrow \quad 16 - (x^2 + y^2) = 0 \quad \Longleftrightarrow \tag{6}$$

$$x^2 + y^2 = 16 \tag{7}$$

According to Pythagoras this is the equation of a circle with radius $R = 4$.

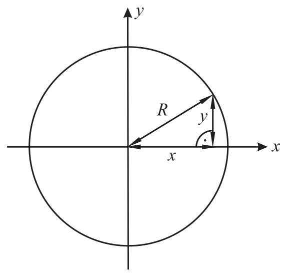


Fig17.7. **2**: Equation of a circle (7) is Pythagoras with r = radius of the circle and the coordinates of a point $P(x, y)$ on the periphery as the base and perpendicular.

REM: intersection at an arbitrary height z gives a circle. Therefore our graph is rotation symmetric about the $z-$axis.

17.7. **c**) Show that the intersection with x-z-plane is a parabola.

(Solution:)

x-z-plane means $y = 0 \iff$

$$z = 16 - x^2 \tag{8}$$

This is the equation of a parabola.

17.7. **d**) At an arbitrary point $P_0(x_0, y_0, z_0)$ with

$$z_0 = z(x_0, y_0) \tag{9}$$

calculate the (complete) differential.
RESULT:

$$dz = -2x_0\, dx - 2y_0\, dy \tag{10}$$

(Solution:)

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \tag{11}$$

$$\frac{\partial z}{\partial x} = -2x = -2x_0 \tag{12}$$

$$\frac{\partial z}{\partial y} = -2y = -2y_0 \tag{13}$$

Thus for the point P_0

$$dz = -2x_0\, dx - 2y_0\, dy \tag{14}$$

17.7. **e**) Calculate the equation of the tangential plane at P_0.
HINT: the differential is the equation of the tangential when you identify dx, dy and dz appropriately.
RESULT:

$$z - z_0 = -2x_0(x - x_0) - 2y_0(y - y_0) \tag{15}$$

(Solution:)

$P(x, y, z)$ is now a point on the tangential plane.
The meaning of the differential is

$$dx = x - x_0, \qquad dy = y - y_0 \tag{16}$$

$$\triangle z = z - z_0 \tag{17}$$

where z is a point on the paraboloid.
In linear approximation

$$\triangle z = dz \tag{18}$$

and z is shifted to a point on the tangential plane. Thus

$$dz = z - z_0 \tag{19}$$

where z is on the tangential plane. With (16) and (19), equation (14) becomes (15), which is the equation of the tangential plane at P_0.

17.7. **f**) Since (15) is linear in x, y, z (i.e. only first powers $x = x^1,\ y = y^1,\ z = z^1$ occur) it is clear (15) is the equation of a plane. Check that it passes through P_0 and that at P_0 it has the same partial derivatives (12) (13) as the paraboloid.

(Solution:)

1) $x = x_0$, $y = y_0$, $z = z_0$ satisfies (15). Thus the tangential plane passes through P_0.
2) From (15) we calculate the partial derivatives as follows (move the constant z_0 to the right hand side of (15)).

$$\frac{\partial z}{\partial x} = -2x_0, \qquad \frac{\partial z}{\partial y} = -2y_0 \tag{20}$$

This is the same as (12) (13), the partial derivatives at P_0 calculated for the paraboloid.

17.7. **g**) Calculate the intersection of the tangential plane (15) with the x-axis.
RESULT:

$$x = \frac{1}{2x_0}(16 + {x_0}^2 + {y_0}^2) \tag{21}$$

(Solution:)

Intersection with the x-axis means: $z = y = 0$, so (15) reads

$$-z_0 = -2x_0(x - x_0) + 2{y_0}^2 \tag{22}$$

This yields

$$2x_0(x - x_0) = z_0 + 2{y_0}^2 \tag{23}$$

$$2x_0 x = z_0 + 2{y_0}^2 + 2{x_0}^2 \overset{(9)(1)}{=} 16 - {x_0}^2 - {y_0}^2 + 2{y_0}^2 + 2{x_0}^2 \tag{24}$$

$$2x_0 x = 16 + {x_0}^2 + {y_0}^2 \tag{25}$$

$$x = \frac{1}{2x_0}(16 + {x_0}^2 + {y_0}^2) \tag{26}$$

17.7. **h**) Calculate the equation of the straight line which is the intersection of the tangential plane with the x-y-plane.
RESULT:

$$2x_0\, x + 2y_0\, y - {x_0}^2 - {y_0}^2 - 16 = 0 \tag{27}$$

(Solution:)

Intersection with the x-y-plane means $z = 0$. Thus (15) reads

$$-z_0 = -2x_0(x - x_0) - 2y_0(y - y_0) \tag{28}$$

By (9) and (1)

$$-16 + ({x_0}^2 + {y_0}^2) = -2x_0 x + 2{x_0}^2 - 2y_0 y + 2{y_0}^2 \tag{29}$$

$$2x_0\, x + 2y_0\, y - {x_0}^2 - {y_0}^2 - 16 = 0 \tag{30}$$

17.7. **i**) Calculate the differential (14) and the equation of the tangential plane (15) for the extremal point of the paraboloid.

(Solution:)

The extremal point of the paraboloid is given by (2) which is here

$$x_0 = y_0 = 0, \qquad z_0 = 16 \tag{2'}$$

Thus (14) reads

$$\boxed{dz = 0} \quad \text{(for the extremum the differential vanishes)} \tag{31}$$

and (15) reads

$$z = 16 \tag{32}$$

which is the equation for the horizontal tangential plane at the top of the paraboloid.

18 ☻ Multiple Integrals

18.T 1: Double integral as an integral of an integral

A double integral is an integral whose integrand is itself an integral, e.g.

$$I = \int_a^b \underbrace{\int_{c(x)}^{d(x)} f(x,y)\,dy}_{\mathfrak{I}(x)}\;dx \tag{1}$$

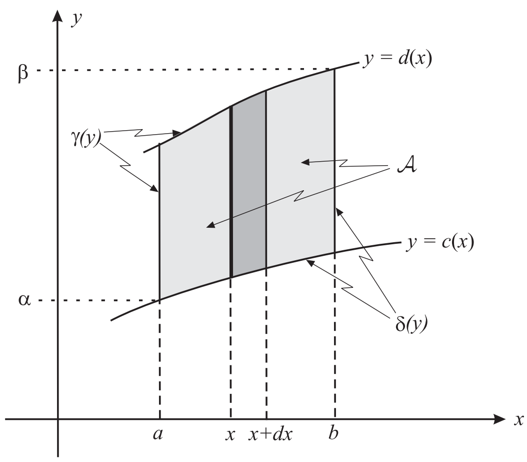


Fig18.1. **1**: Shaded area $\mathcal{A}$ is the integration range of the double integral (1). The function $z = f(x,y)$ can be viewed as the height of mountains over the x-y-plane (with the z-axis upwards). The inner integral (for a fixed x) corresponds to an integral over the solid vertical line. The outer integral is the integral over all darkly shaded subranges. I represents the volume under the mountains.

Rem: Note that an integral and its corresponding differential (e.g. dy) replace an open and closed bracket, i.e. the inner (dy) integral must be performed first, giving a result, say $\mathfrak{I}(x)$. Finally we have to perform the outer (i.e. dx) integral with the integrand $\mathfrak{I}(x)$.

The order of integration can also be interchanged (the dx integral as the innermost) i.e. (1) can also be written as

$$I = \int\limits_{\alpha}^{\beta} \int\limits_{\gamma(y)}^{\delta(y)} f(x,y)\,dx\,dy \tag{1'}$$

with suitable α, β, $\gamma(y)$, $\delta(y)$ to represent the same integration range $\mathcal{A}$. Since the integration range $\mathcal{A}$ gives all the essential information, we can also write

$$I = \iint\limits_{\mathcal{A}} f(x,y)\,dx\,dy \tag{1''}$$

$$(\text{note: } \; dx\,dy = dy\,dx) \tag{2}$$

and leave it to the reader which axes he/she wants to introduce in the x-y-plane and what the boundaries are of the successive simple integrals.
Very often only one integral sign is written (meaning a multiple integral) and the range $\mathcal{A}$ is omitted if it is clear which one has to be taken:

$$I = \int\limits_{\mathcal{A}} f(x,y)\,dx\,dy \tag{1'''}$$

$$I = \int f(x,y)\,dx\,dy \tag{1''''}$$

We can also write

$$I = \int\limits_{\mathcal{A}} f(x,y) d\mathcal{A} \tag{1'''''}$$

with

$$d\mathcal{A} = d^2\mathcal{A} = dx\,dy \tag{3}$$

where $d\mathcal{A}$ is an area element (an area differential). It is a second order differential, as made explicit in the notation $d^2\mathcal{A}$.
A differential of second order (n^{th} order) can be calculated to lowest order i.e. terms of third order or higher ($n + 1^{th}$ order or higher) can be neglected.
Sometimes one says that a second order (n^{th} order) differential is of second order (n^{th} order) small.

18.Ex 2: Area of a triangle calculated as a double integral

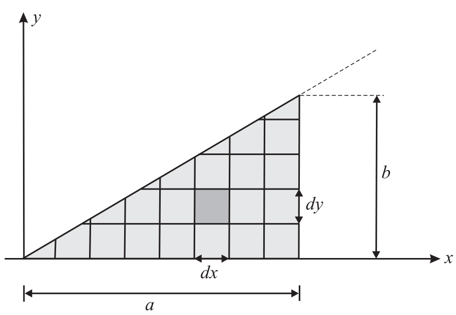


Fig18.2. 1: The shaded area of the triangle is divided into identical small rectangles $dx \cdot dy$. The area is their sum (or integral).

The area A of the shaded triangle can also be expressed as a double integral:

$$A = \int_{\text{triangle}} dA = \int 1\, dx\, dy = \int dx\, dy \tag{1}$$

In this case dA is a second order differential

$$dA = dx\, dy \tag{2}$$

and the integrand is 1 (omitted in the last expression in (1)).

REM: $dA = dx\, dy$ is a second order differential, i.e. the product of two first order differentials dx and dy, or loosely speaking, dA is of second order (infinitesimally) small. To make this explicit, second order differentials are sometimes written with a superscripted 2:

$$d^2A = dx\, dy \tag{2'}$$

18.2. **a**) Evaluate the double integral (1) as a succession of single integrals with the dy integral as the innermost integral, i.e.

$$A = \int \left[\int 1 dy\right] dx \tag{3}$$

We have used brackets, though superfluous, to make it clear that the dy integral has to be performed first. Give the four boundaries of the two integrals.
HINT: the equation of the dotted line (hypotenuse) is

$$y = \frac{b}{a}x \tag{4}$$

RESULT:

$$A = \int_0^a \int_0^{\frac{b}{a}x} 1\ dy\ dx \tag{5}$$

18.2. **b**) Perform the inner integral and then the outer integral. Check that the well-known formula for the area of a right triangle is obtained.

(Solution:)

$$\text{inner integral:}\quad \int_0^{\frac{b}{a}x} dy \overset{\spadesuit}{=} [y]_0^{\frac{b}{a}x} = \frac{b}{a}x \tag{6}$$

♠ The antiderivative of 1 is x, or y here because our integration variable is y.

$$\text{outer integral:}\quad A = \int_0^a \frac{b}{a}x\, dx = \frac{b}{a}\int_0^a x\, dx = \frac{b}{a}\left[\frac{1}{2}x^2\right]_0^a = \tag{7}$$

$$= \frac{1}{2}\frac{b}{a}(a^2 - 0) = \frac{1}{2}\frac{b}{a}a^2 = \frac{1}{2}ba = \tag{8}$$

$$= \frac{1}{2} \cdot \text{ base } \cdot \text{ perpendicular} \tag{9}$$

18.2. **c**) Redo everything by evaluating (1) with the dx integral as the innermost one. Hint: for the lower boundary of the dx integral solve (4) for x:

$$x = \frac{a}{b}y \tag{4'}$$

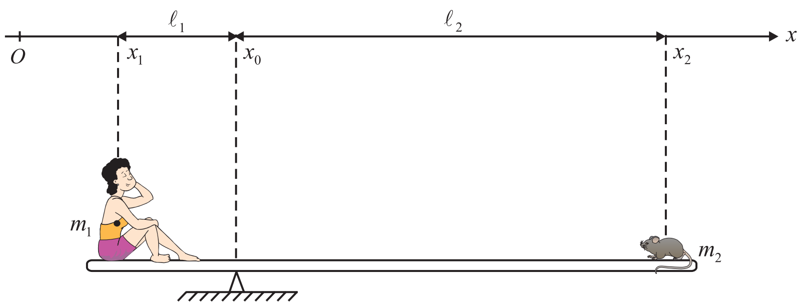
(Solution:)

$$A = \int_0^b \left[\int_{\frac{a}{b}y}^a dx\right] dy = \int_0^b [x]_{\frac{a}{b}y}^a \, dy = \tag{10}$$

$$= \int_0^b (a - \frac{a}{b}y)dy = a\int_0^b dy - \frac{a}{b}\int_0^b y\,dy = \tag{11}$$

$$= ab - \left[\frac{a}{b}\frac{1}{2}y^2\right]_0^b = ab - \frac{1}{2}\frac{a}{b}b^2 = ab - \frac{1}{2}ab = \frac{1}{2}ab \tag{12}$$

18.**Ex 3: Center of mass of a half-moon**



`Fig`18.3. **1**: The see-saw is balanced when it is sustained at the center of mass of a mouse + man. The **beam** [$\overset{\mathbf{G}}{=}$ `Balken`] of the see-saw is treated as massless.

The above **see-saw**[$\stackrel{\mathbf{G}}{=}$ `Wippe, Schaukel`] is balanced when the total **torque**[$\stackrel{\mathbf{G}}{=}$ `Drehmoment`] is zero:

$$m_1 l_1 = m_2 l_2 \qquad \text{(\textbf{lever principle}[}\stackrel{\mathbf{G}}{=}\text{ \texttt{Hebelgesetz}])} \tag{1}$$

It is more systematic to introduce an x-axis (with the origin at an **arbitrary**[$\stackrel{\mathbf{G}}{=}$ `willkürlichen`] point O) whereby m_1 has coordinate x_1, and m_2 has coordinate x_2, and to introduce a point called the **center of mass**[$\stackrel{\mathbf{G}}{=}$ `Schwerpunkt`], ($x_0 = x_{cm} =$ center of mass $= x_s =$ Koordinate des Schwerpunktes), and to express the lever principle by saying:

> the **see-saw's bar**[$\stackrel{\mathbf{G}}{=}$ `Schaukelbalken`] must be **sustained**[$\stackrel{\mathbf{G}}{=}$ `unterstützt`] at the center of mass x_{cm}. (2)

18.3. **a**) What must the definition of the center of mass coordinate x_0 be so that formulation (2) is equivalent to formulation (1)?
HINT: express l_1, l_2 by x_1, x_2, x_0, then formulate (1) and solve for x_0.
RESULT:

$$\boxed{x_0 = x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}} \qquad \text{(\textbf{definition of center of mass})} \tag{3}$$

(Solution:)

$$l_1 = x_0 - x_1 \tag{4}$$

$$l_2 = x_2 - x_0 \tag{5}$$

(1) then reads

$$m_1(x_0 - x_1) = m_2(x_2 - x_0) \tag{1'}$$

then solving for x_0,

$$(m_1 + m_2)x_0 = m_1 x_1 + m_2 x_2 \tag{6}$$

18.3. **b**) Generalize (3) intuitively from two mass-points to N mass-points.
RESULT:

$$\boxed{x_0 = \frac{\sum_{i=1}^{n} m_i x_i}{M}} \qquad \text{(\textbf{definition of center of mass})} \tag{7}$$

$$\boxed{M = \sum_{i=1}^{n} m_i \quad = \quad \text{total mass}} \tag{8}$$

18.3. **c**) Generalize (7) and (8) to a **continuous**[$\stackrel{\mathbf{G}}{=}$ `kontinuierliche`] mass-distribution by replacing the sum by an integral.
RESULT:[16]

$$\boxed{x_0 = \frac{\int x\,dm}{M}} \qquad \textbf{(definition of center of mass)} \tag{7'}$$

$$\boxed{M = \int dm \qquad = \text{total mass}} \tag{8'}$$

18.3. **d**) A symbol for a half-moon is made from **cardboard**[$\stackrel{\mathbf{G}}{=}$ `Karton`] in the form of a half-circle with radius R, see fig.2.

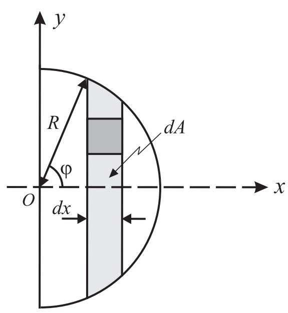


Fig18.3. **2**: The position x_0 of the center of mass of a flat half-moon with radius R is calculated. All area elements (darkly shaded) in the area element (lightly shaded between $x \dots x + dx$) have the same lever arm x, so they can be treated together when evaluating (7′).

Any mass element dm is proportional to its surface element (area element) dA

$$dm = \alpha\, dA \qquad (\alpha = \text{ constant of proportionality}) \tag{9}$$

$$\Big[\; dm = \varrho\, dV \qquad (\varrho = \text{ specific mass} = \textbf{density}[\stackrel{\mathbf{G}}{=} \texttt{Dichte}] \quad \text{e.g. for iron } \varrho = 7.8 \text{ g cm}^{-3}) \tag{10}$$

$$dV = h \cdot dA = \text{ volume element} \qquad (h = \text{thickness of the cardboard}) \tag{11}$$

$$\alpha = \rho h \;\Big] \tag{12}$$

[16]It would be more sytematic to write m instead of M, but M is more common in physics notation.

The constant of proportionality α in (9) drops out of equation (7′) and (8′), so they read

$$x_0 = \frac{\int x\,dA}{A} \quad \text{(definition of the center of mass coordinate)} \tag{7''}$$

$$A = \int dA \qquad \text{(total area)} \tag{8''}$$

Second order area elements d^2A, such as the darkly shaded one in fig.2, having the same x coordinate can be combined (in fact it is integration along the y-coordinate) to a first order area element dA, lightly shaded in fig.2.
Calculate dA expressing it by the angle φ.
HINT 1: Approximate dA as a rectangle.
HINT 2: Express x (the position of the differential dA) by φ; express dx by $d\varphi$ by differentiating.
HINT 3: By area, e.g. dA, we always mean the positive area, so take the absolute value.
RESULT:

$$dA = 2R^2 \sin^2 \varphi \, d\varphi \tag{13}$$

(Solution:)

dA can be calculated in first order as the area of a rectangle. Its width is dx. Its height is $2y$, y being a side-projection with respect to the angle φ, i.e.

$$y = R \sin \varphi$$

The differential dA is situated at $x = R\cos\varphi$ (= projection of R with respect to the angle φ), i.e. at

$$x = R \cos \varphi \tag{13}$$

Differentiating yields

$$dx = -R \sin \varphi \, d\varphi \tag{14}$$

thus

$$dA = 2y\,dx = -2R^2 \sin^2 \varphi \, d\varphi \tag{15}$$

Since we consider area as always being a positive quantitiy, we omit the minus sign; dy is positive in the subsequent integration from $\varphi = 0$ to $\varphi = +\frac{\pi}{2}$.

18.3. **e**) Evaluate integrals (7″) and (8″) using the differential (12).
HINT 1 for (8″): the average of $\sin^2$ is $\frac{1}{2}$ (over a full period, but also over a half one).
HINT 2: express x by φ.
HINT 3: the integration goes from $x = R$ to $x = 0$; what is the corresponding interval for φ?

HINT 4: check result ($8'''$) which must be half the area of a circle.
HINT 5: For difficult integrals consult a formulary
RESULT:

$$A = \frac{1}{2}\pi R^2 \tag{8'''}$$

$$x_0 = \frac{4}{3\pi}R \tag{7'''}$$

(Solution:)

$$1) \quad A = \int dA = 2R^2 \int_0^{+\frac{\pi}{2}} \sin^2\varphi\, d\varphi \tag{16}$$

The lower boundary of the integral is zero because a factor 2 was already introduced in (15).
The integration interval has length $\frac{\pi}{2}$ i.e. is the period of $\sin^2\varphi$. Thus the integral is the average

$$\overline{\sin^2\varphi} = \frac{1}{2} \tag{17}$$

times the interval length $\frac{\pi}{2}$. Thus

$$A = 2R^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{1}{2}\pi R^2 \tag{18}$$

which is half the area of a circle.
2)

$$x_0 \overset{(7'')}{=} A^{-1} \int x\, dA \overset{\spadesuit}{=} A^{-1} 2R^3 \int_0^{\frac{\pi}{2}} \cos\varphi \sin^2\varphi\, d\varphi \tag{19}$$

$$\overset{\clubsuit}{=} \left[2A^{-1}R^3 \cdot \frac{1}{3}\sin^3\varphi\right]_0^{\frac{\pi}{2}} = \frac{2}{3}A^{-1}R^3 = \frac{4}{3\pi}R \tag{20}$$

$$\boxed{\begin{array}{l} \spadesuit\ x = R\cos\varphi \\ \clubsuit\ \text{formulary: } \int \sin^2 x \cos x\, dx = \frac{1}{3}\sin^3 x \end{array}} \tag{21}$$

18.**Ex 4: The cardioid**

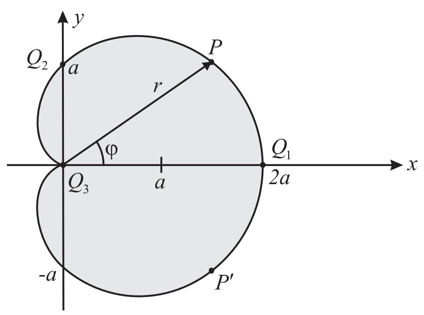


Fig18.4. **1**: This curve given by (1) is called a cardioid because it has a shape similar to a heart.

An **arbitrary**[$\stackrel{G}{=}$ `beliebiger`] point P on the **cardioid**[$\stackrel{G}{=}$ `Herzkurve`] is given in **polar coordinates**[$\stackrel{G}{=}$ `Polarkoordinaten`][17] (r, φ) by

$$r = a(1 + \cos\varphi) \qquad (a = \text{ const.}, \quad -\pi \le \varphi < \pi, \quad a > 0) \tag{1}$$

18.4. **a**) Check that (1) correctly represents the points Q_1, Q_2, and Q_3 of the graph given in fig. 1.
HINT: Q_3 is obtained by $\varphi = \pi$.

(Solution:)

Q_1 has $\varphi = 0$, so by (1):

$$r = a(1 + \underbrace{\cos 0}_{1}) = 2a \tag{1a}$$

Q_2 has $\varphi = \frac{\pi}{2}$, so by (1):

$$r = a\Big(1 + \underbrace{\cos\frac{\pi}{2}}_{0}\Big) = a \tag{2}$$

Q_3 has $\varphi = \pi$, so by (1):

$$r = a(1 + \underbrace{\cos\pi}_{-1}) = 0 \tag{3}$$

18.4. **b**) From (1) show that the cardioid is mirror-symmetric with respect to the x-axis.
HINT 1: show that if $P(r, \varphi)$ fulfills (1) then its mirror-image $P'(r, -\varphi)$ also fulfills (1).
HINT 2: cos is an even function: $\cos(-\varphi) = \cos\varphi$.

[17] Polar coordinates of the point P: r is the distance from P to an origin O on the x-axis (here $O = Q_3$): $r = |\overline{PO}|$, φ is the angle of $\overline{PO}$ with the x-axis.

(Solution:)

$$P: \quad r = a(1+\cos\varphi) \tag{4}$$

$$P': \quad r = a(1+\cos(-\varphi)) \tag{5}$$

Since $\cos(-\varphi) = \cos\varphi$, equation (4) and (5) are **equivalent**:

$$(4) \Longleftrightarrow (5) \tag{6}$$

18.4. **c**)

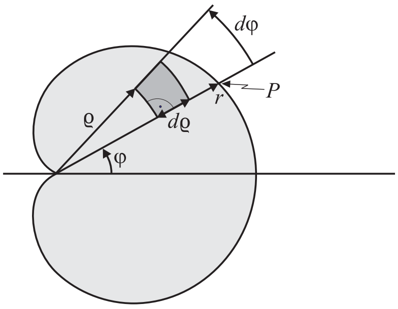


Fig18.4. **2**: The area of the cardioid is calculated here as a double integral, i.e. the 'sum' of all shaded second order differentials at polar coordinate positions ϱ, φ having increments $d\varrho, d\varphi$.

Consider the darkly shaded area element dA at the polar coordinate ϱ, φ, having side length $d\varrho$ and being in the centri-angle $d\varphi$. Calculate dA as a rectangle with $d\varrho$ and the arc length of $d\varphi$ as the side lengths.
Rem 1: It is possible to calculate dA as a rectangle because the r-coordinate line and the $\varphi-$coordinate line intersect intersect at a *right angle* at (ρ, φ), see fig. 2.
Rem 2: A **coordinate line**, is obtained when only that coordinate is varying, while the other coordinates are kept fixed. E.g. the $\varphi-$coordinate line is obtained by fixing $r =$ const. and varying φ.

Result:

$$dA = \varrho\, d\varphi \cdot d\varrho \tag{7}$$

18.4. **d**) Since it is possible to do easily (in this case), calculate (7) exactly (writing ΔA, $\Delta\varphi$, $\Delta\varrho$ instead of dA, $d\varphi$, $d\varrho$ since we will get an exact result).
Result:

$$\Delta A = \varrho\Delta\varphi\Delta\varrho + \frac{1}{2}\Delta\varphi(\Delta\varrho)^2 \tag{8}$$

(Solution:)

(area of a sphere of radius $\varrho + \Delta\varrho$)
− (area of a sphere of radius ϱ) =

$$= \pi(\varrho + \Delta\varrho)^2 - \pi\varrho^2$$

is the area of a circular ring. ΔA is only the fraction $\frac{\Delta\varphi}{2\pi}$ of it. Thus,

$$\Delta A = \underbrace{\frac{1}{2\pi}\Delta\varphi \cdot \pi}_{\frac{1}{2}\Delta\varphi} \Big[\underbrace{(\varrho + \Delta\varrho)^2 - \varrho^2}_{\varrho^2 + 2\varrho\Delta\varrho + (\Delta\varrho)^2 - \varrho^2} \Big] = \varrho\Delta\varphi\Delta\varrho + \frac{1}{2}\Delta\varphi(\Delta\varrho)^2 \qquad (9)$$

18.4. **e)** dA is a second order differential (it would have been better had we denoted it by d^2A instead of dA) so it has to be correct in second order approximation. In view of the exact result (9), is (7) correct as a second order differential?
RESULT: yes.

(Solution:)

(9) contains an additional third order term $\frac{1}{2}\Delta\varphi(\Delta\varrho)^2$ which can be neglected in a second order differential like (7).

18.4. **f)** Using (7) and (1) write the area of a cardioid as a double integral.
HINT 1: perform the $d\varrho$ integral as the innermost integral. (It corresponds to the shaded area element in the centri-angle $d\varphi$ of the figure below.)

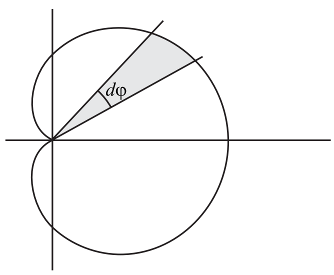


Fig18.4. **3**: The innermost integral of the double integral is the calculation of the shaded first order differential.

HINT 2: the only problem is to identify the four boundaries of the double integral.
RESULT:

$$A = \int dA = \int \varrho \, d\varphi \, d\varrho = \int_0^{2\pi} \left[\int_0^{r=a(1+\cos\varphi)} \varrho \, d\varrho \right] d\varphi \qquad (10)$$

18.4. **g**) Calculate the innermost integral.
RESULT:

$$\int_0^{a(1+\cos\varphi)} \varrho\, d\varrho = \frac{1}{2}a^2(1+\cos\varphi)^2 \tag{11}$$

(Solution:)

$$\int_0^r \varrho\, d\varrho = \left[\frac{1}{2}\varrho^2\right]_0^r = \frac{1}{2}r^2 = \frac{1}{2}a^2(1+\cos\varphi)^2 \tag{12}$$

18.4. **h**) Calculate the area A of the cardioid by evaluating the outermost integral in (10).[18]
HINT 1: expand the integrand leading to a sum of integrands.
HINT 2: geometrically find $\int\limits_0^{2\pi} \cos\varphi\, d\varphi = 0$.
HINT 3: average of $\cos^2 = \frac{1}{2}$; integral = average · integration range.
RESULT:

$$A = \frac{3}{2}\pi a^2 \tag{13}$$

(Solution:)

According to (10)

$$A = \frac{1}{2}a^2 \int_0^{2\pi} (\underbrace{1+\cos\varphi)^2}_{1+2\cos\varphi+\cos^2\varphi} d\varphi = \tag{14}$$

$$= \frac{1}{2}a^2 \left[\int_0^{2\pi} d\varphi + 2\int_0^{2\pi} \cos\varphi\, d\varphi + \int_0^{2\pi} \cos^2\varphi\, d\varphi\right]$$

$$\int_0^{2\pi} 1 d\varphi = [\varphi]_0^{2\pi} = 2\pi \tag{15}$$

---

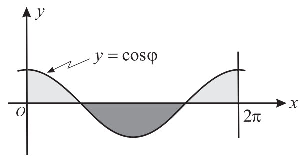
[18] (10) now reads

$$A = \int_0^{2\pi} \frac{1}{2}r\; r\, d\varphi = \int_0^{2\pi} dA \tag{10'}$$

where dA is the shaded (1. order) differential in fig. 3. In first order approximation it can be calculated as a right triangle with base r and perpendicular $r\, d\varphi$, i.e. $dA = \frac{1}{2}rr\, d\varphi$. The experienced mathematician starts immediately from (10′), omitting the innermost integration (11) which only redoes the formula for the area of a triangle.

$$\int_0^{2\pi} \cos\varphi \, d\varphi = 0 \tag{16}$$

Since the shaded area is counted as postive and the darker shaded area is counted as neagative they **cancel each other out**[$\stackrel{\mathbf{G}}{=}$ `sich gegenseitig auslöschen`].



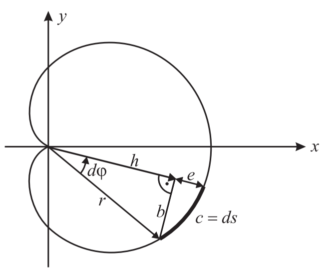
`Fig`18.4. **4**: The darkly and lightly shaded areas under the cosine curve cancel each other out.

$$\int_0^{2\pi} \cos^2\varphi \, d\varphi = \frac{1}{2} \cdot 2\pi = \pi \tag{17}$$

Average value of $\cos^2 = \frac{1}{2}$.
Integration range = (upper boundary) − (lower boundary) = $2\pi - 0 = 2\pi$, thus

$$A = \frac{1}{2}a^2[2\pi + \pi] = \frac{3}{2}a^2\pi \tag{18}$$

18.4. **i**) Calculate the line element ds of the perimeter s of the cardioid.



`Fig`18.4. **5**: A line element ds is calculated as the hypotenuse c of a right triangle with base $b \approx r\,d\varphi$ and perpendicular $e = dr$.

HINT 1: calculate c as the hypotenuse of a right triangle with base b and perpendicular e; use Pythagoras.

e is dr, obtained by differentiating (1).[19]
b is approximately the arc length corresponding to the centri-angle $d\varphi$.
HINT 2: use $\sin^2 + \cos^2 = 1$.
HINT 3: use the half-angle formula

$$\sqrt{2}\cos\frac{\varphi}{2} = \sqrt{1+\cos\varphi}. \tag{19}$$

RESULT:

$$ds = 2a\cos\frac{\varphi}{2}\,d\varphi \tag{20}$$

(Solution:)

According to (1)

$$\frac{dr}{d\varphi} = -a\sin\varphi \tag{21}$$

$$e = -a\sin\varphi\,d\varphi \tag{22}$$

$$d = r\,d\varphi = a(1+\cos\varphi)d\varphi \tag{23}$$

Pythagoras:

$$\begin{aligned} c &= \sqrt{e^2+b^2} = \sqrt{a^2\sin^2\varphi\,d\varphi^2 + a^2(1+\cos\varphi)^2\,d\varphi^2} = \\ &= a\,d\varphi\sqrt{\sin^2\varphi + 1 + 2\cos\varphi + \cos^2\varphi} = \\ &= a\sqrt{2}\sqrt{1+\cos\varphi}\,d\varphi \overset{(19)}{=} 2a\cos\tfrac{\varphi}{2}\,d\varphi \end{aligned} \tag{24}$$

Note: $d\varphi^2$ means $(d\varphi)^2$, not $d(\varphi^2)$.

$$ds = c = 2a\cos\frac{\varphi}{2}\,d\varphi \tag{25}$$

18.4. **j)** Integrate ds in (25) to calculate the perimeter s of the cardioid.
HINT: use the substitution $\alpha = \frac{1}{2}\varphi$.
RESULT:

$$s = 8a \tag{26}$$

[19] In a right triangle with a finite base h and an infinitesimal perpendicular db, the hypothenuse is $r = \sqrt{h^2+(db)^2} = h$, by linear approximation in db since it is a differential.
$h + e = r + e = r + dr \Rightarrow e = dr$
$r = h$ can also been seen geometrically:
The (shortest) distance between two parallels is perpendicular to them and therefore stationary (= extremal) while comparing with slightly rotated (by an angle $d\alpha$) straight connections between the parallels.

(Solution:)

$$s = \int ds = 2a \int\limits_{-\pi}^{\pi} \cos \frac{\varphi}{2} d\varphi \tag{27}$$

With the substitution $\alpha = \frac{1}{2}\varphi$, $d\alpha = \frac{1}{2}d\varphi$ the integral becomes

$$s = 2a \int\limits_{\alpha=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\alpha \cdot 2\ d\alpha = 4a\left[\sin\alpha\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4a(1+1) = 8a \tag{28}$$

19 Differential equations

19.**Q 1: What are differential equations?**
An **algebraic equation** e.g.

$$x + 2 = 5 \tag{1}$$

asks for an (**unknown**) number x, which **satisfies** (**solves**) the equation. In our case we have the **solution** $x = 3$.

It may happen that an equation, e.g.

$$y^2 + 1 = 17 \tag{2}$$

has more than one solution ($y = 4$ and $y = -4$), or none at all, e.g. in case of

$$z + 1 = z \tag{3}$$

Besides the unknown (looked for) number, e.g. x, other **given** (**known**) numbers, e.g. a, b, c, may occur in the equation. E.g. the general **quadratic equation**

$$\boxed{ax^2 + bx + c = 0} \tag{4}$$

has the two solutions:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \tag{5}$$

(5) is the **general solution** of (4), since (4) does not have any other solutions besides (5).

a is called the **coefficient** of the quadratic term, b is called the coefficient of the linear term, and c is called the constant term.

In a **differential equation** we ask for an (unknown) **function** $y = y(x)$, which has to satisfy an equation involving differentials, in most cases, differential quotients, i.e. (higher) derivatives.

Example 1:

$$y' - \frac{4y}{x} = x\sqrt{y} \tag{6}$$

where $y' = dy/dx$ is the differential quotient of the unknown function $y = y(x)$.
SOLUTIONS:

$$y = x^4(\frac{1}{2}\ln x + C)^2 \tag{7}$$

with an **arbitrary** constant C.
TEST:

$$y' = 4x^3(\tfrac{1}{2}\ln x + C)^2 + 2x^4(\tfrac{1}{2}\ln x + C)\cdot\tfrac{1}{2}\cdot\frac{1}{x} \tag{8}$$

$$-\frac{4y}{x} = -4x^3(\tfrac{1}{2}\ln x + C)^2 \tag{9}$$

$$x\sqrt{y} = x^3(\tfrac{1}{2}\ln x + C) \tag{10}$$

q.e.d.

Example 2:

$$\dddot{x} - a^2\dot{x} = 0 \tag{11}$$

Here, the unknown function is denoted by $x(t)$. The coefficient of the first derivative $(\dot{x})$ is $-a^2$, where a is an arbitrary but given (known) constant. The coefficient of the third derivative of the unknown function $(\dddot{x})$ is 1.
SOLUTIONS:

$$x(t) = C_o + C_1e^{at} + C_2e^{-at} \tag{12}$$

with **arbitrary** constants C_o, C_1, C_2.
TEST:

$$\dot{x} = C_1ae^{at} - C_2ae^{-at} \tag{13}$$

$$\dddot{x} = C_1a^3e^{at} - C_2a^3e^{-at} \tag{14}$$

$$-a^2\cdot x = -C_1a^3e^{at} + C_2a^3e^{-at} \tag{15}$$

q.e.d.

Example 3:

$$y' = f(x) \tag{16}$$

where the unknown (looked for) function is denoted by $y = y(x)$, and $f(x)$ is an arbitrary, but given (known) function.
FORMAL SOLUTIONS:

$$y = \int_{x_o}^{x} f(\xi)\, d\xi + C \tag{17}$$

TEST:
The derivative of an integral with respect to its upper boundary (x) is the integrand at the upper boundary: $y' = f(x)$.

REM 1: It seems that the solutions (17) depend upon two arbitrary constants (x_o and C). However, they are not **independent**: **Without loss of generality**, we can choose, e.g. $x_o = 0$, and with arbitrary C (17) is still the *general* solution of (16).

REM 2: We have called (17) a **formal solution** because it is not yet given in **explicit** form, but merely as an integral which has still to be done (which might be possible or not).

REM 3: The differential equation (16) is simply the task of determining the anti-derivative $y(x)$ of the given function $f(x)$.

19.1. **a**) What is the **order** of a differential equation and how is 'order' related to the multitude of solutions (i.e. of the general solution). Explain that for the examples above.

(Solution:)

The order of a differential equation is the 'height' of the highest occuring derivative of the unknown function. E.g., second derivative is order $n = 2$.
The **general solution** of an n-th order differential equation depends upon n arbitrary, **independent** constants, e.g. $C_1, \cdots C_n$.
EXAMPLES:
(6) is of order $n = 1$, the arbitrary constant in the general solution (7) is C.
(11) is of order $n = 3$, arbitrary constants in the general solution (12) are C_o, C_1, C_2.
(16) is of order $n = 1$, the arbitrary constant in solution (17) is C, or alternatively, x_o. The constants C and x_o are not independent, since a change in x_o gives an additional constant only, which can be absorbed in C.

REM: The relation, given here, between order and multitude of solutions is valid only, if so called **Lipshitz-conditions** are satisfied for the known functions occuring in the differential equation. These Lipshitz-conditions are statisfied for most differential equations occuring in physics.

19.1. **b**) Consider the differential equation

$$y' = \lambda y \tag{18}$$

Give a name for that equation, when x is time, and a name for the (known, given) constant λ.

REM 1: In (18), y means an unknown function $y = y(x)$. To look for the general solution of the differential equation (18) means determining all functions $y = y(x)$ for which (18) holds.

(Solution:)

It is called the **growth equation** [$\stackrel{\mathbf{G}}{=}$ `Wachstumsgleichung`], $\lambda =$ growth constant $=$ growth per unit time.

REM 2: When λ is negative (18) it is called a **decay-equation** [$\stackrel{\mathbf{G}}{=}$ `Zerfallsgleichung`] and $\lambda_1 = -\lambda$, which is then positive, is called the **decay-constant** [$\stackrel{\mathbf{G}}{=}$ `Zerfallskonstante`].

19.1. **c**) Give the general solution for (18) and verify it, and give the **particular solution** [$\stackrel{\mathbf{G}}{=}$ `spezielle Lösung`] for the **initial condition** [$\stackrel{\mathbf{G}}{=}$ `Anfangsbedingung`]

$$y(0) = y_0 \tag{19}$$

(Solution:)

GENERAL SOLUTIONS:

$$y = ce^{\lambda x}, \quad c = \text{integration constant} \tag{20}$$

TEST:

$$y' = c\lambda e^{\lambda x} = \lambda y \tag{21}$$

q.e.d.

REM: Since (18) is a first order differential equation ($n = 1$), the general solution depends upon 1 arbitrary constant, c.

Initial condition:

$$y(0) = y_0 = ce^0 = c \tag{22}$$

PARTICULAR SOLUTION:

$$\boxed{y(x) = y_0 e^{\lambda x}} \tag{23}$$

19.1. **d**) What is the differential equation for the growth of a population (e.g. $N(t)$ = number of bacteria) and for radioactive decay ($N(t)$ = number of radioactive atoms)? Give the corresponding solutions.

(Solution:)

Population:

$$\dot{N}(t) = pN(t) \tag{24}$$

$\cdot$ = derivative with respect to t.
REM 1: (24) can also be written as:

$$dN = pN(t)\ dt \tag{24'}$$

i.e. the increase dN in the number N of bacetria in the time interval $(t \cdots t + dt)$ is proportional to the length dt of this interval and to the number $N(t)$ of bacteria already present at time t.
Note that this must be true only in linear approximation in dt, since dN is a differential.
The constant of proportionality (growth constant p) may depend e.g. on the temperature and concentration of the **nutrient solution**[$\stackrel{\mathbf{G}}{=}$ `Nährlösung`], in which the bacteria are living, but also upon how efficiently waste, accumulated by the bacteria, is removed. p will be constant only if these conditions are kept constant in a particular experiment.

SOLUTION:

$$N(t) = N_0 e^{pt} \tag{25}$$

Radioactive decay:

$$\dot{N}(t) = -\lambda N(t), \qquad (\lambda = \text{decay-constant}) \tag{26}$$

REM 2: (26) can also be written as

$$dN = -\lambda N(t)\ dt \tag{26'}$$

Unlike the case of the bacteria, λ does not depend on the conditions in the stone (mineral), in which the radiactive atoms are immersed, e.g. not on the number of already decayed or of other non-radioactive atoms. Neither does λ depend on the age of the radioactive atoms. This is unlike the case of animals or men, where the probability of dying increases with age.

SOLUTION:

$$N(t) = N_0 e^{-\lambda t} \tag{27}$$

REM 3: In both cases N_0 is the arbitrary constant upon which the general solution (25) or (26) depend. At the same time N_0 has the meaning of the initial condition $N_0 = N(0)$, i.e. the initial number of bacteria or radioactive atoms.

19.Q 2: Separation of variables

REM: This is the simplest method for solving differential equations and should always be tried first.

Solve the differential equation

$$y' = \frac{e^x}{y^2} \tag{1}$$

by **separation of variables**, and explain the method in words. Give the **general solution** and then the **particular solution** for the initial condition

$$y_0 = y(x_0). \tag{2}$$

HINT: We have to solve the following problem: Find all functions $y = y(x)$, i.e. the general solution, so that (1) is satisfied, where y' is the derivative of the function $y = y(x)$. Then, select a particular solution of these functions statisfying (2), where x_0 and y_0 are given constants.

(Solution:)

$$\frac{dy}{dx} = \frac{e^x}{y^2} \tag{3}$$

We try to place the x- variables (i.e. x and dx, i.e. the independent variable and the independent increment) on the one side of the equation and the y-variables (i.e. y and dy, i.e. the dependent variable and the dependent increment) on the other side of the equation.

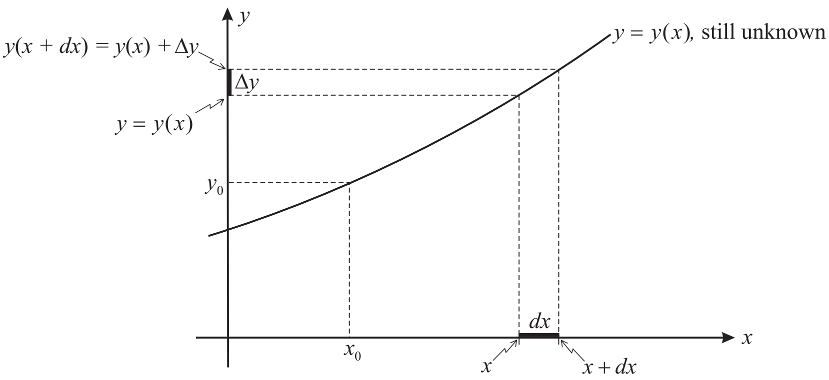


Fig19.2. **1**: An unknown function $y = y(x)$ has initial values $y_0 = y(x_0)$. The final (arbitrary) values (x, y) are obtained by integration of the corresponding differentials dx and dy.

This is possible here:

$$y^2\ dy = e^x\ dx \tag{4}$$

(i.e. separation of variables was successful)
Integrating (4) leads to

$$\int y^2\ dy = \int e^x\ dx \tag{5}$$

[Further explanation: (4) is valid for each interval dx from an initial value x_0 to a final value x. Integration is just summing all these cases of (4).]

Performing the integrals in (5) gives

$$\frac{1}{3}y^3 = c + e^x \tag{6}$$

[Since no boundaries are specified for the integrals, both sides lead to integration constants c_1 and c_2, which we unify $c = c_2 - c_1$]
Solving for y gives

$$y = \sqrt[3]{3(c + e^x)} \tag{7}$$

which is the general solution for the differential equation (1)

The initial condition for (6)

$$\frac{1}{3}y_0^3 = c + e^{x_0} \tag{8}$$

leads to the calculation of c for the particular solution. This c must be inserted into (7) and we obtain the particular solution (9).

Alternatively we could write (5) with definite integrals:

$$\int_{y_0}^{y} \eta^2 \, d\eta = \int_{x_0}^{x} e^{\xi} \, d\xi \tag{5'}$$

leading to

$$\frac{1}{3}(y^3 - y_0^3) = e^x - e^{x_0} \tag{6'}$$

and for the particular solution in explicit form:

$$y = \sqrt[3]{3(e^x - e^{x_0}) + y_0^3} \tag{9}$$

19.Ex 3: Growth equation solved again by separation of variables

19.3. **a**) Write the growth equation

$$y' = \lambda y \qquad (\lambda = \text{const.}) \tag{1}$$

as a differential equation with variables separated.

Result:

$$\frac{dy}{y} = \lambda \, dx \tag{2}$$

(Solution:)

$$\frac{dy}{dx} = \lambda y \qquad \Rightarrow \quad (2) \tag{3}$$

19.3. **b**) Integrate (2) indefinitely.

Hint: The integral of $1/x$ is $\ln|x|$.

Result:

$$\ln|y| = \lambda x + c \tag{4}$$

(Solution:)

$$\int \frac{dy}{y} = \int \lambda \, dx \tag{5}$$

REM: This is a short form for

$$\int_{y_o}^{y} \frac{d\eta}{\eta} = \int_{x_o}^{x} \lambda \, d\xi \tag{6}$$

Note that the name of the integration variables (η, ξ) are irrelevant, but should be different from symbols already used (x and y for upper boundaries).
In (6) boundaries correspond, i.e.

$$\begin{aligned} y &= y(x) \\ y_o &= y(x_o) \end{aligned} \tag{7}$$

where $y(\ )$ is the searched function. The y on the left hand side of (7) is a variable, not the name of a function. By integrating (5) we could find a special solution satisfying the initial condition (7b). But the present task was to integrate *indefinitely*, i.e. taking arbitrary upper boundaries (variables y and x), but unspecified lower boundaries leading to different values of the integration constant c.
This is the meaning of (5) with boundaries omitted.

Integration of (5) leads to

$$\ln|y| = \lambda x + c \tag{8}$$

We need only one integration constant c.

19.3. **c)** Solve (4) for y

HINT 1: Exponentiate both sides of (4), i.e. take both sides as exponents of e.

HINT 2: For reasons of simplicity ignore the absolute value symbol: $|x| = x$. In the next step try to understand the reasoning in the following solution concerning the absolute sign.

RESULT:

$$y = Ce^{\lambda x} \tag{9}$$

(Solution:)

$$e^{\ln|y|} = e^{\lambda x + c} \tag{10}$$

$$|y| = e^{\lambda x} e^{c} = Ce^{\lambda x} \tag{11}$$

with a new arbitrary integration constant

$$C = e^{c} \tag{12}$$

$e^{\lambda x}$ in (11) is positive definite. When $C = 0$ we obtain (9). When $C \neq 0$ then $|y| \neq 0$ everywhere. Since $y(x)$ is a continuous function, it must either be positive everywhere (leading to (9)) or it is negative everywhere, so with a new integration constant C (the negative of the previous C), we obtain again (9).

19.3. **d**) Give the special solution for the initial condition

$$y = y_o \quad \text{for} \quad x = 0 \tag{13}$$

(Solution:)

$$y_o \overset{(13)}{=} y(0) \overset{(9)}{=} Ce^{\lambda 0} = C \cdot 1 = C \tag{14}$$

so the special solution of the differential equation (1) with initial condition (13) is

$$y = y_o e^{\lambda x} \tag{15}$$

19.**Ex 4: Further examples for separation of variables**

Solve the following differential equation by the method of separation of variables:

19.4. **a**)

$$y' = \frac{y}{x} \tag{1}$$

(Solution:)

$$\frac{dy}{y} = \frac{dx}{x} \tag{2}$$

$$\ln|y| = \ln|x| + c \tag{3}$$

$$e^{\ln|y|} = |y| = e^{\ln|x|+c} = e^{\ln|x|}e^c = C|x| \quad \Longrightarrow \quad |y| = C|x| \tag{4}$$

Distinguishing the cases $x > 0$ and $x < 0$ leads to

$$y = Cx \tag{5}$$

possibly with different C's in both cases. But since $y(x)$ has to be differentiable at $x = 0$, both C's must be the same, leading to (5).

With more details: For $C = 0$ in (4) we immediately conclude (5).
Otherwise, consider first the subregion $x < 0$ and we have from (4): $|y| = -Cx$ and thus $y = \pm Cx$. Since $y(x)$ has to be a differentiable (and thus continuous) function of x, the sign ($\pm$) cannot

change in the subregion $x < 0$, i.e. that sign can be absorbed into the constant C, forming a new constant C used in (5), possibly different from the constant C in (4), differing by a sign.
Similarly for the subregion $x > 0$ we also obtain (5), possibly with a different constant C. But because $y(x)$ must be differentiable at $x = 0$, both constants C must be equal, leading to (5) for $-\infty < x < +\infty$.

19.4. **b)**

$$y' = \frac{x}{y} \tag{6}$$

(Solution:)

$$y\ dy = x\ dx \tag{7}$$

$$\tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + c \tag{8}$$

$$y^2 = x^2 + C \tag{9}$$

$$y = \sqrt{C + x^2} \tag{10}$$

19.4. **c)**

$$y' = \frac{1 \pm y}{1 \pm x} \tag{11}$$

Rem: These are two exercises, one for the upper sign and one for the lower sign.

Hint: While integrating use the substitution

$$u = 1 \pm x \tag{12}$$

(Solution:)

$$\int \frac{dy}{1 \pm y} = \int \frac{dx}{1 \pm x} \tag{13}$$

Both integrals have the same form. The substitution (12) gives

$$du = \pm dx \tag{14}$$

so for the second integral (13)

$$\int^x \frac{dx}{1 \pm x} = \pm \int^u \frac{du}{u} = [\pm \ln |u|]^u = \pm \ln |u| + c = \pm \ln |1 \pm x| + c \tag{15}$$

We have not to consider lower boundaries since that would influence the integration constant c only.
Thus (13) reads

$$\pm \ln |1 \pm y| = \pm \ln |1 \pm x| + c_1 \tag{16}$$

where c_1 contains contributions from the integrations constants c from both similar integrals in (13). Multiplying by ± 1 gives

$$\ln |1 \pm y| = \ln |1 \pm x| + C \tag{17}$$

where $C = \pm c_1$. Exponentiating both sides of (17), i.e. taking both sides of (17) as exponents to the base e gives

$$e^{\ln |1 \pm y|)} = |1 \pm y| = e^C e^{\ln |1 \pm x|} = a|1 \pm x| \tag{18}$$

i.e.

$$|1 \pm y| = a|1 \pm x| \tag{19}$$

because e and ln cancel each other and putting $a = e^C$, where a is an arbitrary positive constant.
The same reasoning as in a) leads to[20]

$$1 \pm y = a(1 \pm x) \tag{29}$$

possibly with a new integration constant a.

[20] E.g. for the case of the upper sign in (19), we have

$$|1 + y| = a|1 + x| \tag{20}$$

and we consider first the subregion $x > -1$:

$$|1 + y| = a(1 + x) \quad \Rightarrow \quad 1 + y = \pm a(1 + x) \tag{21}$$

Both signs in (21) refer to the upper sign in(19). From (21) we deduce

$$y = -1 \pm a(1 + x) \tag{22}$$

Since $a(1 + x) > 0$ and y is a continuous function of x, the sign ($\pm$) cannot change in the whole subregion $x > -1$. So, possibly with a new constant a ($a \mapsto \pm a$), we obtain

$$1 + y = a(1 + x) \tag{23}$$

i.e. (29).
For the other subregion ($x < -1$) we have from (20)

$$|1 + y| = a(-1)(1 + x) \quad \Rightarrow \quad 1 + y = \pm a(-1)(1 + x) \tag{24}$$

Since $a(-1)(1+x) > 0$ and since y is a continuous function of x, the sign ($\pm$) in (24) cannot change in the whole subregion $x < -1$. So possibly with a new constant a ($a \mapsto \pm a$), we have obtained:

$$1 + y = a(1 + x) \tag{25}$$

i.e. again (29). But possibly the two constants a in (23) and (25), refering to the two subregions might differ by a sign.
Differentiuating (23) and (25) leads to

$$y' = a \tag{26}$$

This is valid also at the seperation of the subregions (i.e. at $x = -1$). Since $y(x)$ should be differentiable there, the two a's must be equal. Thus we obtain for both subregions (i.e. for $-\infty < x < +\infty$)

$$1 + y = a(1 + x) \tag{27}$$

Similar reasoning can be performed for the lower sign in (19) leading to

$$1 - y = a(1 - x) \tag{28}$$

(27) and (28) can be combined and written as (29).

Multiplying both sides of (29) by ± 1 gives:

$$y = \pm a(1 \pm x) \mp 1 \tag{30}$$

$$y = \pm a \mp 1 + ax \tag{31}$$

$$y = \pm(a - 1) + ax \tag{32}$$

with an integration constant a.

19.Ex 5: The oscillation equation

One of the most important differential equations in physics is the **oscillation equation** [$\stackrel{\mathbf{G}}{=}$ `Schwingungsgleichung`]

$$\boxed{\ddot{x} = -kx, \qquad (k = \text{ const.}, \quad k > 0)} \tag{1}$$

written here for an unknown function $x = x(t)$.

Rem: Therefore, while solving a differential equation, it is good advice to check first if it is an oscillation equation (possibly in disguised form).

Physical Application:

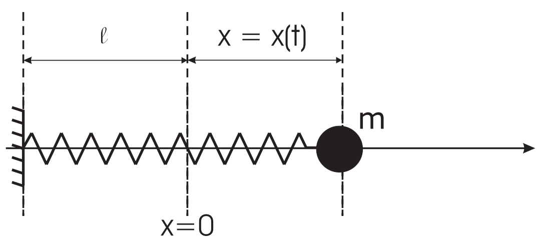


Fig19.5. **1**: Simplest model for an harmonic oscillator: An elastic spring acts on a mass m with a force proportional to the elongation $x = x(t)$, leading to the oscillation equation.

An elastic spring acts on a mass m. No other forces (in the x-direction) should act, i.e. m moves on a frictionless horizontal rail. A spring is characterized by a **resting length** [$\stackrel{\mathbf{G}}{=}$ `Ruhelänge`] (also called **slack length** [$\stackrel{\mathbf{G}}{=}$ `entspannte Länge`]) l, when the spring does not excert any force on m. x is measured from the resting position ($x = 0$).
In a general position $x = x(t)$ the spring acts with the force

$$\boxed{F = -kx} \qquad (\textbf{spring law} [\stackrel{\mathbf{G}}{=} \texttt{Federgesetz}]) \tag{2}$$

When x is positive, i.e. m is to the right of the resting position, the force is negative, i.e. acting to the left in the above figure. k is called the **spring constant** [$\stackrel{\mathbf{G}}{=}$ `Federkonstante`].
Thus the **equation of motion** [$\stackrel{\mathbf{G}}{=}$ `Bewegungsgleichung`] of the mass m is

$$m\ddot{x} = -kx \tag{3}$$

which is an oscillation equation with $k \mapsto k/m$.

19.5. **a**) What is the general solution of the oscillation equation?

REM:

$$\boxed{\ddot{x}(t) = -\omega^2\, x(t)} \qquad \textbf{(oscillation equation)}$$

(Solution:)

$$\boxed{x(t) = A\sin(\omega t) + B\cos(\omega t)} \tag{4}$$

(**General solution of the oscillation equation**)

with integration constants A and B.

REM: A system governed by the oscillation equation is called an **harmonic oscillator**.

19.5. **b**) Check the solution (4) and determine the **angular frequency** [$\stackrel{\text{G}}{=}$ `Kreisfrequenz`] ω

(Solution:)

We insert (4) into (1) and therefore calculate:

$$\begin{aligned} \dot{x} &= A\omega\ \cos(\omega t) - B\omega\ \sin(\omega t) \\ \ddot{x} &= -A\omega^2 \sin(\omega t) - B\omega^2 \cos(\omega t) = -\omega^2 x \end{aligned} \tag{5}$$

Thus (1) is satisfied for

$$k = \omega^2 \tag{6}$$

REM: Thus it is very convenient to write the oscillation equation as

$$\boxed{\ddot{x} = -\omega^2 x} \qquad \textbf{(oscillation equation)} \tag{1'}$$

which at the same time ensures $k > 0$.

19.**Ex 6: Constant velocity**

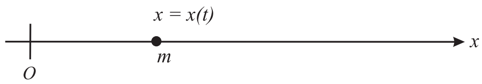


Fig19.6. **1**: A body m is at position $x = x(t)$ at time t.

A point-mass m moves along the x-axis with constant velocity[21] v_0 (e.g. $v_0 = 1$ m sec^{-1}). We will calculate its position

$$x = x(t) \tag{1}$$

at an arbitrary time t.
19.6. **a**) Use the definition of velocity as the derivative with respect to time (in this case we use a dot instead of a prime to denote differentiation):

$$\dot{x}(t) = v_0 \tag{2}$$

Determine $x(t)$ as the antiderivative of the constant v_0.
Result:

$$x(t) = v_0 t + c \qquad (c = \text{ integration constant}) \tag{3}$$

(Solution:)

That (3) is the antiderivative of (2) can be checked by the following test.

$$\dot{x}(t) = (v_0 t + c)^{\cdot} = (v_0 t)^{\cdot} + \dot{c} = v_0 \dot{t} + 0 = v_0 \qquad \text{q.e.d.} \tag{4}$$

19.6. **b**) The information from (2) was not sufficient enough to determine $x(t)$ uniquely since the antiderivative was indefinite due to the integration constant c.
Determine $x(t)$ uniquely by imposing the
initial condition[$\stackrel{\mathbf{G}}{=}$ `Anfangsbedingung`][22] (for a certain time t_0).

$$x(t_0) = x_0 \tag{5}$$

(typically $t_0 = 0,\ x_0 = 0$)
Result:

$$x(t) = x_0 + v_0(t - t_0) \tag{6}$$

[21]This is the case when no force is acting on the body m, e.g. in cosmic space. In the laboratory we let the body move on a frictionless horizontal rail, e.g. on ice.

[22]Typically t_0 is the *initial* time t_0 when the motion started. However, t_0 can be any given time. So, we could alternatively call (5) a final condition, or an intermediate condition.

(Solution:)

In view of (3) our initial condition (5) reads

$$x(t_0) = v_0 t_0 + c = x_0 \quad \Rightarrow \quad c = x_0 - v_0 t_0 \tag{7}$$

so (3) becomes (6).

19.6. **c)**☻ In equivalent but slightly different notation we write (2) as

$$\frac{dx}{dt} = v_0 \quad \Longleftrightarrow \quad dx = v_0\, dt \tag{2'}$$

and think of x(t) as its initial value x_0 plus the sum (integral) of all increments dx:

$$x(t) = x_0 + \int_{t_0}^{t} dx = x_0 + \int_{t_0}^{t} v_0\, d\tau \tag{8}$$

(We have changed the name of the integration variable from t to τ since t was already used as the upper boundary.) During integration τ moves from t_0 to t:

$$t_0 \le \tau \le t \tag{9}$$

For an illustration of (8) and (9) see fig. 2. Similarly we write (2′) as

$$d\xi = v_0\, d\tau \tag{2''}$$

as x is already used for $x = x(t)$ at the final time t.)
For the range (9) we have

$$x_0 \le \xi \le x \tag{10}$$

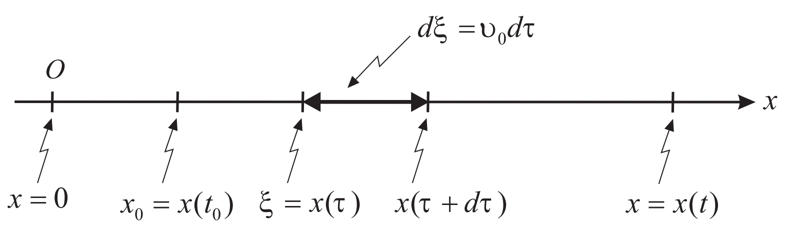


Fig19.6. **2**: The final position $x = x(t)$ is the initial position $x_0 = x(t_0)$ plus the sum (integral) of all increments $d\xi = v_0\, d\tau$, while τ goes from t_0 to t.

Evaluate (8) to obtain (6).

(Solution:)

(8) reads

$$x(t) = x_0 + [v_0 \tau]_{t_0}^{t} = x_0 + v_0(t - t_0) \tag{11}$$

19.Ex 7: Constant acceleration

19.7. **a**) The **acceleration**[$\stackrel{G}{=}$ `Beschleunigung`] is the derivative of the velocity, i.e. the second derivative of the position. For constant acceleration

$$\ddot{x}(t) = g \qquad (g = \text{ constant acceleration}) \tag{1}$$

In the case of a **free fall**[$\stackrel{G}{=}$ `freier Fall`] on the earth

$$g = 9.81\text{m sec}^{-2} = \text{ gravitational acceleration of the earth}$$

and x points vertically downwards towards the center of the earth.

Rem 1: A body is called **free** if no force is acting upon it. The expression '**free fall**' means that no force (e.g. no air resistance) is acting except gravitational attraction by the earth leading to the constant acceleration g.

Integrate (1) under the intial condition

$$\dot{x}(t_0) = v_0 \tag{2}$$

to get the **first integral** $\dot{x}(t) \equiv v(t)$.
Result:

$$\dot{x}(t) \equiv v(t) = v_0 + g(t - t_0) \tag{3}$$

Rem 2: (3) is called a **first integral** because we have only integrated once (resulting in only one integration constant v_0), and we have not yet found the final solution (8), requiring an additional (i.e. second) integration. (8) is thus called a **second integral**, depending on *two* integration constants (v_0 and x_0).

(Solution:)

The antiderivative of (1) is

$$\dot{x}(t) = gt + c \tag{4}$$

Test: $\ddot{x}(t) = (gt + c)^{\cdot} = g$ (5)

The initial condition (2) yields

$$\dot{x}(t_0) = v_0 = g\,t_0 + c_1 \quad \Rightarrow \quad c_1 = v_0 - g\,t_0 \tag{6}$$

so (4) becomes (3).

19.7. **b**) Integrate (3) under the initial condition

$$x(t_0) = x_0 \tag{7}$$

to get the **second integral** of (1), i.e. $x(t)$.
RESULT:

$$\boxed{x(t) = x_0 + v_0(t - t_0) + \frac{1}{2}g(t - t_0)^2} \tag{8}$$

(free fall under the initial condition $x(t_0) = x_0,\ \dot{x}(t_0) = v_0$)

(Solution:)

The antiderivative of (3) is

$$x(t) = v_0 t + \frac{1}{2}g(t - t_0)^2 + c_2 \tag{9}$$

$$\text{Test: } \dot{x}(t) = v_0 + \frac{1}{2}g \cdot 2(t - t_0) \tag{10}$$

where we have used the chain rule with

$$z = t - t_0, \quad \frac{dz}{dt} = 1 \tag{11}$$

The initial condition (7) yields

$$x(t_0) = x_0 = v_0 t_0 + c_2 \quad \Rightarrow \quad c_2 = x_0 - v_0 t_0 \tag{12}$$

so (9) becomes (8).

19.7. **c**) Calculate the maximum height x_m of a free fall and the time $t = t_m$ when x_m is reached.

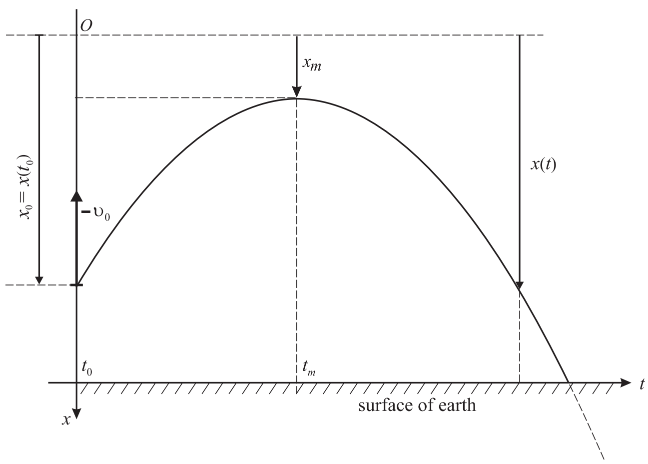


Fig19.7. **1**: 1-dimensional free fall $x = x(t)$. We calculate $x(t)$ from an arbitrary origin O, while x is pointing downwards. In the 2-dimensional free fall the body m has constant velocity in the horizontal direction. So the graph $x = x(t)$ is also the **trajectory** [$\stackrel{G}{=}$ Bahnkurve] of the 2-dimensional free fall, which is a parabola.
In the figure we have assumed that v_0 is negative, so $-v_0$ points upwards.

HINT: Since x points downwards, the maximum height above the earth is a minimum of $x(t)$. The height above the earth cannot be calculated because we did not specify the origin O of x relative to the earth, see fig. 3.
RESULT:

$$t_m = t_0 - \frac{v_0}{g}, \qquad x_m = x_0 - \frac{{v_0}^2}{2g} \tag{13}$$

(Solution:)

1) The extremum of $x(t)$ is where the derivative $\dot{x}(t)$ vanishes, i.e. according to (3)

$$0 = v_0 + g(t_m - t_0) \tag{14}$$

which yields (13)

2) $$x_m = x(t_m) \overset{(8)}{=} x_0 + v_0\left(-\frac{v_0}{g}\right) + \frac{1}{2}g\left(-\frac{v_0}{g}\right)^2 = x_0 - \frac{{v_0}^2}{g} + \frac{1}{2}\frac{{v_0}^2}{g} \tag{15}$$

20 ☻ Binomial theorem

20.Q 1: Binomial theorem

20.1. **a**) What is a monomial, **binomial**, trinomial?

(Solution:)

A binomial is an expression of the form $a + b$ (*'bi'* from Latin 'bis' = twice, *'nom'* from Latin 'nomen' = name, or from Greek '$\nu o \mu o \sigma$' = range) i.e. the sum of two terms.
$a + b + c$ is a trinomial, though that word is rarely used.
A binomial is the sum of two monomials.
a, b, c can also be complicated expressions. So, $e^x + \ln x$ is also a binomial.

20.1. **b**) What is the (first) *binomial formula*?

(Solution:)

$$\boxed{(a + x)^2 = a^2 + 2ax + x^2} \quad \text{first binomial formula} \tag{1}$$

[We have written x instead of b in a very popular formula.]

20.1. **c**) Derive the second and third binomial formula.

(Solution:)

$$\boxed{(a - x)^2 = a^2 - 2ax + x^2} \quad \text{second binomial formula} \tag{2}$$

[Can be derived from (1) by $x \to -x$.]

$$\boxed{(a + x)(a - x) = a^2 - x^2} \quad \text{third binomial formula} \tag{3}$$

[PROOF: $(a + x)(a - x) = a^2 - ax + xa - x^2$]

20.1. **d**) Ex: Calculate $(a + x)^3$ by direct expansion.

(Solution:)

$(a + x)^3 = (a + x)(a^2 + 2ax + x^2) =$
$= a^3 + 2a^2x + ax^2 + a^2x + 2ax^2 + x^3$

$$\boxed{(a + x)^3 = a^3 + 3ax^2 + 3a^2x + x^3} \tag{4}$$

20.1. **e**) Formulate the **binomial theorem**.
What are the **binomial coefficients**?
What is the meaning of 0! ?

(Solution:)

$$\boxed{(a + x)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k} \quad \text{binomial theorem} \tag{5}$$

valid for $n \in \mathbb{N}$.

The **binomial coefficients** are defined as

$$\boxed{\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-(k-1))}{k!} = \frac{n!}{k!(n-k)!}} \tag{6}$$

One adopts the definition

$$\boxed{0! = 1} \tag{7}$$

REM 1: For $n \in \mathbb{N}$, the numerator and the denominator of (6) have the same number of factors.

20.1. **f**) Calculate again $(a+x)^3$ using the binomial theorem and read off the occuring binomial coefficients by comparing with d).

(Solution:)

For $n = 3$ the binomial theorem reads:

$$(a+x)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2x + \binom{3}{2}ax^2 + \binom{3}{3}x^3 \tag{8}$$

Comparing with (4) we read off

$$\binom{3}{0} = 1, \quad \binom{3}{1} = 3, \quad \binom{3}{2} = 3, \quad \binom{3}{3} = 1 \tag{9}$$

REM 2: For $n \in \mathbb{N}$ the binomial series (5) is a finite sum.

20.1. **g**) Give the symmetry formula for binomial coefficients.

(Solution:)

$$\boxed{\binom{\alpha}{\beta} = \binom{\alpha}{\alpha-\beta}} \tag{10}$$

21 Introduction of vectors

21.**Q 1: Introduction of vectors**
What is a (2-dimensional) **vector**?
21.1. **a**) geometrically

(Solution:)

An **arrow**[$\stackrel{\mathbf{G}}{=}$ `Pfeil`], or an oriented **rod**[$\stackrel{\mathbf{G}}{=}$ `Stab`] in a plane. ('Oriented' means: it is known what is the tip (= end-point) and what is the starting-point of the rod.)

REM 1: We say: a vector has a length, direction and orientation. Sometimes, the term 'direction' is meant to imply orientation. Then we can say: a vector has length and orientation.

REM 2: Two arrows with the same length, direction and orientation but different starting points are *different* arrows, but they are the *same* vector. Thus we should say more exactly: a vector is an **equivalence class**[$\stackrel{\mathbf{G}}{=}$ `Äquivalenzklasse`] of arrows, whereby two arrows are called **equivalent** (with respect to the concept of vectors) if they differ only by their starting points (or in other words: if they can be brought to coincidence by a parallel-transport).

21.1. **b**) algebraically

(Solution:)

A **2-tuple** of numbers:

$$\vec{a} = (a_1, a_2) \tag{1}$$

The $a_i; \quad i = 1, 2$ are called the **components** of the vector. a_1 is the *first* component, etc.

REM: As here, it is usual to denote a vector by a **kernel symbol**[$\stackrel{\mathbf{G}}{=}$ `Kernsymbol`] (in this case a) with an arrow over it, to make manifest the symbolised quantity is a vector. Alternatively, a bar under the kernel-symbol, i.e. underlining it,

$$\underline{a} \tag{2}$$

can be used, or simply a bold kernel symbol:

$$\mathbf{a} \tag{3}$$

is used to qualify a as a vector.

21.1. **c**) What is the connection between a) and b) ?

(Solution:)

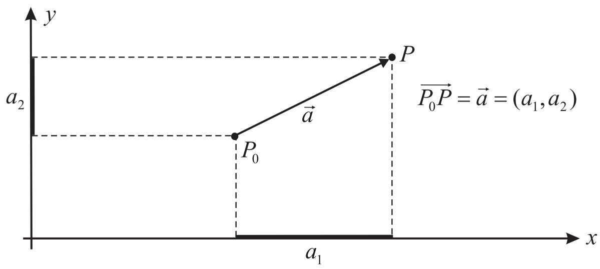


Fig21.1. **1**: Algebraic components a_1 and a_2 of a vector $\vec{a}$

Introducing a **Cartesian system of coordinates** (x, y) the components are given by **(orthogonal) projections** of the arrow to the x- and y-axis.

REM:

$$\overrightarrow{P_0P} \tag{4}$$

is a notation for a vector when the end points of the arrow are given.

21.1. **d**) What is the length of the vector?

(Solution:)

According to the **Pythagorean theorem**, the length of the vector is

$$\boxed{a = |\vec{a}| = \sqrt{a_1^2 + a_2^2}} \tag{5}$$

Algebraically, instead of length, we say **absolute value** of the vector.

REM 1: It is usual to denote the length (absolute value) of a vector with the kernel symbol only, i.e. omitting the arrow symbol, the underlining or the bold type.

REM 2: In mathematics 'vector' is a concept more general than introduced here by the model of arrows having a definite length.
In mathematical terminology our vectors having length are called 'vectors with a (Euclidean) scalar product'.

21.1. **e**) What is the multiplication of a vector by a number (geometrically and algebraically)?

(Solution:)

For

$$\lambda \,\epsilon\, \mathbb{R}, \quad \vec{b} = \lambda\vec{a} = (\lambda a_1, \lambda a_2) \tag{6}$$

i.e.

$$\boxed{b_i = \lambda a_i, \quad i = 1, \ldots n} \quad \text{scalar multiplication} \tag{7}$$

$(n = 2)$.
Thus, algebraically, 'multiplication of a vector by a number λ' means to multiply **componentwise**[$\stackrel{\mathbf{G}}{=}$ `komponentenweise`], i.e. each component is multiplied by λ.

Geometrically, it means *streching* the arrow by the factor λ. For $\lambda < 0$ the resulting vector points into the opposite direction.

REM: 3-vectors are very analogous to 2-vectors. They are arrows not necessarily restricted to lie in a particular plane. Algebraically they are given by n-components, $n = 3$, and we can identify the vector by the **triple** and in general n**-dimensional spaces** by an **n-tuple** of its components:

$$\vec{a} = (a_1, a_2, \ldots, a_n) \tag{1'}$$

Thus **vector calculus** gives us the possibility to deal with n-dimensional spaces for which $(n \geq 4)$ we have no intuitive geometrical insight.

$\boxed{n = \textbf{dimension of vector space}}$

21.1. **f)** What is a **scalar** in contrast to a number and in contrast to a component? Give an example of a scalar and an alternative word for 'scalar'.

(Solution:)

The length of a vector is a scalar, because it is independent of the choice (orientation) of the cartesian coordinate system. A synonymous word for 'scalar' is '**invariant**' (i.e. it does not vary when the cartesian coordinate system is changed).

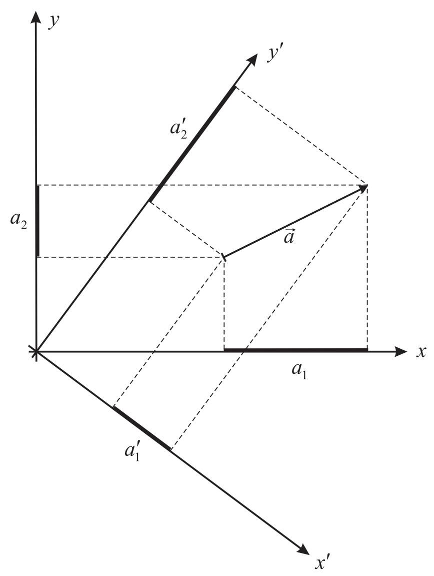


Fig21.1. **2**: The same vector $\vec{a}$ has different components (a_1, a_2) and (a_1', a_2'), respectively, when the frames of reference (i.e. the coordinate axes) are changed from (x, y) to (x', y').

A component of a vector, e.g. a_1 is not a scalar, because it depends on the choice of the (cartesian) coordinate system: it is a_1 for (x, y) and a_1' for (x', y').

Rem: 'Number' is a neutral expression, irrespective of questions of **invariance** or **covariance** (i.e. variability together with the coordinate system). The length but also the components of a vector are numbers, but only the length is a scalar (invariant). Therefore (7) is called 'scalar multiplication' or 'multiplication by a scalar λ' and not only 'multiplication by a number λ'.

21.1. **g**) What is the **null-vector** (geometrically and algebraically)?

(Solution:)

The null-vector (**0-vector**) denoted by $\vec{0}$ or simply by 0, e.g.

$$\vec{a} = 0 \tag{8}$$

is geometrically an arrow of length zero, i.e. a point. Since $-\vec{0} = \vec{0}$, the orientation of that point is irrelevant (undefined).

Algebraically, it is a vector with all its componets zero:

$$\vec{a} = (a_1, \ldots, a_n) = (0, \ldots, 0) = 0 \quad \text{null vector} \tag{9}$$

or

$$\boxed{a_i = 0} \tag{10}$$

where, as usual, we have omitted the range $i = 1, \ldots, n$.
Rem: Only in case of the null-vector, the components of a vector are themselves

invariants.

21.1. **h**) Give the **associative law** for multiplication by a scalar λ.

(Solution:)

For $\lambda, \mu \in \mathbb{R}$

$$\lambda(\mu\vec{a}) = (\lambda\mu)\vec{a} \quad =: \lambda\mu\vec{a} \quad \text{(associative law for scalar multiplication)} \tag{11}$$

REM 1: Because of the associative law it is possible without **ambiguity**[$\stackrel{\text{G}}{=}$ `Zweideutigkeit`] to omit brackets alltogether, as is done on the rightmost side of (11).

REM 2: There is also a **commutative** law

$$\lambda\vec{a} = \vec{a}\lambda \tag{12}$$

which can be and is avoided in mathematical literature, if one adopts the convention that a scalar is always written to the left of the vector.

21.1. **i**) What is a **unit vector**[$\stackrel{\text{G}}{=}$ `Einheits-Vektor`]? Give a notation for it.

(Solution:)

It is a vector of length 1

$$|\vec{a}| = 1 \tag{13}$$

Usual notations for unit-vectors are:

$$\vec{n}, \quad \vec{e}, \quad \hat{a} \tag{14}$$

21.1. **j**) What means 'division of a vector by a scalar'?

(Solution:)

$$\frac{\vec{a}}{\lambda} = \frac{1}{\lambda}\vec{a} \qquad (\lambda \neq 0) \tag{15}$$

21.1. **k**) What is the meaning of $\hat{a}$?

(Solution:)

$\hat{a}$ is a unit vector with the same direction (and orientation, i.e. sign) as $\vec{a}$

$$\boxed{\hat{a} = \frac{\vec{a}}{|\vec{a}|}} \quad (\text{for } \vec{a} \neq 0) \tag{16}$$

REM: The hat implies the arrow symbol.

21.1. **l**) Give the representation of an arbitrary vector as a scalar times a unit vector.

(Solution:)

$$\boxed{\vec{a} = |\vec{a}|\,\hat{a} = a\hat{a}} \qquad (a \neq 0) \tag{17}$$

21.Q 2: Addition of vectors

21.2. **a**) What means addition of vectors (geometrically and algebraically)?

(Solution:)

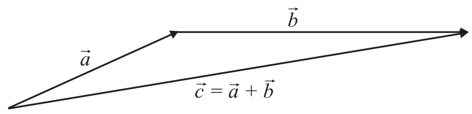


Fig21.2. **1**: $\vec{c} = \vec{a} + \vec{b}$ constructed by the parallelogram rule

$$\vec{c} = \vec{a} + \vec{b} \quad \text{vector addition} \tag{1}$$

is geometrically defined by the so called **parallelogramm construction** (see figure, in fact it is only half of a parallelogramm plus its diagonal): Transport the vector $\vec{b}$ parallely so that its starting point coincides with the tip of $\vec{a}$. The vector $\vec{c}$ (sum of $\vec{a}$ plus $\vec{b}$) is the arrow from the starting point of $\vec{a}$ to the tip of $\vec{b}$.

Algebraically, addition of vectors is performed **component-wise**:

$$\boxed{c_i = a_i + b_i} \quad \text{addition of vectors} \tag{2}$$

21.2. **b**) Give the commutative law of vector addition.

(Solution:)

$$\boxed{\vec{a} + \vec{b} = \vec{b} + \vec{a}} \quad \text{commutative law for vector addition} \tag{3}$$

21.2. **c**) Give the associative law of vector addition.

(Solution:)

$$\boxed{\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} =: \vec{a} + \vec{b} + \vec{c}} \tag{4}$$

associative law for vector addition

REM 1: All these laws follow immediately from the representation of vectors by its components.

REM 2: Because of that law, brackets are not necessary, which is the third expression in (4).

REM 3: In (4) we have used **=:** similar to **:=** meaning that something is defined. The colon is on the side of the expression which is defined, e.g.
elephant := animal with a trunk ...

21.2. **d**) Give the distributive law for vectors.

(Solution:)

$$\boxed{\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}} \quad \text{distributive law for vectors} \tag{5}$$

REM: The usual priority rules, with multiplication having higher priority than addition is used also here. Therefore, on the right hand side of (5) brackets in $(\lambda\vec{a}) + (\lambda\vec{b})$ can be omitted.

21.2. **e**) What means subtraction of vectors.

(Solution:)

$$\vec{c} = \vec{a} - \vec{b} := \vec{a} + (-1)\vec{b} \tag{6}$$

i.e. subtraction is reduced to a scalar multiplication by (-1) followed by a vector addition.

REM: $\vec{c}$ is called the difference vector.

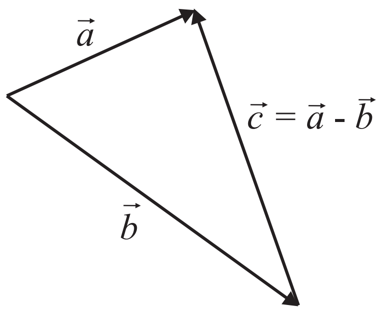


Fig21.2. **2**: $\vec{c} = \vec{a} - \vec{b}$ goes from the tip of the subtrahend ($\vec{b}$) to the tip of the minuend ($\vec{a}$). Test: $\vec{a} = \vec{b} + \vec{c}$

It goes from the tip of $\vec{b}$ to the tip of $\vec{a}$.
(Test: $\vec{a} = \vec{b} + \vec{c}$)

21.T 3: Computer graphics

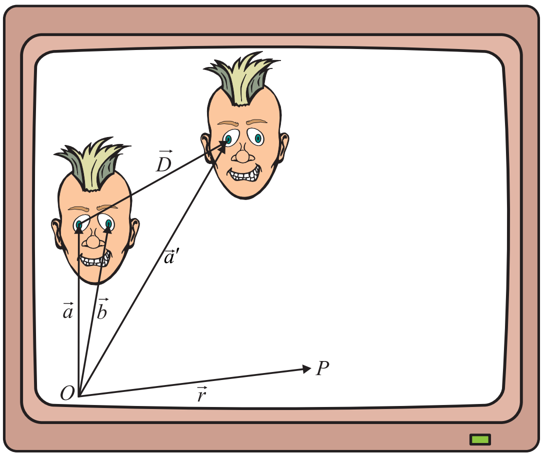


Fig21.3. **1**: $\vec{r}$ is the position vector from the origin O to an arbitrary point P. When the elements of a figure (e.g. eyes of the face) have position vectors $\vec{a}, \vec{b}$, the shifted figure has position vectors obtained by addition of a displacement vector $\vec{D}$

Vector calculus is a means to do **analytic geometry**, namely to describe geometric objects algebraically, e.g. by vectors given as n-tuples (2-tuples for 2-vectors, e.g. $\vec{a} = (a_1, a_2)$).

O, called the **origin**, is an arbitrary point of the plane. In computer graphics mostly the lower-left corner of the screen is used as the origin.

Each point of the human face is given by a so called **position vector**[$\stackrel{\mathbf{G}}{=}$ `Ortsvektor`] (also called: **radius vector**[$\stackrel{\mathbf{G}}{=}$ `Ortsvektor`]). So $\vec{a}$ is the position vector for the left eye.

Position vectors all have their starting points at a common, arbitrarily chosen point, called the origin.

On the other hand, the *usual* interpretation of vectors is

vector = displacement

Displacing the face by the displacement vector $\vec{D}$ we get a new face more to the right of the screen. The new left eye has position vector

$$\vec{a}' = \vec{a} + \vec{D} \tag{1}$$

Position vectors, like $\vec{a}$, can also be conceived as displacement vectors, displacing from the origin to the intended object (e.g. the left eye).

An arbitrary point P of the plane is given by a radius vector usually denoted by $\vec{r}$. Thus, $\vec{r}$ is a **vectorial variable**[$\stackrel{\mathbf{G}}{=}$ `Vektorvariable = vektorwertige Variable`] ranging over all vectors, i.e. over the whole plane (for $n = 2$) or over the whole space (for $n = 3$).

For the plane, the vectorial variable is equivalent to two *numerical* variables:

$$\vec{r} = (x, y) = (x_1, x_2) = (r_1, r_2) = (r_x, r_y) = \vec{x} = (x_i) = x_i \tag{2}$$

giving some usual notations.
The kernel symbol x is as usual as r.
Components are denoted by indices, the so called **vector indices** as in (x_1, x_2), or by using different letters: (x, y).
In (x_i) the index i is a so called an **enumeration index**[$\stackrel{\mathbf{G}}{=}$ `Aufzählungsindex`] $(i = 1, \ldots, n)$ and () is called the **tuple bracket**, i.e. (x_i) is a shorthand for

$$(x_i) = (x_1, \ldots, x_n) \qquad \text{i is an } \textbf{enumeration index} \tag{3}$$

Sometimes the tuple bracket is also omitted: x_i
(depending of the degree of sloppiness of the author).

The displacement $\vec{D}$ gives a **mapping**[$\stackrel{\mathbf{G}}{=}$ `Abbildung`] of the plane unto itself, i.e. each point P (with position vector $\vec{r}$) is mapped (displaced) to a point P' (with position vector $\vec{r}\,'$) given by

$$\vec{r}\,' = \vec{r} + \vec{D} \tag{4}$$

Thus $\vec{D}$ can also be viewed as an **increment vector**

$$\vec{D} = \Delta\vec{r} = (\Delta x, \Delta y) \tag{5}$$

being equivalent to two numerical increments $\Delta x, \Delta y$. Then (4) reads

$$\vec{r}\,' = \vec{r} + \Delta\vec{r} \tag{4'}$$

21.**Ex 4: $\vec{a}+\vec{b}$, $\lambda\vec{a}$ and $\hat{a}$**

Consider 3 vectors

$$\begin{aligned} \vec{a} &= (1,\ 0,\ 2) \\ \vec{b} &= (-1,\ 3,\ 1) \end{aligned} \tag{1}$$

21.4. **a**) Calculate $\vec{a}+\vec{b}$.

RESULT:

$$\vec{a}+\vec{b} = (0,\ 3,\ 3) \tag{2}$$

21.4. **b**) Calculate $a = |\vec{a}|$.

RESULT:

$$a = \sqrt{5} \tag{3}$$

(Solution:)

$$a = |\vec{a}| = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{1+4} = \sqrt{5} \tag{4}$$

21.4. **c**) Calculate $\hat{a}$.

RESULT:

$$\hat{a} = \frac{1}{\sqrt{5}}(1,\ 0,\ 2) \tag{5}$$

(Solution:)

$$\hat{a} = \frac{\vec{a}}{a} = \frac{1}{\sqrt{5}}\vec{a} = \frac{1}{\sqrt{5}}(1,\ 0,\ 2) \tag{6}$$

21.4. **d**) Show: $\vec{a} \nparallel \vec{b}$ i.e. the vectors $\vec{a}$ and $\vec{b}$ are not parallel.

HINT: assume parallelity $\vec{a} \parallel \vec{b}$, i.e.

$$\vec{a} = \lambda\vec{b} \tag{7}$$

Write (7) componentwise and derive a contradiction.

(Solution:)

$$(1,\, 0,\, 2) = \lambda(-1,\, 3,\, 1) \tag{8}$$

means

$$\left|\begin{array}{l} 1 = -\lambda \\ 0 = 3\lambda \\ 2 = \lambda \end{array}\right. \tag{9}$$

These equations are contradictory.

21.**Ex 5: Vector addition by parallelogram construction**
Given two vectors

$$\begin{aligned} \vec{a} &= (4,\, 2) \\ \vec{b} &= (2,\, 3) \end{aligned} \tag{1}$$

21.5. **a**) Calculate $\vec{c} = \vec{a} + \vec{b}$ algebraically, i.e. componentwise.
RESULT:

$$\vec{c} = (6,\, 5) \tag{2}$$

(Solution:)

$$\vec{c} = \vec{a} + \vec{b} = (4,\, 2) + (2,\, 3) = (4+2,\, 2+3) = (6,\, 5) \tag{3}$$

21.5. **b**) Draw $\vec{a}$ and $\vec{b}$ on a sheet of **graph paper**[$\stackrel{\mathbf{G}}{=}$ `kariertes Papier`]; construct $\vec{a}+\vec{b}$ by the parallelogram construction (using a ruler, triangle and compass). Verify

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} = (6,\, 5) \tag{4}$$

(Solution:)

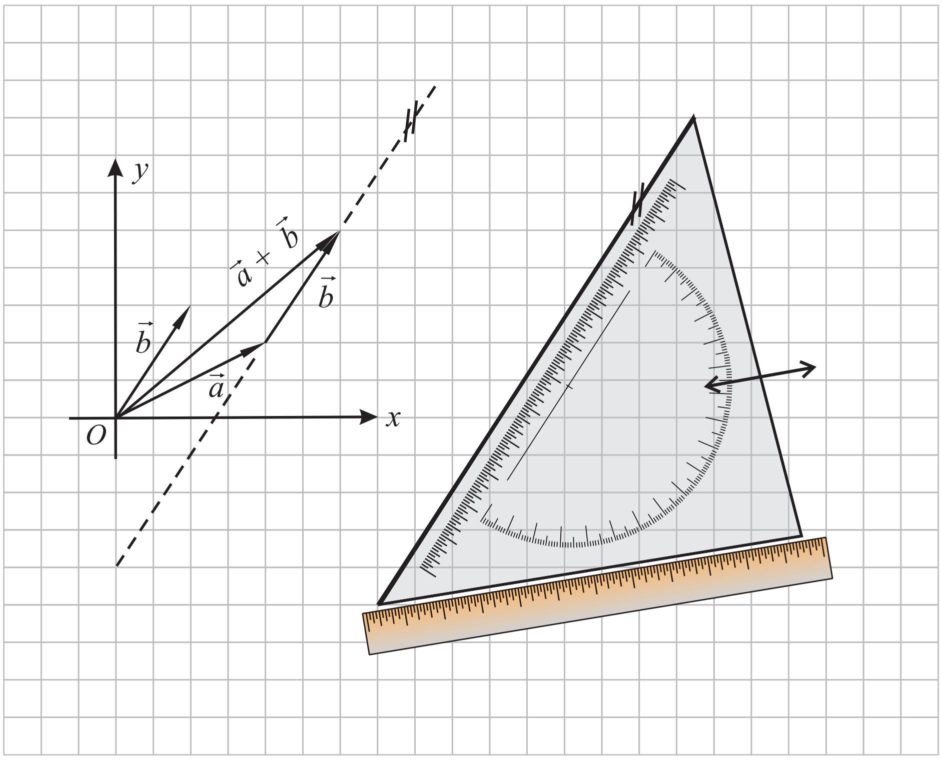


Fig21.5. **1**: Vector addition $\vec{c} = \vec{a} + \vec{b}$ can be done graphically by parallelogram construction. By sliding a solid triangle along a ruler we obtain a series of parallel lines. We adjust the ruler so that these are parallel to $\vec{b}$ (with starting point at the origin O). Thus we construct the parallel dotted line through the tip of $\vec{a}$. With the compass we construct another copy of $\vec{b}$ on the dotted line with the correct length $b = |\vec{b}|$. The tip of the new $\vec{b}$ is the tip of $\vec{a} + \vec{b}$.

21.5. **c**) Calculate the length of $\vec{a}$ algebraically and verify the result graphically using a **compass**[$\stackrel{\mathbf{G}}{=}$ `Zirkel`] to draw the length of $\vec{a}$ along the x-axis.
RESULT:

$$a = \sqrt{20} \tag{5}$$

(Solution:)

$$a = |\vec{a}| = \sqrt{{a_1}^2 + {a_2}^2} = \sqrt{4^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20} \approx 4.47 \tag{6}$$

21.5. **d**) Draw the position vectors

$$\vec{r} = \lambda \vec{a} \tag{7}$$

with $\lambda = 2,\ 1,\ -1,\ 0$, and verify the graphical results algebraically.

(Solution:)

E.g. $\lambda = -1$:

$$\vec{r} = -\vec{a} = (-4,\ -2) \tag{8}$$

21.5. **e**) Make it obvious to yourself that $\vec{r}$ is the equation for a straight line through O in the direction of $\vec{a}$ while λ is considered to be a parameter

$$-\infty < \lambda < +\infty \tag{9}$$

whereby the tips of $\vec{r}$ are points of that straight line.

21.5. **f**) What is the straight line

$$\vec{r} = \vec{a} + \lambda\vec{b}\ ? \tag{10}$$

RESULT: The dotted line in fig. 1.

21.**Ex 6: Equation of a sphere**

Let $\vec{r} = (x, y, z)$ be a point on the surface of a sphere[23] with center $\vec{a} = (1,\ 0,\ 2)$ and radius 1.

21.6. **a**) Derive the x-y-z-equation for that sphere.

HINT: use

$$|\vec{r} - \vec{a}| = 1 \tag{1}$$

RESULT:

$$(x-1)^2 + y^2 + (z-2)^2 = 1 \tag{2}$$

(Solution:)

$$\vec{r} - \vec{a} = (x-1,\ y,\ z-2) \tag{3}$$

$$|\vec{r} - \vec{a}|^2 = \sqrt{(x-1)^2 + y^2 + (z-2)^2}^{\,2} \tag{4}$$

gives (2).

21.6. **b**) At which point(s) does the z-axis intersect that sphere?

RESULT:

$$P(0,\ 0,\ 2) \tag{5}$$

[23] i.e. $\vec{r}$ is a position vector and its tip is a point on the sphere

(Solution:)

In (2) we have to put

$$x = y = 0 \tag{6}$$

which gives

$$1 + (z-2)^2 = 1 \quad \Rightarrow \quad (z-2)^2 = 0 \quad \Rightarrow \quad z = 2 \tag{7}$$

21.**Ex 7: Construction of a regular tetrahedron**

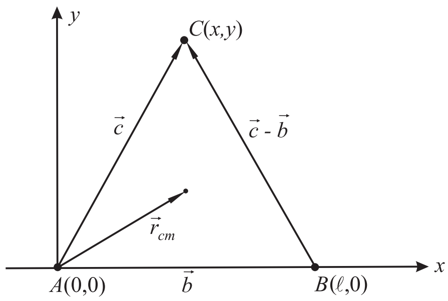


Fig21.7. **1**: Equilateral triangle ABC in the x-y-plane as the base of a regular tetrahedron.

Given two points with their x-y-coordinate (see fig. 1):

$$A(0,\,0), \quad B(\ell,\,0) \tag{1}$$

21.7. **a**) Calculate the vector $\vec{b} = \overrightarrow{AB}$, i.e. the vector whose starting point is A and whose tip is B.
RESULT:

$$\vec{b} = (l,\,0) \tag{2}$$

21.7. **b**) Calculate the vector $\vec{b}' = \overrightarrow{BA}$.
RESULT:

$$\vec{b}' = (-\ell,\,0) \tag{3}$$

21.7. **c**) Express $\vec{b}'$ as a scalar λ multiplied by $\vec{b}$.
RESULT:

$$\vec{b}' = -\vec{b} \tag{4}$$

(Solution:)

$$\vec{b}' = \lambda\vec{b} \tag{5}$$

This is true for $\lambda = -1$:

$$\lambda\vec{b} = \lambda(\ell,\, 0) = (\lambda\ell,\, 0) = (-\ell,\, 0) = \vec{b}' \qquad \text{q.e.d.} \tag{6}$$

21.7. **d**) Calculate the lengths of the vectors $\vec{b}$ and $\vec{b}'$ according to the formula for the length of a vector.
RESULT:

$$|\vec{b}| = |\vec{b}'| = \ell \tag{7}$$

(Solution:)

$$\begin{aligned} |\vec{b}| &= \sqrt{b_x^2 + b_y^2} = \sqrt{\ell^2} = \ell \\ |\vec{b}'| &= \sqrt{(-\ell)^2} = \ell \end{aligned} \tag{8}$$

21.7. **e**) Let $C(x, y)$ be an arbitrary point in the plane. Calculate the vector $\vec{c} = \overrightarrow{AC}$.
RESULT:

$$\vec{c} = (x, y) \tag{9}$$

21.7. **f**) Check that the following equation is true.

$$\vec{c} - \vec{b} = \overrightarrow{BC} \tag{10}$$

(Solution:)

According to the parallelogram rule, we must have

$$\vec{c} = \vec{b} + \vec{c} - \vec{b} \tag{11}$$

which is true.

21.7. **g**) Determine the point C so that ABC becomes an **equilateral triangle**[$\stackrel{\mathbf{G}}{=}$ `gleichseitiges Dreieck`].
HINT: the x-component of $\frac{1}{2}\vec{b}$ and the $x-$component of C, denoted by x, must be the same. The length of $\vec{c}$ must be ℓ. Remove the square root by squaring.
RESULT:

$$C\left(\tfrac{1}{2}\ell,\ \tfrac{\sqrt{3}}{2}\ell\right), \quad \vec{c} = \left(\tfrac{1}{2}\ell,\ \tfrac{\sqrt{3}}{2}\ell\right) \tag{12}$$

(Solution:)

$$\tfrac{1}{2}\vec{b} = \left(\tfrac{1}{2}\ell,\ 0\right) \quad \Rightarrow \quad x = \tfrac{1}{2}\ell \tag{13}$$

$$|\vec{c}| = \sqrt{x^2 + y^2} = \ell \quad \Rightarrow \quad \sqrt{\frac{1}{4}\ell^{\,2} + y^2} = \ell \tag{14}$$

squaring:

$$\frac{1}{4}\ell^2 + y^2 = \ell^2 \quad \Rightarrow \quad y^2 = \frac{3}{4}\ell^2 \quad \Rightarrow \quad y = \frac{\sqrt{3}}{2}\ell \tag{15}$$

21.7. **h**) Check that the length of $\vec{c} - \vec{b}$ is again ℓ.

(Solution:)

$$\vec{c} - \vec{b} = (x, y) - (\ell,\ 0) = (x - \ell,\ y) = \left(\tfrac{1}{2}\ell - \ell,\ \tfrac{\sqrt{3}}{2}\ell\right) = \left(-\tfrac{1}{2}\ell,\ \tfrac{\sqrt{3}}{2}\ell\right) \tag{16}$$

$$|\vec{c} - \vec{b}| = \sqrt{\frac{1}{4}\ell^2 + \frac{3}{4}\ell^2} = \sqrt{\ell^2} = \ell \quad \text{q.e.d.} \tag{17}$$

21.7. **i**) The **center of mass**[$\stackrel{\mathbf{G}}{=}$ `Schwerpunkt`] $\vec{r}_{cm}$ of n mass points m_α at positions $\vec{r}_\alpha,\ \alpha = 1, 2, \cdots n$ is given by

$$\vec{r}_{cm} = \frac{\sum\limits_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha}{\sum\limits_{\alpha=1}^{n} m_\alpha} \tag{18}$$

REM: $\vec{r}_\alpha$, $\vec{r}_{cm}$ are position vectors from an origin, taken here as point A. Specialize that formula for three equal masses.
RESULT:

$$\vec{r}_{cm} = \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3) \tag{19}$$

(Solution:)

$$m_1 = m_2 = m_3 = m \tag{20}$$

$$\vec{r}_{cm} = \frac{m(\vec{r}_1 + \vec{r}_2 + \vec{r}_3)}{m + m + m} \tag{21}$$

21.7. **j**) Assuming that the masses at the corners of the triangle A, B, C are equal, calculate their center of mass.
HINT: $\vec{r}_1$ is the null vector $\vec{r}_2 = \vec{c}$, $\vec{r}_3 = \vec{b}$.
RESULT:

$$\vec{r}_{cm} = \tfrac{1}{2}\ell(1,\ \tfrac{1}{\sqrt{3}}) \tag{22}$$

(Solution:)

$$\vec{r}_1 = (0,0), \quad \vec{r}_2 = \vec{c} \overset{(12)}{=} (\tfrac{1}{2}\ell,\ \tfrac{\sqrt{3}}{2}\ell), \quad \vec{r}_3 = \vec{b} \overset{(2)}{=} (\ell,\ 0) \tag{23}$$

$$\vec{r}_{cm} = \tfrac{1}{3}(\tfrac{3}{2}\ell,\ \tfrac{\sqrt{3}}{2}\ell) = (\tfrac{1}{2}\ell,\ \tfrac{1}{2\sqrt{3}}\ell) \tag{24}$$

21.7. **k**) By introducing a z-axis upward, give the 3-dimensional coordinates of the points A, B, C and the 3 components of the vectors $\vec{b}$, $\vec{c}$, $\vec{r}_{cm}$.

(Solution:)

$$A(0,0,0), \quad B(\ell,0,0), \quad C(\tfrac{1}{2}\ell,\ \tfrac{\sqrt{3}}{2}\ell,\ 0) \tag{25}$$

$$\vec{b} = (\ell,0,0), \quad \vec{c} = (\tfrac{1}{2}\ell,\ \tfrac{\sqrt{3}}{2}\ell,\ 0) \tag{26}$$

$$\vec{r}_{cm} = \tfrac{1}{2}\ell(1,\ \tfrac{1}{\sqrt{3}},\ 0) \tag{27}$$

21.7. **l**) Construct a regular tetrahedron by constructing a fourth point D at height z, so that

$$|\overrightarrow{AD}| = \ell$$

HINT: D has the z-coordinate z and its x-y-coordinates are the same as the x-y-components of $\vec{r}_{cm}$.
RESULT:

$$D(\tfrac{1}{2}\ell,\ \tfrac{\ell}{2\sqrt{3}},\ \sqrt{\tfrac{2}{3}}\ell) \tag{28}$$

(Solution:)

$$D(\tfrac{1}{2}\ell,\ \tfrac{\ell}{2\sqrt{3}},\ z) \tag{29}$$

$$\overrightarrow{AD} = (\tfrac{1}{2}\ell,\ \tfrac{\ell}{2\sqrt{3}},\ z) \tag{30}$$

$$|\overrightarrow{AD}| = \sqrt{\frac{1}{4}\ell^2 + \frac{\ell^2}{4\cdot 3} + z^2} \overset{!}{=} \ell \quad \overset{\text{squaring}}{\Rightarrow} \tag{31}$$

$$\frac{4}{3}\frac{1}{4}\ell^2 + z^2 = \ell^2 \quad \Rightarrow \quad z^2 = \frac{2}{3}\ell^2 \quad \Rightarrow \quad z = \sqrt{\frac{2}{3}}\,\ell \tag{32}$$

We have taken the positive sign of the root since D has to lie *above* the x-y-plane.

21.7. **m**) Check that all **edges**[$\overset{\mathbf{G}}{=}$ `Kanten`] of our tetrahedron have equal length, i.e. that we have obtained a regular tetrahedron.

(Solution:)

We still have to prove

$$\overrightarrow{BD} = |\overrightarrow{CD}| = \ell \tag{33}$$

we have

$$\overrightarrow{BD} \overset{(28)(25)}{=} (-\tfrac{1}{2}\ell,\ \tfrac{\ell}{2\sqrt{3}},\ \sqrt{\tfrac{2}{3}}\,\ell) \tag{34}$$

$$\overrightarrow{CD} \overset{(28)(25)}{=} (0,\ -\tfrac{\ell}{\sqrt{3}},\ \sqrt{\tfrac{2}{3}}\,\ell) \tag{35}$$

where we have used

$$\sqrt{3} = \frac{3}{\sqrt{3}} \tag{36}$$

$$|\overrightarrow{BD}|^2 = \frac{1}{4}\ell^2 + \frac{\ell^2}{4\cdot 3} + \frac{2}{3}\ell^2 = \frac{\ell^2}{4\cdot 3}(3 + 1 + 2\cdot 4) = \ell^2 \tag{37}$$

$$|\overrightarrow{CD}|^2 = \frac{\ell^2}{3} + \frac{2}{3}\ell^2 = \ell^2 \qquad \text{q.e.d.} \tag{38}$$

21.7. **n)** Give the coordinates of the corners A', B', C', D' of a new tetrahedron obtained from the old one by applying a mirror-symmetry with respect to the x-y-plane (i.e. the x-y-plane is the mirror).

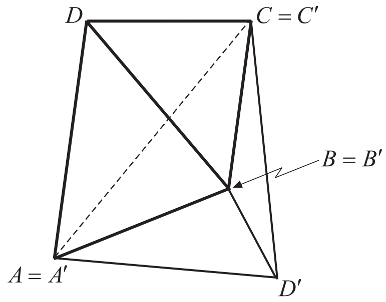


Fig21.7. **2**: Tetrahedron A', B', C', D' is obtained from the tetrahedron A, B, C, D by a mirror-symmetry with respect to one of its **faces**[$\stackrel{G}{=}$ `Flächen`].

(Solution:)

$$\begin{aligned} &A' = A = (0,0,0), \qquad B' = B = (\ell, 0, 0), \qquad C' = C = (\tfrac{1}{2}\ell,\ \tfrac{\sqrt{3}}{2}\ell,\ 0) \\ &D'(\tfrac{1}{2}\ell,\ \tfrac{\ell}{2\sqrt{3}},\ -\sqrt{\tfrac{2}{3}}\,\ell) \end{aligned} \tag{39}$$

since D' is obtained from D by changing the sign of the z-coordinate.

21.7. **o)**

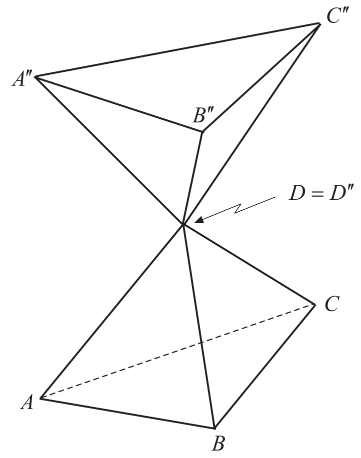


Fig21.7. **3**: By applying a parallel transport to the tetrahedron A', B', C', D' of fig. 2 we bring it to a position A'', B'', C'', D'' so that it is tip to tip above the old (A, B, C, D) one.

Apply a parallel transport (= **displacement**[$\stackrel{\mathbf{G}}{=}$ Verschiebung]) to the tetrahedron A', B', C', D' so that in the new position A'', B'', C'', D'' we have

$$D'' = D \tag{40}$$

HINT: when $\vec{d}$ is the displacement vector, any P'' is obtained from P' by adding the components of $\vec{d}$ (for all $P = A, B, C, D$).
Choose $\vec{d}$ so that (40) holds.

(Solution:)

Condition (40) yields

$$D'' = D \stackrel{(28)}{=} \left(\tfrac{1}{2}\ell,\ \tfrac{\ell}{2\sqrt{3}},\ \sqrt{\tfrac{2}{3}}\ell\right) \stackrel{(\text{HINT})}{=} D' + \vec{d} \stackrel{(39)}{=} \left(\tfrac{1}{2}\ell,\ \tfrac{\ell}{2\sqrt{3}},\ -\sqrt{\tfrac{2}{3}}\ell\right) + (d_1, d_2, d_3) \tag{41}$$

whereby we have indentified a point (e.g. D) with its position vector (with A as the origin). Vectorial equation (41) must hold componentwise, i.e. we have

$$\begin{aligned} \frac{1}{2}\ell &= \frac{1}{2}\ell + d_1 \quad \Rightarrow \quad d_1 = 0 \\ \frac{\ell}{2\sqrt{3}} &= \frac{\ell}{2\sqrt{3}} + d_2 \quad \Rightarrow \quad d_2 = 0 \\ \sqrt{\frac{2}{3}}\,\ell &= -\sqrt{\frac{2}{3}}\,\ell + d_3 \quad \Rightarrow \quad d_3 = 2\sqrt{\frac{2}{3}}\,\ell \end{aligned} \tag{42}$$

i.e. the displacement vector is

$$\vec{d} = (0, 0, 2\sqrt{\tfrac{2}{3}}\,\ell) \tag{43}$$

Applying it to all points we obtain

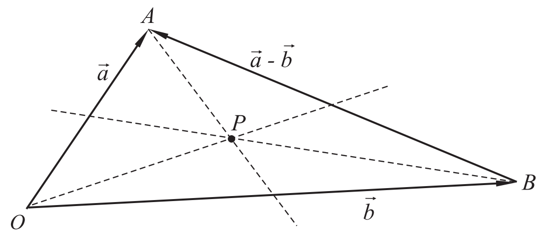
$$
\begin{aligned}
&A''(0, 0, 2\sqrt{\tfrac{2}{3}}\ell) \\
&B''(\ell, 0,\ 2\sqrt{\tfrac{2}{3}}\ell) \\
&C''(\tfrac{1}{2}\ell,\ \tfrac{\sqrt{3}}{2}\ell,\ 2\sqrt{\tfrac{2}{3}}\ell) \\
&D''(\tfrac{1}{2}\ell,\ \tfrac{\ell}{2\sqrt{3}},\ \sqrt{\tfrac{2}{3}}\ell)
\end{aligned}
\tag{44}
$$

21.**Ex 8: Bisectors intersect at a single point**

Prove the following well-known theorem of plane trigonometry:

$$\boxed{\text{In an arbitrary triangle, the } \textbf{bisectors of the sides} [\overset{\mathbf{G}}{=} \texttt{Seitenhalbierenden}] \text{ intersect at one point.}} \tag{1}$$

by the following procedure.



`Fig`21.8. **1**: The dotted lines bisect the sides, i.e. they pass through a corner and the middle of the opposite side. The three bisectors intersect at a single point.

21.8. **a**) The bisector of O has the parameter representation

$$\vec{r} = \frac{1}{2}(\vec{a} + \vec{b})\tau \tag{2}$$

where $\vec{r}$ is an arbitrary point on the bisector and τ $(-\infty < \tau < \infty)$ is the parameter. Rem: all points on the plane are identified with their position vectors with respect to the origin O, e.g.

$$A = \vec{a} \tag{3}$$

etc.

Check parameter representation (2) by showing that for certain values of the parameter τ you get $\vec{r} = O =$ the null vector and also the middle of the side opposite

O.

(Solution:)

For $\tau = 0$ we get $\vec{r} = \vec{0} = O$.
For $\tau = 1$ we get

$$\vec{r} = \frac{1}{2}(\vec{a} + \vec{b}) = \vec{b} + \frac{1}{2}(\vec{a} - \vec{b}) \qquad \text{q.e.d.} \tag{4}$$

21.8. **b**) Find the parameter representation of the remaining bisectors.
HINT:

$$\boxed{\vec{r} = \vec{r}_1 + (\vec{r}_2 - \vec{r}_1)t} \qquad \text{(parameter representation of a straight line passing through } \vec{r}_1 \text{ and } \vec{r}_2\text{)} \tag{5}$$

RESULT:

$$\vec{r} = \vec{b} + \left(\frac{1}{2}\vec{a} - \vec{b}\right)\lambda \tag{6}$$

$$\vec{r} = \vec{a} + \left(\frac{1}{2}\vec{b} - \vec{a}\right)\mu \tag{7}$$

21.8. **c**) Find where lines (6) and (7) intersect.
HINT: equalize the right hand sides of (6) and (7); write as a linear combination of $\vec{a}$ and $\vec{b}$.
RESULT:

$$c_1\vec{a} + c_2\vec{b} = 0 \tag{8}$$

with

$$\begin{aligned} c_1 &= 1 - \mu - \tfrac{1}{2}\lambda \\ c_2 &= -1 + \lambda + \tfrac{1}{2}\mu \end{aligned} \tag{9}$$

(Solution:)

$$\vec{b} + \left(\frac{1}{2}\vec{a} - \vec{b}\right)\lambda = \vec{a} + \left(\frac{1}{2}\vec{b} - \vec{a}\right)\mu \tag{10}$$

$$\vec{b}\left(1 - \lambda - \frac{1}{2}\mu\right) = \vec{a}\left(1 - \mu - \frac{1}{2}\lambda\right) \tag{11}$$

21.8. **d**) For a proper triangle $\vec{a}$ and $\vec{b}$ are **linearly independent** i.e. they span a plane, i.e.

$$\vec{a} \neq 0, \quad \vec{b} \neq 0, \quad \vec{a} \nparallel \vec{b} \tag{12}$$

Therefore, from (8) we can conclude

$$c_1 = c_2 = 0 \tag{13}$$

Prove (13) by showing that any of the following cases are impossible.

$$c_1 \neq 0,\ c_2 \neq 0 \tag{14a}$$

$$c_1 = 0,\ c_2 \neq 0 \tag{14b}$$

$$c_1 \neq 0,\ c_2 = 0 \tag{14c}$$

(Solution:)

(14a) $\Rightarrow$ $\vec{a} = -\frac{c_2}{c_1}\vec{b}$ contradicts $\vec{a} \nparallel \vec{b}$ in (12).
(14b) $\Rightarrow$ $c_2\vec{b} = 0 \quad \Rightarrow \quad \vec{b} = 0$ contradicts $\vec{b} \neq 0$ in (12).
(14c) $\Rightarrow$ $c_1\vec{a} = 0 \quad \Rightarrow \quad \vec{a} = 0$ contradicts $\vec{a} \neq 0$ in (12).

21.8. **e**) Now find the intersection point P and check that it lies on both lines (6) and (7).
HINT: use (13), (9), (6) and (7).
RESULT:

$$\overrightarrow{OP} = \frac{1}{3}(\vec{a} + \vec{b}) \tag{15}$$

(Solution:)

$$\left| \begin{array}{l} c_1 = 1 - \mu - \frac{1}{2}\lambda = 0 \\ c_2 = -1 + \lambda + \frac{1}{2}\mu = 0 \end{array} \right. \tag{16}$$

Adding theses equations gives

$$(\lambda - \mu) - \frac{1}{2}(\lambda - \mu) = \frac{1}{2}(\lambda - \mu) = 0 \quad \Rightarrow \quad \lambda = \mu \tag{17}$$

Then the first equation (16) gives

$$1 - \frac{3}{2}\mu = 0 \quad \Rightarrow \quad \mu = \frac{2}{3} = \lambda \tag{18}$$

Then (6) gives

$$\overrightarrow{OP} = \vec{r} = \vec{b} + \frac{1}{3}\vec{a} - \frac{2}{3}b = \frac{1}{3}(\vec{a} + \vec{b}) \tag{19}$$

(7) gives the same:

$$\vec{r} = \vec{a} + \frac{1}{3}\vec{b} - \frac{2}{3}\vec{a} = \frac{1}{3}(\vec{a} + \vec{b}) \tag{20}$$

i.e. P lies on both lines (6) and (7).

21.8. **f**) Check that P also lies on the bisector (2), which proves our theorem.

(Solution:)

For $\tau = \frac{2}{3}$ we obtain $\vec{r} = \overrightarrow{OP}$.

22 Vectors in physics. Linear combinations

22.Q 1: Forces as vectors

22.1. **a**) What is the zeroth **Newtonian axiom** in physics.

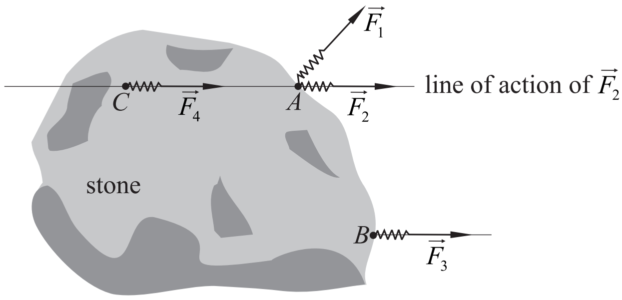


Fig22.1. **1**: Forces as vectors. In physics most vectors are **fixed vectors**, i.e. they must be considered as different when they act (i.e. start) at different points, though they have the same components (e.g. $\vec{F}_2, \vec{F}_3, \vec{F}_4$). For a rigid body (stone) $\vec{F}_2, \vec{F}_4$ are identical because they are on the same line of action. When in a certain application it does not matter where the vector starts, we call them **free vectors**.

(Solution:)

Forces are vectors (= zeroth Newtonian axiom)

This means the following:

- The force can be represented as an arrow. The direction of the force being the direction of the arrow, and the intensity (strength) of the force represented by the length of the arrow.
- When two forces $\vec{F}_1$ and $\vec{F}_2$ are acting on a body (e.g. on a stone or on a **custard**[$\overset{\text{G}}{=}$ `Pudding`]) pulling at the same material point A (e.g. by attaching springs) that is equivalent to a single force

  $$\vec{F} = \vec{F}_1 + \vec{F}_2 \tag{1}$$

  acting on the same point A.

Rem: When representing physical vectors, e.g. forces, geometrically as arrows, as was done in fig.1 , that presupposes the choice of a unit of length corresponding to the physical unit, e.g. 1 cm $\hat{=}$ 1 Newton.

22.1. **b)**☻☻ In what sense is it physically equivalent or not, when the same force $\vec{F}_3 = \vec{F}_2$ is acting on a different point B.
Discuss the particular case B = C, i.e. $\vec{F}_4 = \vec{F}_2$ is on the same **line of action**[$\stackrel{\mathbf{G}}{=}$ `Wirkungslinie`] as $\vec{F}_2$.
Explain the following notions:
fixed vector (French: vecteur fixe),
gliding vecor (French: vecteur glissant),
free vector (French: vecteur libre).

(Solution:)

In general, e.g. in case of a custard, it is not equivalent. $\vec{F}_2$ produces local deformations at A, whereas $\vec{F}_3$ produces local deformations at B. Therefore, forces are fixed vectors.

Even when the stone is approximated as a **rigid body**, it is not equivalent because $\vec{F}_3$ exerts a different **torque**[$\stackrel{\mathbf{G}}{=}$ `Drehmoment`], thus producing a different rotation of the body.

However, for rigid bodies, it does not matter if $\vec{F}_2$ is transported along its line of action (dotted line of figure). Thus for rigid bodies, forces are gliding vectors, i.e. $\vec{F}_4$ is equivalent to $\vec{F}_2$.

When the stone is a point mass, we have trivially $A = B = C$ and the question becomes meaningless.

When we are only interested in the **center of mass**[$\stackrel{\mathbf{G}}{=}$ `Schwerpunkt`] P_{CM} of the body, the famous **law of the center of mass**[$\stackrel{\mathbf{G}}{=}$ `Schwerpunktsatz`] holds:

The center of mass P_{CM} moves as if the vectorial sum of all forces (acting on the body) is acting on P_{CM}.

Thus when we are interested in the center of mass only, forces are free vectors.

22.1. **c)** Discuss free-vector and fixed-vector in the example of computer graphics.

(Solution:)

Position vectors $\vec{r}$ are fixed vectors because their starting points are fixed at the origin O. (Position vectors are meaningful only when the adopted origin O is known.)

The displacement vector $\vec{D}$ is a free vector because $\vec{D}$ is everywhere the same, so it does not matter at which point it acts.

22.**Q 2: Vectors depending on a scalar variable**
Explain what is a vector valued function

$$\vec{r} = \vec{r}(t) \tag{1}$$

depending on a scalar variable (often called a parameter, because x, y, z are the principal variables) t by giving an example and by explaining it algebraically.

(Solution:)

$\vec{r} = \vec{r}(t)$ can e.g. denote the position vector of a moving point-mass at time t.

Algebraically:

$$\vec{r} = \vec{r}(t) = (x(t), y(t), z(t)) \tag{2}$$

is equivalent to three number valued (i.e. ordinary) functions of one variable t.

REM: As an example we can think of $\vec{r}(t)$ as the position of a mosquito at time t.

22.Q 3: Velocity as a vector

Explain why velocity is a vector. Use the position vector $\vec{r}(t)$.

(Solution:)

When the point mass has position $\vec{r}(t)$ at time t and $\vec{r}(t + \Delta t)$ at a later time, then it has made the positional displacement

$$\Delta\vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t) \tag{1}$$

during that time interval. Thus its velocity is

$$\vec{v} = \frac{\Delta\vec{r}}{\Delta t} \tag{2}$$

This is valid only for **uniform velocity** (= constant velocity).

For arbitrary motion, in analogy to the definition of the derivative y' of a function $y(x)$, we have

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) = (\dot{x}, \dot{y}, \dot{z}) \tag{3}$$

REMAINDER: Differentials $d\vec{r}, dt$ are increments in lowest order of approximation, becoming exact in the limit $\Delta t \to 0$.

REM: $d\vec{r}$ is a **differential vector**, being equivalent to a tuple of numerical differentials, e.g.

$$d\vec{r} = (dx, dy, dz) \tag{4}$$

22.Q 4: Vector fields

Explain (algebraically) the notion of a **vector field**.

(Solution:)

A vector field is a vector valued function of a vectorial variable, e.g.

$$\vec{v} = \vec{v}(\vec{r}) = (v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)) \tag{1}$$

i.e. it is equivalent to n ordinary functions of n variables.

REM 1: In general, velocities are not constant in time, then a fourth independent variable t occurs:

$$\vec{v} = \vec{v}(\vec{r}, t) = \vec{v}(x, y, z, t) = \dots \tag{1'}$$

It is sometimes usual to suppress writing down the variable t, which is then called a parameter.

REM 2: The velocity field of a liquid can be made visible by inserting a blob of ink into the liquid (e.g. water). In a short time interval dt, the blob remains pointlike and performs a path to be approximated by the vector $d\vec{r}$. Dividing that vector by dt gives the velocity of the liquid at that point.

22.Q 5: Vector spaces

What is a **vector-space**[$\stackrel{\mathbf{G}}{=}$ `Vektor-Raum`]. Give examples for $n = 3, 2, 1, 4, 0$ and then for general n.

(Solution:)

A vector space is a collection (or to use another word: a **set**[$\stackrel{\mathbf{G}}{=}$ `Menge`] of vectors **closed** under the operation of addition and scalar multiplication.
In other words: If $\vec{a}$ and $\vec{b}$ belong to the vector space V, then $\vec{a} + \vec{b}$ and $\lambda\vec{a}$ (for all $\lambda \,\epsilon\, \mathbb{R}$) belong to V. In formulae

$$\begin{aligned} &\vec{a} \,\epsilon\, V, \quad \vec{b} \,\epsilon\, V \qquad \Rightarrow \qquad \vec{a} + \vec{b} \,\epsilon\, V \\ &\vec{a} \,\epsilon\, V, \quad \lambda \,\epsilon\, \mathbb{R} \qquad \Rightarrow \qquad \lambda\vec{a} \,\epsilon\, V \end{aligned} \tag{1}$$

EXAMPLES:

- $n = 3$: all arrows in 3-space ($V = V_3$)
- $n = 2$: all arrows in a definite plane ($V = V_2$)
  REM: In this case V_2 is a **subspace** of V_3:
  $$V_2 \subset V_3 \tag{2}$$
- $n = 1$: all arrows lying on a definite straight line ($V = V_1$)
- $n = 0$: the null vector ($V = V_0$)
- $n = n$: all n-tuples $(a_1, \dots, a_n)$, i.e. ($V = V_n$)

22.Q 6: ☻ Vector space versus geometrical space

22.6. **a**) What is the difference between a 2-dim. vector space (V_2) and a (geometrical) plane?

(Solution:)

From a pragmatic point of view, there is no difference: Each point P of the plane corresponds to a position vector $\vec{r}$, and vice versa.

However, there is a **subtle**[$\stackrel{G}{=}$ spitzfindig] conceptual difference: A plane is completely smooth, having no distinguished point, i.e. all points are equivalent. A plane becomes a vector space by giving the plane an additional structure, namely by selecting a point (e.g. by dropping a blob of ink onto it) and declaring it as the origin. The origin then corresponds to the null vector.

REM: In a vector space, the null vector can be found immediately, by starting from any vector $\vec{a}$ and by multiplying it by the scalar 0:

$$0\vec{a} = \vec{0} = 0 \tag{1}$$

In a smooth plane there is no such operation.

22.6. **b**) Similarly, what is the difference between a V_1 and $\mathbb{R}$

(Solution:)

There is almost no difference, since the number axis $\mathbb{R}$ has a distinguished origin, the number 0, which corresponds to the null-vector. The number 1 corresponds to a vector with length 1. In V_1, there are two vectors with length 1, differing by a sign. Thus there is still a small difference between a V_1 and $\mathbb{R}$:
$\mathbb{R}$ has an orientation (from 0 to 1), while a V_1 has no (defined) orientation.
REM: According to the terminology used in (pure) mathematics, vectors do not (necessarily) have a (defined) length. Thus, in mathematics, we could say: $\mathbb{R}$ is a V_1 together with a definition for length and orientation.

22.6. **c**) What's the difference between $\mathbb{R}$ and a straight line?
REM: That's the same question as a) for the 1-dimensional case.

(Solution:)

In $\mathbb{R}$ each element is a unique individual, which can be distinguished from any other element. (E.g. the number 1.482 has certain properties which no other number has.) On the other hand all elements on a straigth line g are equivalent. (The points on g are indistinguishable from each other: **translation invariance** of the straight line). By selecting a point on g (denoted by O and called the origin) g becomes (almost) $\mathbb{R}$, because, now each element of g is unique, distinguishable by its distance from O. (We can identify the origin O with the number 0, a point P on g with distance d by the number $d \in \mathbb{R}$.) We have said 'almost' since our g together with O still has no orientation, because there are two point on g having distance d. By selecting one of them as positive, *this* point is identified with the number d. Then g has an orientation and g with O and that orientation is (**isomorphic** to) $\mathbb{R}$.

22.Q 7: Linear combinations, linear dependence

22.7. **a**) What is a **linear combination of two vectors** $\vec{a}$ and $\vec{b}$? Give some trivial examples.

(Solution:)

It is a vector $\vec{c}$ of the form

$$\vec{c} = \lambda\vec{a} + \mu\vec{b} \quad \text{with} \quad \lambda,\ \mu \in \mathbb{R} \tag{1}$$

Trivial examples:

- $\vec{a}$ is such a linear combination $(\lambda = 1, \mu = 0)$
- the null vector is one $(\lambda = \mu = 0)$
- $\vec{a} + \vec{b}$ is one $(\lambda = \mu = 1)$, etc.

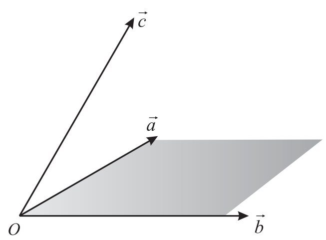
Rem: The generalization to a linear combination of k vectors is

$$\vec{b} = \sum_{i=1}^{k} \lambda_i \vec{a}_i \tag{2}$$

22.7. **b**) (For $n = 3, \vec{a} \neq 0, \vec{b} \neq 0, \vec{a} \nparallel \vec{b}$)
give a geometric description of all linear combinations of $\vec{a}$ and $\vec{b}$. What means 'a plane spanned by $\vec{a}$ and $\vec{b}$' ?

(Solution:)

Considering position vectors for $n = 3$, all linear combinations of $\vec{a}$ and $\vec{b}$ form a plane through the origin O, with $\vec{a}$ and $\vec{b}$ lying in the plane. We say 'the plane is **spanned**[$\stackrel{\mathbf{G}}{=}$ `aufgespannt`] by $\vec{a}$ and $\vec{b}$ '.



`Fig`22.7. **1**: All **linear combinations** of $\vec{a}, \vec{b}$ are the position vectors whose end-points lie on the shaded plane. $\vec{c}$ is not such a linear combination of $\vec{a}, \vec{b}$, but is **linearly independent** from $\vec{a}, \vec{b}$.

22.7. **c**) Give an example of a vector $\vec{c}$ not being a linear combination of $\vec{a}$ and $\vec{b}$.

(Solution:)

Any vector not lying in the plane, e.g. $\vec{c}$ chosen perpendicular to the plane.

22.7. **d**) what means that a vector $\vec{c}$ is **linearly dependent** on two vectors $\vec{a}$ and $\vec{b}$?

(Solution:)

'linear dependent' is synonymous with 'being a linear combination of'. E.g. a vector lying in the plane spanned by $\vec{a}$ and $\vec{b}$ is linearly dependent on $\vec{a}$ and $\vec{b}$.
A vector $\vec{c}$ not lying in that plane, e.g. perpendicular to it, is linearly independent of $\vec{a}$ and $\vec{b}$.

22.Q 10: Components of vectors in some directions

For $n = 2$: What does it mean 'to decompose a vector $\vec{c}$ into a component in the direction of $\vec{a}$ and into a component in the direction of $\vec{b}$.

(Solution:)

It means writing

$$\vec{c} = \lambda\hat{a} + \mu\hat{b} \tag{1}$$

with suitable λ, μ, i.e. as a sum of a vector $\lambda\hat{a}$ (= component of $\vec{c}$ in the direction of $\vec{a}$) plus a vector $\mu\hat{b}$ (= component of $\vec{c}$ in the direction of $\vec{b}$).

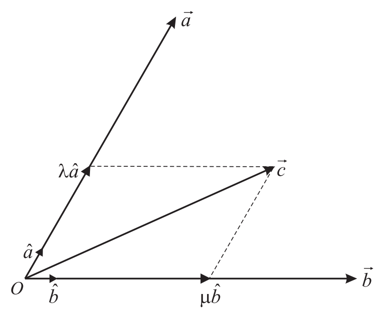


Fig22.10. **1**: For an arbitrary $\vec{a}$, $\hat{a}$ is the unit vector in the direction of $\vec{a}$, $\lambda\hat{a}$ is the component of $\vec{c}$ in the direction of $\vec{a}$.

Rem 1: Verify (1) by using the properties of a parallelogramm and the definition of vector addition.

Rem 2: For the exercise (1) to be meaningful, it is assumed $\vec{a} \neq 0, \vec{b} \neq 0$ (otherwise $\hat{a}$ and/or $\hat{b}$ is not defined) and that

$$\boxed{\vec{a} \nparallel \vec{b}} \quad \text{not parallel} \tag{2}$$

Rem 3: $\hat{a}, \hat{b}$ are called the base vectors of the decomposition of $\vec{c}$ into components. When the base vectors are not orthogonal, $\lambda\hat{a}$, $\lambda\hat{b}$ are called *vectorial components*, in contrast to *normal components* obtained by a normal projection (i.e. projecting at right angles) unto the axes $\vec{a}, \vec{b}$.

22.**Ex 8: Linear combinations**

Consider three vectors

$$\begin{aligned} \vec{a} &= (1,\, 0,\, 2) \\ \vec{b} &= (-1,\, 3,\, 1) \\ \vec{c} &= (0,\, 1,\, 1) \end{aligned} \tag{1}$$

22.8. **a**) Show that $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

(Solution:)

The proposition is

$$\vec{c} = \lambda \vec{a} + \mu \vec{b} \tag{2}$$

i.e.

$$(0,\, 1,\, 1) = \lambda(1,\, 0,\, 2) + \mu(-1,\, 3,\, 1) \tag{3}$$

or componentwise

$$\left| \begin{array}{l} 0 = \lambda - \mu \\ 1 = 3\mu \\ 1 = 2\lambda + \mu \end{array} \right. \tag{4}$$

$\mu = \frac{1}{3}, \quad \lambda = \mu = \frac{1}{3}$

Since the third equation of (4) is then also fulfilled:

$$1 = \frac{2}{3} + \frac{1}{3}$$

22.8. **b**) What is the component of $\vec{c}$ in the direction of $\vec{a}$?

RESULT:

$$\lambda \vec{a} = \frac{1}{3}\vec{a} = \frac{1}{3}(1,\, 0,\, 2)$$

23 Scalar product

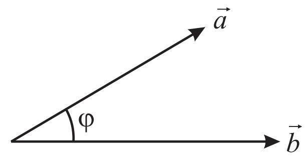
23.Q 1: Scalar product (= dot product)

23.1. **a**) What is the **scalar product** of two vectors (geometrical definition)?

(Solution:)

$$\boxed{\vec{a}\,\vec{b} = ab\cos\varphi} \tag{1}$$

where φ is the angle between both vectors.



`Fig`23.1. **1**: The scalar product $\vec{a}\,\vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ is the length of $\vec{a}$ times the length of $\vec{b}$ times the cosine of the enclosed angle φ.

In words: the scalar product of two vectors $\vec{a}$ and $\vec{b}$ is the length of $\vec{a}$ times the length of $\vec{b}$ times the cosine of the intermediate angle φ.

Rem 1: Since $\cos(2\pi - \varphi) = \cos\varphi$ it is irrelevant which of two possible 'enclosed angles' you take. It is usual to take the smaller one: $0 \le \varphi \le \pi$.

Rem 2: Sometimes, instead of (1), we write

$$\boxed{\vec{a}\ \cdot\ \vec{b} = ab\cos\varphi} \tag{1'}$$

i.e. with a **dot** [$\stackrel{\mathbf{G}}{=}$ `Tupfen`] between $\vec{a}$ and $\vec{a}$, to emphasize it is not ordinary multiplication by numbers. Thus, instead of 'scalar product' we can also say **dot product**. This is advantageous, because 'scalar product' can easily be confused with 'multiplication by a scalar': $\lambda\vec{a}$.

23.1. **b**) Why the word *scalar* product?

(Solution:)

$\vec{a}\vec{b}$ is not a vector but a number, and that number is an **invariant** (i.e. a **scalar**): moving both vectors (translation and rotation by an angle χ)

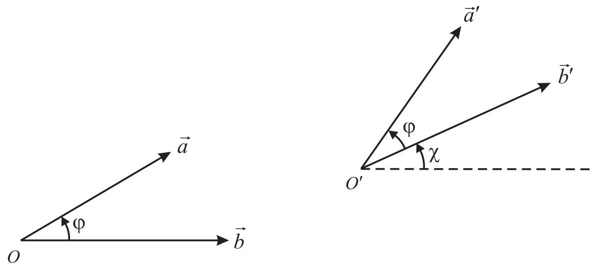


Fig23.1. **2**: When the pair $(\vec{a}, \vec{b})$ is translated (from O to O') and rotated by an angle χ the scalar product remains invariant (i.e. is unchanged: $\vec{a}\vec{b} = \vec{a}'\vec{b}'$).

all quantities on the right hand side of (1) are invariants

$$a^{'} = a, \quad b^{'} = b, \quad \varphi^{'} = \varphi \tag{2}$$

thus $\vec{a}^{\,'}\vec{b}^{\,'} = \vec{a}\,\vec{b}$, i.e. the scalar product is an invariant (=scalar).

23.1. **c**) Express the length of a vector with the help of the scalar product.

(Solution:)

$$a = |\vec{a}| = \sqrt{\vec{a}\vec{a}} = \sqrt{\vec{a}^{\,2}} \tag{3}$$

PROOF: In (1) put $b = a, \ \varphi = 0, \ \cos 0 = 1$

23.1. **d**) Express the normal component of a vector $\vec{a}$ in the direction of a unit-vector with the help of the scalar product.

(Solution:)

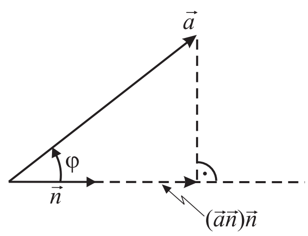


Fig23.1. **3**: When $\vec{a}$ is an arbitrary vector and $\vec{n}$ is an arbitrary unit-vector, the normal projection of $\vec{a}$ unto $\vec{n}$ (in other words: the component of $\vec{a}$ in the direction of $\vec{n}$) is $(\vec{a}\vec{n})\vec{n}$, i.e.can be expressed by a scalar product.

$\vec{a}\vec{n} = a \cdot 1 \cos\varphi =$ normal projection of the rod $\vec{a}$ onto the line $\vec{n}$. Thus $(\vec{a}\vec{n})\vec{n}$ is the component of $\vec{a}$ in the direction of $\vec{n}$.

REM: When $\vec{n}$ is not a unit-vector but an arbitrary vector $\vec{b}$, the component of $\vec{a}$ in the direction $\vec{b}$ is

$$(\vec{a}\,\hat{b})\,\hat{b} \tag{4}$$

23.1. **e**) Express the orthogonality of two vectors with the help of the scalar product.

(Solution:)

$$\boxed{\vec{a} \perp \vec{b} \quad \Longleftrightarrow \quad \vec{a}\,\vec{b} = 0} \tag{5}$$

Orthogonality means vanishing scalar product

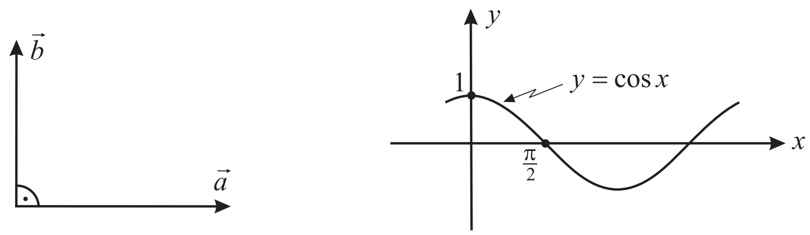


Fig23.1. **4**: When two vectors $\vec{a}$ and $\vec{b}$ are orthogonal (i.e. their intermediate angle is right) they have vanishing scalar product (because the cosine of a right angle is zero).

REM 1: We adopt the convention that **the null-vector is orthogonal to any vector**.

REM 2: $\vee$ denotes the **logical OR**[$\stackrel{\mathbf{G}}{=}$ `logisches ODER`]. The symbol $\vee$ reminds us of the Latin *vel* = or.

PROOF OF (5):
$\vec{a} \perp \vec{b} \iff (\vec{a} = 0 \vee \vec{b} = 0 \vee \varphi = \frac{\pi}{2}) \iff (a = 0 \vee b = 0 \vee \cos\varphi = 0) \iff$
$\iff ab\cos\varphi = 0 \iff \vec{a}\vec{b} = 0$

23.1. **f**) Give the algebraic formula for the scalar product of two vectors, i.e. in terms of their cartesian components a_i and b_i.

(Solution:)

$$\boxed{\vec{a}\vec{b} = \sum_{i=1}^{n} a_i b_i} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (\text{for} \quad n = 3) \tag{6}$$

23.1. **g**) The same but formulated using **Einstein's summation convention**. What does the latter mean?

(Solution:)

$$\vec{a}\,\vec{b} = a_i\; b_i \tag{7}$$

Einstein introduced the convention, that when the same index occurs twice in a term, the formula should be read with an additional summation symbol, i.e. (7) is a shorthand for (6). Einstein used $n = 4$. In classical physics one has $n = 3$.

23.1. **h**) Give the commutative law for the scalar product and an associative law valid for scalar multiplication.

(Solution:)

$$\boxed{\vec{a}\vec{b} = \vec{b}\vec{a}} \quad \text{commutative law} \tag{8}$$

$$\boxed{\vec{a}(\lambda\vec{b}) = (\lambda\vec{a})\vec{b} = \lambda(\vec{a}\vec{b}) = \lambda\vec{a}\vec{b}} \tag{9}$$

associative law for scalar multiplication

i.e. brackets can be omitted

$$\vec{a}\lambda\vec{b} = \lambda\vec{a}\vec{b} = \vec{a}\vec{b}\lambda \tag{10}$$

23.1. **i**) Why is the expression

$$\vec{a}\vec{b}\vec{c} \tag{11}$$

in general meaningless?

(Solution:)

The expression could either mean $\vec{a}(\vec{b}\vec{c})$ or $(\vec{a}\vec{b})\vec{c}$. Since the brackets are scalars, the first is a vector in the direction of $\vec{a}$, the second a vector in the direction of $\vec{c}$, and both cases are in general unequal. In other words: There is no associative law for the scalar product.

23.1. **j**) What does it mean that the scalar product is **bilinear**?

(Solution:)

It means that the distributive law is valid in both [bi = twice] factors

$$\vec{a}(\lambda_1\vec{b_1} + \lambda_2\vec{b_2}) = \lambda_1\vec{a}\vec{b_1} + \lambda_2\vec{a}\vec{b_2} \tag{12}$$

in words: the scalar product of a vector with a linear combination of vectors is the linear combination of the individual scalar products.

The linearity in the first factor

$$(\lambda_1 \vec{a_1} + \lambda_2 \vec{a_2})\vec{b} = \lambda_1 \vec{a_1}\vec{b} + \lambda_2 \vec{a_2}\vec{b} \tag{13}$$

now follows from the commutative law.

23.**Ex 2: Angles in an equilateral triangle**
Prove: in an equilateral triangle any angle is 60°.

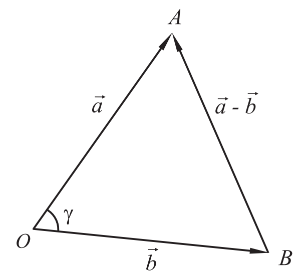


Fig23.2. **1**: In an equilateral triangle $|\vec{a}| = |\vec{b}| = |\vec{a} - \vec{b}|$ we conclude $\gamma = 60°$.

HINT 1: in an **equilateral triangle**[$\stackrel{G}{=}$ `gleichseitiges Dreieck`], by definition, each side has the same length, say ℓ.
HINT 2: use

$$\vec{a}^2 = \vec{a}\vec{a} = \ell^2, \quad \vec{b}^2 = \ell^2, \quad (\vec{a} - \vec{b})^2 = \ell^2 \tag{1}$$

In the last condition use the bilinearity and the symmetry of the scalar product. Finally express the scalar product $\vec{a}\vec{b}$ with the help of $\cos\gamma$. From $\cos\gamma = \frac{1}{2}$ conclude $\gamma = 60°$ since $\gamma \geq 0$ and $\gamma \leq \pi$.

(Solution:)

$$(\vec{a} - \vec{b})^2 = (\vec{a} - \vec{b})(\vec{a} - \vec{b}) \stackrel{\spadesuit}{=} \vec{a}(\vec{a} - \vec{b}) - \vec{b}(\vec{a} - \vec{b}) = \tag{2}$$

$$\stackrel{\clubsuit}{=} \vec{a}\vec{a} - \vec{a}\vec{b} - \vec{b}\vec{a} + \vec{b}\vec{b} \stackrel{\spadesuit\spadesuit}{=} \vec{a}^2 + \vec{b}^2 - 2\vec{a}\vec{b} \stackrel{(1)}{=} 2\ell^2 - 2\vec{a}\vec{b} \stackrel{(1)}{=} \ell^2 \tag{3}$$

$$\Rightarrow \quad \frac{1}{2}\ell^2 = \vec{a}\vec{b} = ab\cos\gamma = \ell^2\cos\gamma \quad \Rightarrow \quad \cos\gamma = \frac{1}{2} \tag{4}$$

♠ linearity of the scalar product in the first factor
♣ linearity of the scalar product in the second factor
♠♠ symmetry of the scalar product

(5)

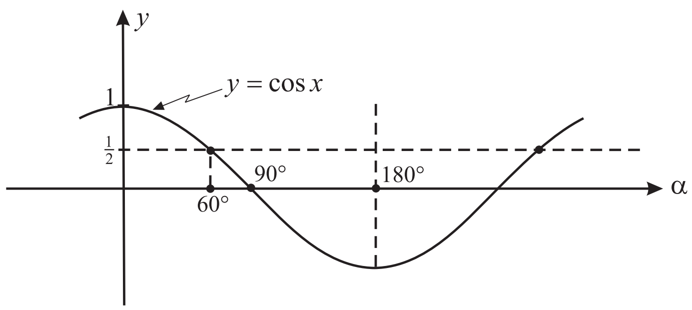


Fig23.2. **2**: Graph of $y = \cos\gamma$. $\cos\gamma = \frac{1}{2}$ only has the solution $\gamma = 60°$ in the interval $[0, 180°]$.

One solution is $\gamma = 60°$, as can be seen from fig. 2. This is the only solution for a triangle.
REM 1: In this proof O is an arbitrary corner, so any angle is $60°$.
REM 2: All angles are equal due to symmetry: $\alpha = \beta = \gamma$. Because the sum of the angles in a triangle is π, we immediately have $\gamma = \frac{\pi}{3} = 60°$. Thus our procedure was much too complicated, but it was useful as an exercise and it is necessary when the side lengths of the triangle are not equal.

23.Ex 3: Shortest distance from a straight line

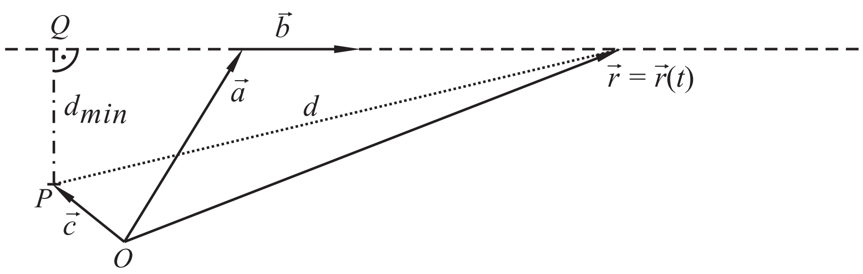


Fig23.3. **1**: $\vec{r}(t) = \vec{a} + t\vec{b}$ is a parameter representation of a straight line (dashed line), where t is the parameter ($-\infty < t < \infty$). The straight line passes through $\vec{a}$ and has direction $\vec{b}$. We determine the point Q which has the shortest distance from a point P given by its position vector $\vec{c}$.

Determine the point Q on the straight line

$$\vec{r} = \vec{a} + t\vec{b} \tag{1}$$

having the shortest distance from the point P, see fig. 1.

23.3. **a**) By minimizing d^2, where d is the distance (dotted line in figure) of P to an arbitrary point $\vec{r}$ of the straigth line (dashed line).
RESULT:

$$Q: \quad \overrightarrow{OQ} = \vec{r} = \vec{a} + \frac{[(\vec{c}-\vec{a})\,\vec{b}\,]}{\vec{b}^2}\,\vec{b} \tag{2}$$

(Solution:)

$$d^2 = (\vec{r}-\vec{c})^2 = \Big[(\vec{a}-\vec{c}) + t\,\vec{b}\Big]^2 = (\vec{a}-\vec{c})^2 + 2t(\vec{a}-\vec{c})\,\vec{b} + t^2\vec{b}^2 \tag{3}$$

$$0 \overset{!}{=} \frac{d}{dt}\,d^2 = 2(\vec{a}-\vec{c})\vec{b} + 2t\vec{b}^2 \tag{4}$$

$$t = \frac{(\vec{c}-\vec{a})\vec{b}}{\vec{b}^2} \tag{5}$$

$$Q: \quad \vec{r} = \vec{a} + t\vec{b} = \vec{a} + \frac{[(\vec{c}-\vec{a})\vec{b}]}{\vec{b}^2}\,\vec{b} \tag{6}$$

ATTENTION: there is no associative law for scalar product, thus (6) cannot be simplified to

$$\vec{a} + \frac{(\vec{c}-\vec{a})(\vec{b}\,\vec{b})}{\vec{b}^2} = \vec{a} + \vec{c} - \vec{a} = \vec{c} \tag{7}$$

23.3. **b**) By the condition that $\overrightarrow{PQ}$ is perpendicular to $\vec{b}$.

(Solution:)

Let Q be given by $\vec{r}$, i.e. $\overrightarrow{OQ} = \vec{r}$, then we have $\overrightarrow{PQ} = \vec{r} - \vec{c}$ and

$$O \overset{!}{=} (\vec{r}-\vec{c})\vec{b} = (\vec{a}-\vec{c}+t\vec{b})\vec{b} \tag{8}$$

$$(\vec{c}-\vec{a})\vec{b} = t\vec{b}^2 \quad \Rightarrow \quad t = \frac{(\vec{c}-\vec{a})\vec{b}}{\vec{b}^2} \tag{9}$$

We have again obtained (5) and further calculation preceeds as before.

23.3. **c**) For the special case $\vec{c} = 0$, $\vec{a} \perp \vec{b}$ calculate d and verify $d = a$.

(Solution:)

(2) gives $\vec{r} = \vec{a}$, thus $d^2 = \vec{r}^{\,2} = \vec{a}^{\,2} \quad \Rightarrow \quad d = a$.

23.Ex 4: Shortest distance from a plane

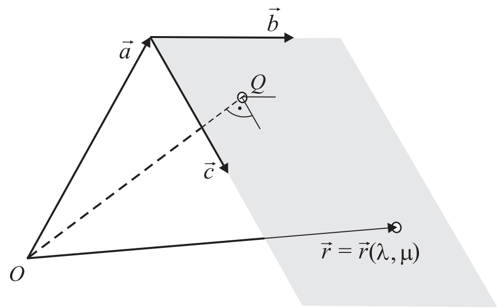


Fig23.4. **1**: $\vec{r}(\lambda,\mu) = \vec{a} + \vec{b}\lambda + \vec{c}\mu$ is a parameter representation of the shaded plane where $\vec{r}$ is a general point on that plane. $\lambda \in (-\infty,+\infty)$, $\mu \in (-\infty,+\infty)$ are the parameters. The plane is spanned by $\vec{b}$ and $\vec{c}$ and it passes through $\vec{a}$. We determine the point Q on the plane having the shortest distance from the origin.

Determine the point Q on the plane

$$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c} \tag{1}$$

having the shortest distance d from the origin, see fig. 1. Calculate d as well.
To save calculation time we only consider the special case

$$\vec{a} \perp \vec{b}, \quad \vec{a} \perp \vec{c}, \quad \vec{b} \perp \vec{c}, \quad \vec{b} \neq 0, \quad \vec{c} \neq 0 \tag{2}$$

23.4. **a**) Give the answer geometrically without any calculation.
RESULT:

$$d = a, \qquad \overrightarrow{OQ} = \vec{a} \tag{3}$$

(Solution:)

$\overrightarrow{OQ}$ must be perpendicular to the plane, i.e. to $\vec{b}$ and $\vec{c}$. This is already the case for $\vec{r} = \vec{a}$.

23.4. **b**) ☻ By minimizing $d^2 = r^2$.
HINT: both partial derivative must be zero.

(Solution:)

$$d^2 = \vec{r}^{\,2} = (\vec{a} + \lambda\vec{b} + \mu\vec{c}\,)^2 = \tag{4}$$

$$= \vec{a}^2 + \lambda^2 \vec{b}^{\,2} + \mu^2 \vec{c}^{\,2} + 2\lambda \underbrace{\vec{a}\,\vec{b}}_{0} + 2\mu \underbrace{\vec{a}\,\vec{c}}_{0} + 2\lambda\mu \underbrace{\vec{b}\,\vec{c}}_{0} \tag{5}$$

We must have

$$\left| \begin{array}{l} O \stackrel{!}{=} \frac{\partial}{\partial\lambda} d^2 = 2\lambda\vec{b}^2 \\ O \stackrel{!}{=} \frac{\partial}{\partial\mu} d^2 = 2\mu\vec{c}^{\,2} \end{array} \right. \tag{6}$$

$$\Rightarrow \quad \lambda = \mu = 0 \quad \Rightarrow \quad \vec{r} = \vec{a}, \qquad d = r = a \tag{7}$$

23.Ex 5: Invariance of the scalar product under rotations

23.5. **a**) Calculate the vector $\overrightarrow{AB}$ in fig 1.

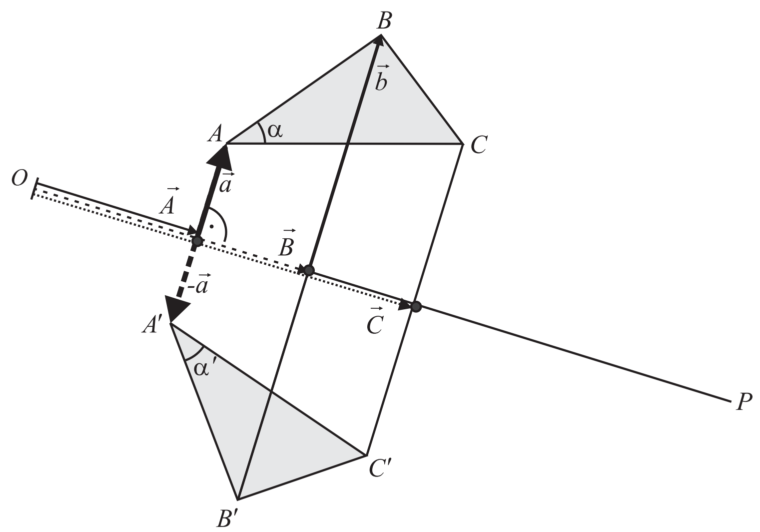


Fig23.5. **1**: The triangle A', B', C' is obtained from the triangle A, B, C by a mirror-symmetry where P is the plane of the mirror. The position vector of A is obtained by $\overrightarrow{OA} = \vec{A} + \vec{a}$, etc, while $\overrightarrow{OA'} = \vec{A} - \vec{a}$, where $\vec{a}\vec{A} = 0$ and $\vec{A}$ is in the mirror plane.

Result:

$$\overrightarrow{AB} = -\vec{A} + \vec{B} - \vec{a} + \vec{b} \tag{1}$$

(Solution:)

$$\overrightarrow{OA} = \vec{A} + \vec{a}, \quad \overrightarrow{OB} = \vec{B} + \vec{b} \tag{2}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{A} + \vec{B} - \vec{a} + \vec{b} \tag{3}$$

23.5. **b**) Similarly we want $\overrightarrow{AC}$. Find it directly from (1) by applying a formal symmetry.

RESULT:

$$\overrightarrow{AC} = -\vec{A} + \vec{C} - \vec{a} + \vec{c} \tag{4}$$

(Solution:)

We obtain (4) from (1) by formal symmetry

$$B \to C, \quad b \to c \tag{5}$$

23.5. **c**) Calculate the scalar product

$$\overrightarrow{AB} \cdot \overrightarrow{AC} \tag{6}$$

HINT: use symmetry and bilinearity of the scalar product. Use orthogonality, i.e. all vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are orthogonal to the mirror plane, i.e. to all $\vec{A}$, $\vec{B}$, $\vec{C}$.

RESULT:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = A^2 - \vec{A}\vec{C} - \vec{A}\vec{B} + \vec{B}\vec{C} + \vec{a}\vec{a} - \vec{a}\vec{c} - \vec{b}\vec{a} + \vec{b}\vec{c} \tag{7}$$

(Solution:)

$$\begin{aligned} \overrightarrow{AB} \cdot \overrightarrow{AC} &= (-\vec{A} + \vec{B} - \vec{a} + \vec{b})(-\vec{A} + \vec{C} - \vec{a} + \vec{c}\,) = \\ &= A^2 - \vec{A}\vec{C} + \underbrace{\vec{A}\vec{a}}_{0} - \underbrace{\vec{A}\vec{c}}_{0} - \vec{A}\vec{B} + \vec{B}\vec{C} - \underbrace{\vec{B}\vec{a}}_{0} + \underbrace{\vec{B}\vec{c}}_{0} + \\ &\quad + \underbrace{\vec{a}\vec{A}}_{0} - \underbrace{\vec{a}\vec{C}}_{0} + \vec{a}\vec{a} - \vec{a}\vec{c} - \underbrace{\vec{b}\vec{A}}_{0} + \underbrace{\vec{b}\vec{C}}_{0} - \vec{b}\vec{a} + \vec{b}\vec{c} \end{aligned} \tag{8}$$

8 terms are zero because of orthogonality.

23.5. **d**) By applying a formel symmetry to (7) calculate $\overrightarrow{A'B'} \cdot \overrightarrow{A'C'}$ and show that scalar products are invariant under mirror-symmetry.

RESULT:

$$\overrightarrow{A'B'} \cdot \overrightarrow{A'C'} = \text{ same as (7)} \tag{9}$$

(Solution:)

The formal symmetry to be applied to (7) is

$$\vec{a} \to -\vec{a}, \qquad \vec{b} \to -\vec{b} \tag{10}$$

while the capital letters remain unchanged. Thus the right-hand side of (7) remains unchanged, i.e. scalar products are invariant under mirror-symmetry.

23.5. **e**) Show that lengths and angles are invariant under mirror-symmetry, e.g.

$$|\overrightarrow{AB}| = |\overrightarrow{A'B'}|, \quad \alpha = \alpha' \tag{11}$$

(Solution:)

Lengths and angles are given by scalar products which are invariant.

23.5. **f**)

$$\boxed{\text{A rotation can be obtained by a succession of mirror symmetries.}} \tag{12}$$

Visualize this for a plane triangle

$$A(2,\,2), \quad B(5,\,2), \quad C(5,\,4)$$

and apply two mirror symmetries, one with respect to the x-axis and the other one with respect to the line

$$y = -x \tag{13}$$

(Solution:)

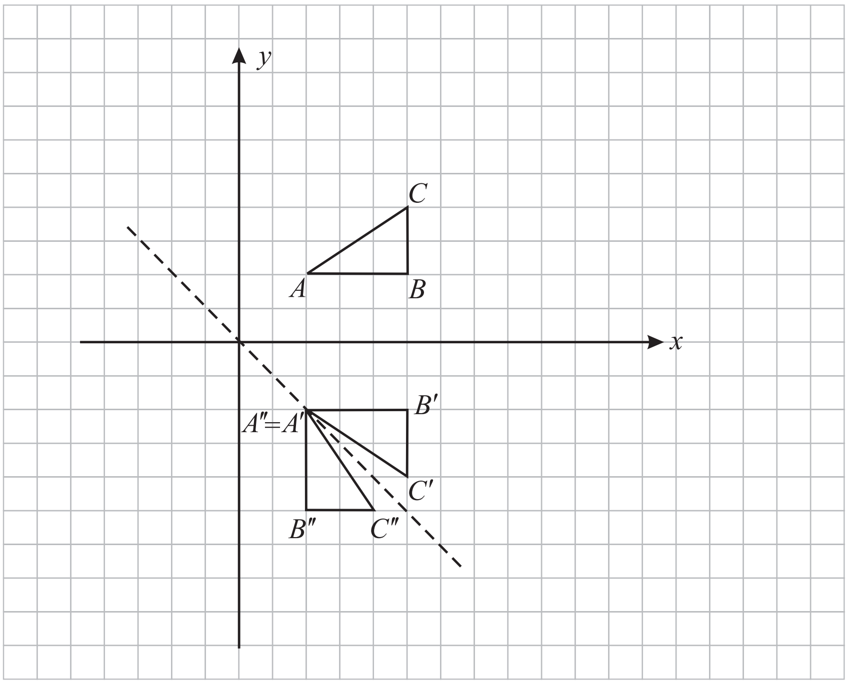


Fig23.5. **2**: Any rotation is obtained by a succession of mirror symmetries. The triangle A', B', C' is obtained from A, B, C with the x-axis as the mirror-symmetry axes (x-z-plane as the mirror) and A'', B'', C'' is obtained by the dotted line as the second mirror-symmetry axis. The resulting rotation is around the intersection of both mirror-symmetry axes (the origin in our case) and the angle of rotation is twice the angle between both mirror-symmetry axes (a rotation by $\frac{1}{2}\pi$ in our case).

FACIT:

$$\boxed{\text{Scalar products, lengths and angles are invariant under translation, mirror-symmetries and rotations.}} \qquad (14)$$

24 Vector product

24.**Q 1: Vector product**

REM: The vector product in the form below is possible only for $n = 3$.

24.1. **a**) Give the geometric definition of the **vector product**.

(Solution:)

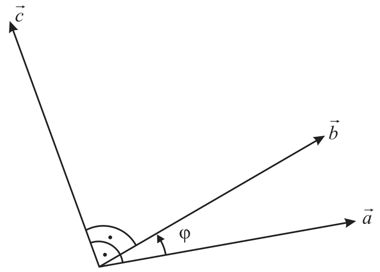


Fig24.1. **1**: The vector product $\vec{c} = \vec{a} \times \vec{b}$ is orthogonal to both of its factors $\vec{a}$ and $\vec{b}$. Its length is the length of $\vec{a}$ times the length of $\vec{b}$ times the sine of the enclosed angle φ $(0 \leq \varphi \leq \pi)$.

$\vec{c} = \vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ (or: perpendicular to the plane spanned by $\vec{a}$ and $\vec{b}$). Its length is given by

$$c = ab \sin \varphi \tag{1}$$

where φ is the (shorter, positive, i.e. $0 \leq \varphi \leq \pi$) angle between $\vec{a}$ and $\vec{b}$. The orientation of $\vec{c}$ is the direction of forward movement of a **right screw**[$\stackrel{\mathbf{G}}{=}$ `Rechtsschraube`] (or **corkscrew**[$\stackrel{\mathbf{G}}{=}$ `Korkenzieher`]) when it is rotated like $\vec{a}$ is rotated into the direction of $\vec{b}$ (via the shorter angle, i.e. via φ).

REM 1: In case $\vec{a} = 0$ or $\vec{b} = 0$, we have $\vec{c} = 0$, thus the above definition is unique even in these cases, although every vector is perpendicular to the null-vector.

REM 2: The above definition presupposes that space has an **orientation**, e.g. by selecting a particular corkscrew and defining it as a right screw.
If we were exactly mirror-symmetric, we could not distinguish between left and right.

REM 3: A vector product can be defined only for 3-dimensional vector spaces ($n = 3$).

24.1. **b**) Give an alternative name for the vector product and explain both names.

(Solution:)

Cross-product, because a cross ($\times$) is used (instead of the multiplication point) to distinguish it from the scalar product. It is called *vector* product, because the result $\vec{c}$ is a vector.

REM: More exactly it is only a pseudo-vector, differing from a true vector when reflexions are considered. For more details, see Ex.5.

24.1. **c**) Express parallelity (including antiparallelity) of two vectors with the help of the vector product.

(Solution:)

$$\boxed{\vec{a}\|\vec{b} \quad \Leftrightarrow \quad \vec{a}\times\vec{b}=0} \tag{2}$$

parallelity means vanishing vector product

where we adopt the convention that **the null vector is parallel to any vector**.

PROOF :
$\vec{a}\|\vec{b} \iff (\vec{a}=0 \vee \vec{b}=0 \vee \varphi=0 \vee \varphi=\pi) \iff$
$\iff (a=0 \vee b=0 \vee \sin\varphi=0) \iff ab\sin\varphi=0 \iff \vec{a}\times\vec{b}=0$

24.1. **d**) Express the area A of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ with the help of the vector product.

(Solution:)

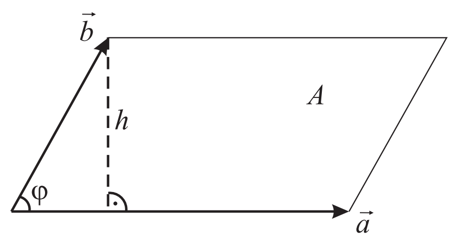


Fig24.1. **2**: $\vec{a}$ and $\vec{b}$ span a parallelogram. The length of the vector product $\vec{c}=\vec{a}\times\vec{b}$ is identical to the area of the parallelogram.

A is the base line a times height h, i.e.

$$A = ah = ab\sin\varphi = |\vec{a}\times\vec{b}| \tag{3}$$

REM:All points of the parallelogram are given as $\vec{r} = \lambda\vec{a} + \mu\vec{b}$ with $0\le\lambda\le1, \quad 0\le\mu\le1$.

24.1. **e**) Give the modified commutative law.

(Solution:)

$$\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}} \quad \text{anticommutative law} \tag{4}$$

24.1. **f)** Give an associative law.

(Solution:)

$$\lambda(\vec{a} \times \vec{b}) = (\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b}) \tag{5}$$

for which reason we can omit bracket e.g.

$$\lambda\vec{a} \times \vec{b} = \vec{a} \times \lambda\vec{b} = \vec{a} \times \vec{b}\lambda \tag{6}$$

24.1. **g)** Prove that in general

$$\vec{a} \times (\vec{b} \times \vec{c}) \quad \neq \quad (\vec{a} \times \vec{b}) \times \vec{c} \tag{7}$$

(Hint: use $\vec{a} \perp \vec{b}$, $\vec{b} = \vec{c}$, $a = b = c = 1$ as a counter example.)

(Solution:)

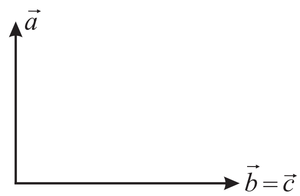


Fig24.1. **3**: A simple example of three vectors for which the associative law for vector products does not hold: $\vec{a} \times (\vec{b} \times \vec{c}) \quad \neq \quad (\vec{a} \times \vec{b}) \times \vec{c}$

$$\vec{b} \times \vec{c} = 0 \tag{8}$$

so the left hand side of (7) is zero.
On the other hand: $\vec{a} \times \vec{b} \perp \vec{c}$, $\quad |\vec{a} \times \vec{b}| = 1$, therefore the right hand side of (7) is $-\vec{a}$.

Rem 1: Thus an associative law for vector multiplication does not hold.

REM 2: Thus $\vec{a}\times\vec{b}\times\vec{c}$ is meaningless (ambiguous) except if we adopt the convention that with equal priority of operation, priority is from left to right

$$\vec{a}\times\vec{b}\times\vec{c} := (\vec{a}\times\vec{b})\times\vec{c} \tag{9}$$

but which is not usual

24.1. **h**) Explain why the vector product is bilinear.

(Solution:)

It is linear in both factors, i.e. the vector product of a vector with a linear combination of vectors is the linear combination of the individual vector products.

$$\vec{a}\times(\lambda_1\vec{b_1}+\lambda_2\vec{b_2}) = \lambda_1\vec{a}\times\vec{b_1}+\lambda_2\vec{a}\times\vec{b_2} \tag{10}$$

$$(\lambda_1\vec{a_1}+\lambda_2\vec{a_2})\times\vec{b} = \lambda_1\vec{a_1}\times\vec{b}+\lambda_2\vec{a_2}\times\vec{b} \tag{11}$$

REM: The last formula follows from the anticommutative law.

24.1. **i**) By writing $\vec{a}=\vec{a}_\perp+\vec{a}_{||}$ prove

$$\boxed{\vec{a}\times\vec{b} \quad = \quad \vec{a}_\perp\times\vec{b} \quad = \quad \vec{a}\times\vec{b}_\perp} \tag{12}$$

where $\vec{a}_{||}$ is component of $\vec{a}$ in the direction of $\vec{b}$ and $\vec{a}_\perp$ is the orthogonal component.

(Solution:)

$$\vec{a}\times\vec{b} = (\vec{a}_\perp+\vec{a}_{||})\times\vec{b} = \vec{a}_\perp\times\vec{b}+\vec{a}_{||}\times\vec{b} \tag{13}$$

with the help of linearity. The last term is zero.

24.1. **j**) Give the generation rule for the calculation of the components of $\vec{a}\times\vec{b}$ (algebraic definition of the vector product)

(Solution:)

To calculate $\vec{a}\times\vec{b}=\vec{c}=(c_1,c_2,c_3)$ in cartesian components: write down both factors above each other in two lines:

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array} \tag{14}$$

You obtain c_i by deleting the i-th column and by calculation of the remaining determinant[24] (with an additional -1 in the case $i = 2$):

$$c_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2 b_3 - a_3 b_2$$

$$c_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -(a_1 b_3 - b_1 a_3) \tag{15}$$

$$c_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

in summary

$$\boxed{\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, \quad -a_1 b_3 + b_1 a_3, \quad a_1 b_2 - a_2 b_1)} \tag{16}$$

REM 1: You should not learn by hard that formula, but instead the procedure how it was generated.

REM 2: (15)(16) are valid only in a **right-handed**[$\stackrel{\mathbf{G}}{=}$ `rechtshändig`] Cartesian coordinate system, i.e. when a right screw which is rotated as $\vec{e_x} \mapsto \vec{e_y}$ moves in the direction of $\vec{e_z}$ (and *not* in the direction of $-\vec{e_z}$) or in other words, if

$$\vec{e_x} \times \vec{e_y} = \vec{e_z}$$

.

Without strong reasons for the contrary, only **right handed Cartesian coordinate systems** are used in physics.

24.Ex 2: Products of coordinate unit vectors

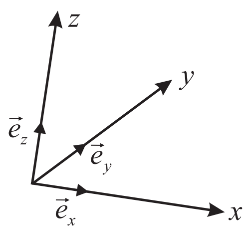


Fig24.2. **1**: 3-dimensional cartesian coordinate system with unit vectors $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ in the direction of the axes.

24.2. **a**) Calculate the components of the unit vectors along the coordinate axes, see

[24] The definition of the determinant is given in Ex 3 (3).

fig. 1.

RESULT:

$$\begin{aligned}\vec{e}_x &= (1,\,0,\,0)\\ \vec{e}_y &= (0,\,1,\,0)\\ \vec{e}_z &= (0,\,0,\,1)\end{aligned} \tag{1}$$

24.2. **b**) Show algebraically that those vectors form an orthonormalized **reference frame**[$\stackrel{\mathbf{G}}{=}$ `Bezugssystem`] i.e. that they are normalized, i.e. have lenght 1 and are orthogonal in pairs.

(Solution:)

E.g.

$$\vec{e}_x^{\,2} = \vec{e}_x\vec{e}_x = (1\cdot 1 + 0\cdot 0 + 0\cdot 0) = 1 \tag{2}$$

$$\vec{e}_x\vec{e}_y = (1\cdot 0 + 0\cdot 1 + 0\cdot 0) = 0 \tag{3}$$

24.2. **c**) Verify algebraically.

$$\vec{e}_x \times \vec{e}_y = \vec{e}_z \tag{4}$$

(Solution:)

We use the scheme

$$\begin{matrix} 1 & 0 & 0\\ 0 & 1 & 0\end{matrix} \tag{5}$$

to obtain the components of

$$\vec{e}_x \times \vec{e}_y = (0,\,0,\,1) = \vec{e}_z \tag{6}$$

24.2. **d**) Write (4) down again and all equations obtained from (4) by **cyclic permutation**.

$$\begin{tikzcd} x \arrow[rr] & & y \arrow[dl] \\ & z \arrow[ul] & \end{tikzcd} \tag{7}$$

RESULT:

$$\boxed{\begin{aligned}\vec{e}_x \times \vec{e}_y &= \vec{e}_z\\ \vec{e}_y \times \vec{e}_z &= \vec{e}_x\\ \vec{e}_z \times \vec{e}_x &= \vec{e}_y\end{aligned}} \qquad \text{(vector products of coordinate unit vectors)} \tag{8}$$

24.2. **e**) Verify all 3 equations from (8) geometrically.

(Solution:)

The right-hand side is orthogonal to both factors on the left-hand side. Since both factors on the left-hand side are orthogonal and have unit length, the right-hand side must have unit length.
That the signs of the right-hand sides are correct must be checked for each equation separately by applying the right-screw rule.

24.2. **f**) Check the last equation of (8) algebraically.

(Solution:)

We apply the scheme

$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} \tag{9}$$

to obtain

$$\vec{e}_z \times \vec{e}_x = (0,\, 1,\, 0) = \vec{e}_y \qquad \text{q.e.d.}$$

24.Ex 3: Area of a parallelogram expressed by a determinant

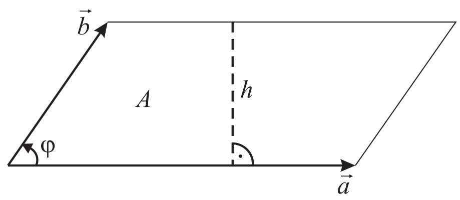


Fig24.3. **1**: Area A of parallelogram spanned by $\vec{a}$ and $\vec{b}$ will be expressed by a determinant. From elementary planimetry it is known that the area is the base × the height, i.e. $A = a \cdot h$.

We will show that the area A of a parallelogram spanned by the vectors

$$\begin{aligned} \vec{a} &= (a_1,\, a_2) \\ \vec{b} &= (b_1,\, b_2) \end{aligned} \tag{1}$$

(see fig. 1) can be expressed by the determinant

$$\boxed{A = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} \qquad \text{((oriented) area } A \text{ of a parallelogram spanned by } \vec{a} \text{ and } \vec{b}\text{)} \tag{2}$$

Determinants are defined for arbitrary quadratic matrices. For a 2×2 matrix that definition is

$$\boxed{\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \beta\gamma} \qquad \text{(definition of a 2×2 determinant)} \tag{3}$$

REM 1: the vertical bars cannot be confused with an absolute value because 1×1 matrices are rarely used. To avoid possible confusion, there is an alternative notation for determinants, e.g.

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \beta\gamma \tag{3'}$$

REM 2: (2) gives an oriented area changing its sign when the vectors are interchanged.

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} \tag{4}$$

When A is understood to be the ordinary (i.e. unoriented, positive definite) area, (2) should read

$$A = \left\| \begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix} \right\| = \left| \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \right| \tag{2'}$$

Here in the first expression the outer bars denote an absolute value while the inner bars denote the determinant.
REM 3: in $n = 3$ and in higher dimensional spaces ($n \geq 4$) the determinant also gives the (oriented) volume of a parallelepipedon (hyperparallelepipedon for $n \geq 4$) spanned by n vectors.
24.3. **a**) First verify that the absolute value of the vector product gives the area A in keeping with the rule

$$\boxed{\text{area} = \text{base} \times \text{height}} \tag{5}$$

$$A = ah = |\vec{a} \times \vec{b}| \tag{6}$$

(Solution:)

$$|\vec{a} \times \vec{b}| = ab \sin\varphi \tag{7}$$

See fig. 1.
Relative to the angle φ, the height h is the side-projection of b:

$$h = b \sin\varphi \tag{8}$$

which proves (6).

24.3. **b**) Assume that there is a third axis (i.e. an upward z-axis), we enlarge (1) to $n = 3$ dimensional vectors

$$\begin{aligned} \vec{a} &= (a_1,\, a_2,\, 0) \\ \vec{b} &= (b_1,\, b_2,\, 0) \end{aligned} \tag{9}$$

calculate its vector product and prove (2′).

(Solution:)

Using the scheme

$$\begin{matrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{matrix} \tag{1}$$

we obtain the components of

$$\vec{c} = \vec{a} \times \vec{b} = \left(0,\, 0,\, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right) \tag{11}$$

Since the other components are zero, the third component is the absolute value, except for a possible minus sign. Together with (6) this proves (2′). q.e.d.

24.**Ex 4: Linear (in)dependence expressed by vector product**

We will prove the following equivalences.

$$\boxed{\vec{a},\, \vec{b} \ \text{ linearly dependent} \quad \Longleftrightarrow \quad \vec{a} \times \vec{b} = 0} \tag{1}$$

$$\boxed{\vec{a},\, \vec{b} \ \text{ linearly independent} \quad \Longleftrightarrow \quad \vec{a} \times \vec{b} \neq 0} \tag{2}$$

REM: the zero on the right-hand side of (1) and (2) is the null-*vector*, so a more exact representation of it would be $\vec{0}$.

24.4. **a**) Prove $\Rightarrow$ in (1).

HINT: $\vec{a}$, $\vec{b}$ linear dependent means one of the following cases:

$$\vec{a} = 0 \quad \text{or} \quad \vec{b} = 0 \quad \text{or} \quad \vec{a} = \lambda \vec{b} \tag{3}$$

These three cases can be reduced to the following two cases:

$$\vec{a} = \lambda \vec{b} \quad \text{or} \quad \vec{b} = \lambda \vec{a} \tag{4}$$

(including the probability that $\lambda = 0$)

(Solution:)

$$\begin{aligned} &\text{For } \vec{a} = \lambda \vec{b} \quad \Rightarrow \quad \vec{a} \times \vec{b} = \lambda \underbrace{\vec{b} \times \vec{b}}_{0} = 0 \\ &\text{For } \vec{b} = \lambda \vec{a} \quad \Rightarrow \quad \vec{a} \times \vec{b} = \vec{a} \times \lambda \vec{a} = \lambda(\vec{a} \times \vec{a}) = 0 \end{aligned} \tag{5}$$

24.4. **b**) Prove $\Leftarrow$ in (1).
HINT: consider the absolute value of the vector product.

(Solution:)

$$\vec{a} \times \vec{b} = 0 \quad \Rightarrow \quad |\vec{a} \times \vec{b}| = 0 \quad \Rightarrow \quad ab \sin\varphi = 0 \tag{6}$$

From this we have three possibilities: $a = 0$ (i.e. $\vec{a} = 0$), or $b = 0$ (i.e. $\vec{b} = 0$), or $\sin\varphi = 0$ i.e. either $\varphi = 0$ or $\varphi = \pi$ i.e. $\vec{a} = \lambda\vec{b}$.

24.4. **c**) Prove (2).
HINT: use the following logical equivalence.

$$(\mathcal{A} \Longleftrightarrow \mathcal{B}) \quad \Longleftrightarrow \quad (\neg\mathcal{A} \Longleftrightarrow \neg\mathcal{B}) \tag{7}$$

$\mathcal{A}$ and $\mathcal{B}$ are any statements, $\neg\mathcal{A}$ is the negation of the statement $\mathcal{A}$.

(Solution:)

$$\begin{aligned} \mathcal{A} &\equiv (\vec{a},\, \vec{b} \text{ are linearly dependent}) \\ \neg\mathcal{A} &\equiv (\vec{a},\, \vec{b} \text{ are linearly independent}) \\ \mathcal{B} &\equiv (\vec{a} \times \vec{b} = 0) \\ \neg\mathcal{B} &\equiv (\vec{a} \times \vec{b} \neq 0) \end{aligned} \tag{8}$$

24.Ex 5: ☻ The vector product as a pseudo-vector; axial and polar vectors

Consider two position vectors

$$\begin{aligned} \vec{a} &= (a_1,\, a_2,\, a_3) \\ \vec{b} &= (b_1,\, b_2,\, b_3) \end{aligned} \tag{1}$$

24.5. **a**) Calculate the position vectors $\vec{a}'$, $\vec{b}'$ obtained from $\vec{a}$, $\vec{b}$ by a mirror-symmetry with respect to the x-y-plane.
RESULT:

$$\begin{aligned} \vec{a}' &= (a_1,\, a_2,\, -a_3) \\ \vec{b}' &= (b_1,\, b_2,\, -b_3) \end{aligned} \tag{2}$$

(Solution:)

The z-component of the vector changes sign, while the x and y-components remain unchanged.

24.5. **b**) Calculate the vector product.

$$\vec{c} = \vec{a} \times \vec{b} \tag{3}$$

RESULT:

$$\vec{c} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1) = (c_1,\ c_2,\ c_3) \tag{4}$$

(Solution:)

Use the scheme

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \tag{5}$$

24.5. **c**) By applying a formal symmetry to (4), calculate the components of

$$\vec{c}' = \vec{a}' \times \vec{b}' \tag{6}$$

RESULT:

$$\vec{c}' = (-a_2b_3 + a_3b_2,\ -a_3b_1 + a_1b_3,\ a_1b_2 - a_2b_1) = (-c_1,\ -c_2,\ c_3) \tag{7}$$

(Solution:)

The formal symmetry to be applied is changing the sign of a_3 and b_3.

24.5. **d**) Apply mirror-symmetry directly to $\vec{c}$, assuming (incorrectly) that $\vec{c}$ is an ordinary vector.
RESULT:

$$\vec{c}' = (c_1,\ c_2,\ -c_3) \tag{8}$$

in analogy with the result (2).
FACIT: The vector $\vec{c}$ obtained by forming a vector product $\vec{c} = \vec{a} \times \vec{b}$ from ordinary (e.g. position) vectors $\vec{a}$ and $\vec{b}$ does not behave like an ordinary vector under a mirror-symmetry. The correct result is (7), which compared to an ordinary vector (with result (8)) assumes an additional factor of -1.
(2) and (8) are true if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are ordinary vectors. (7) is true, as assumed in (1)(3), that $\vec{c}$ is the vector product of two ordinary vectors.

24.5. **e**) Visualize this behaviour graphically using the example

$$\begin{aligned} \vec{a} &= (4,\ 0,\ 0) \\ \vec{b} &= (2,\ 3,\ 0) \end{aligned} \tag{9}$$

by drawing $\vec{a}'$, $\vec{b}'$, $\vec{c} = \vec{a} \times \vec{b}$ and $\vec{c}' = \vec{a}' \times \vec{b}'$.

(Solution:)

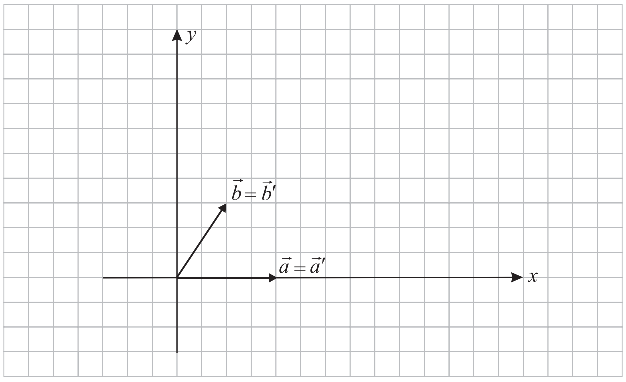


Fig24.5. **1**: Under a mirror-symmetry with respect to the x-y-plane we have $\vec{a}' = \vec{a}$, $\vec{b}' = \vec{b}$ (since these vectors lie in the mirror plane). The vector products $\vec{c} = \vec{a} \times \vec{b}$ and $\vec{c}' = \vec{a}' \times \vec{b}'$ are equal, namely upwards along the z-axis with the length $4 \cdot 3 = 12$ units (3 units = projection of $\vec{b}$ to the y-axis). Therefore, the vector product yields a pseudo-vector, assuming an additional -1 under a mirror-symmetry, since an ordinary vector would behave like $\vec{c}' = -\vec{c}$.

The vector $\vec{c} = \vec{a} \times \vec{b}$ points upwards (along the positive x-axis) since by moving $\vec{a}$ into the direction of $\vec{b}$ via the shorter angle a right-screw moves upwards.

24.5. **f**) The **mapping** [$\stackrel{\mathbf{G}}{=}$ `Abbildung`]

$$\boxed{\vec{r} \mapsto \vec{r}\,' = -\vec{r}} \qquad \begin{array}{l}\text{(point-symmetry)}\\ \text{(central-symmetry)}\end{array} \tag{10}$$

is called a **point-symmetry** [$\stackrel{\mathbf{G}}{=}$ `Punkt-Symmetrie`] or a **central-symmetry** [$\stackrel{\mathbf{G}}{=}$ `Zentralsymmetrie`].

REM: In the case of (10), the 'point' = the 'center' is the origin ($\vec{r} = 0$).

Show that the point-symmetry (10) can be obtained by applying 3 mirror symmetries in succession, e.g. a first one with the x-y-axis as the mirror, a second one with the x-z-axis as the mirror and a last one with the y-z-axis as the mirror.
HINT: write $\vec{r} = (x,\, y,\, z)$. In a mirror-symmetry that component changes sign which are *not* the axes of the mirror.

(Solution:)

$\vec{r} = (x,\, y,\, z)$. Denoting by M_{xy} the mirror-symmetry with the x-y-axis as the mirror, we obtain

$$\vec{r}\,'' = M_{xy}\vec{r} = (x,\, y,\, -z) \tag{11}$$

$$\begin{aligned} \vec{r}\,''' &= M_{xz}\vec{r}\,'' = M_{xz}M_{xy}\vec{r} = (x,\, -y,\, -z) \\ \vec{r}\,'''' &= M_{yz}\vec{r}\,''' = M_{yz}M_{xz}M_{xy}\vec{r} = (-x,\, -y,\, -z) = -(x,\, y,\, z) = -\vec{r} = \vec{r}\,' \end{aligned} \tag{12}$$

q.e.d.

24.5. **g**) Since a point symmetry is obtained by a succession of three mirror-symmetries whereby the vector product $\vec{c} = \vec{a} \times \vec{b}$ assumes three times an additional factor of -1, we expect that a vector product assumes the additional factor

$$(-1)^3 = -1 \tag{13}$$

compared to an ordinary (i.e. position) vector. Validate that statement by applying point-symmetry (10) to the general vectors (1).

(Solution:)

$$\begin{aligned} \vec{a}' &= -\vec{a} \\ \vec{b}' &= -\vec{b} \end{aligned} \quad \Rightarrow \quad \vec{c}' = \vec{a}' \times \vec{b}' = (-\vec{a}) \times (-\vec{b}) = (-1)^2 \vec{a} \times \vec{b} = \vec{a} \times \vec{b} = \vec{c} \tag{14}$$

$$\vec{c}' = \vec{c} \tag{15}$$

If $\vec{c}$ were an ordinary (i.e. position) vector we would have

$$\vec{c}' = -\vec{c} \tag{16}$$

instead of (15).

24.5. **h**) Validate the last statement graphically by using the vectors $\vec{a}$ and $\vec{b}$ in fig. 1.

(Solution:)

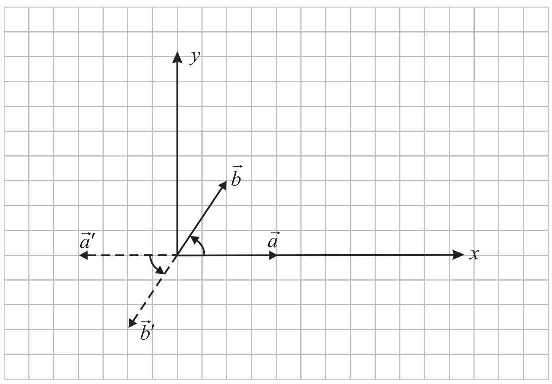


Fig24.5. **2**: The vectors $\vec{a}'$, $\vec{b}'$ are obtained from $\vec{a}$, $\vec{b}$ by a point-symmetry (with respect to the origin). Forming the vector products $\vec{c} = \vec{a} \times \vec{b}$ and $\vec{c}' = \vec{a}' \times \vec{b}'$ we have to move the first factor ($\vec{a}$ or $\vec{a}'$) via the shorter angles into the direction of the second factor ($\vec{b}$ or $\vec{b}'$). In both cases this means the same sense of rotation (arc arrows in figure) so $\vec{c}$ and $\vec{c}'$ both point upwards: $\vec{c}' = \vec{c}$, while for an ordinary vector we would have $\vec{c}' = -\vec{c}$.

FACIT: a position vector $\vec{r}$ is the prototype of an **ordinary vector**. It behaves like (10) under a point-symmetry. Thus an ordinary vector is also called a **polar vector** since it changes polarity (i.e. sign) under a point-symmetry,

$$\vec{r}\,' = -\vec{r} \tag{17}$$

while $\vec{c}$ does not:

$$\vec{c}\,' = \vec{c} \tag{18}$$

24.5. **i)** Because the vector $\vec{c}$ obtained by a vector product from ordinary vectors, $\vec{c} = \vec{a} \times \vec{b}$, assumes an additional (-1) under a mirror-symmetry (compared to an ordinary vector) it is called a **pseudo-vector**. A second example of a pseudo-vector is the **angular velocity vector** [$\stackrel{\mathbf{G}}{=}$ `Winkelgeschwindigkeitsvektor`] $\vec{\omega}$ of a rotation

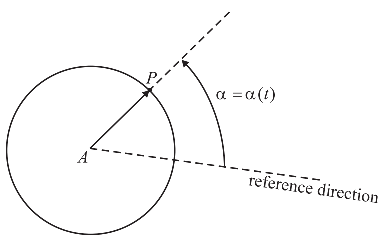
$$\boxed{\vec{\omega} = \text{ angular velocity vector}} \tag{19}$$

defined as follows:[25]

- $\vec{\omega}$ has the direction of the axis of rotation.
- $\vec{\omega}$ has the same orientation as a right-screw when it is rotated in the same way as the body.
- $\vec{\omega}$ has magnitude

$$|\vec{\omega}| = \omega = \dot{\alpha} \tag{20}$$

For the definition of α see fig. 3.



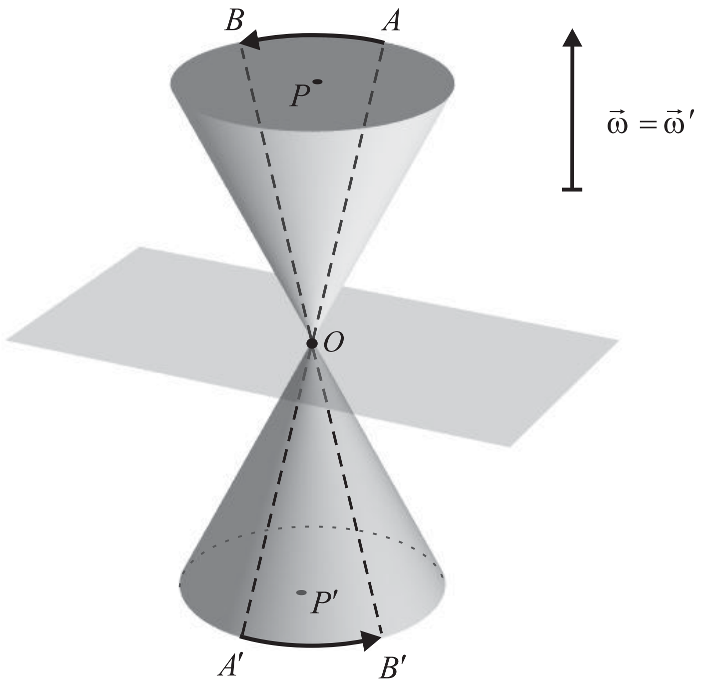
`Fig`24.5. **3**: A rotating body (e.g. a ball) rotating about the axis A (asssumed here to be perpendicular to the sheet of the figure). A physical mark P on the body has angular position $\alpha = \alpha(t)$ with respect to an arbitrary (but fixed) dotted reference line. When α is increasing $\vec{\omega}$ points upwards.

---

[25]It is defined here only for a fixed axis.

Assume that a **conical**[$\stackrel{\mathbf{G}}{=}$ `kegelförmig`] top (**spinning top**[$\stackrel{\mathbf{G}}{=}$ `Kreisel`]) is standing (rotating) upright. Construct the point-symmetric top and show that both have the same angular velocity. Do the same for a mirror-symmetry with the sustaining plane as the mirror. Perform that experiment with a real top and mirror.

(Solution:)



`Fig`24.5. **4**: A spinning top rotating with angular velocity $\vec{\omega}$ as also indicated by the arc's arrow $\overrightarrow{AB}$ on the top. The dotted top is obtained by a point-symmetry with respect to O. Though both tops are not identical they have the same angular velocity $\vec{\omega}' = \vec{\omega}$. Thus angular velocity is a pseudo-vector (= axial vector) since an ordinary vector, e.g. $\overrightarrow{OP}$, would behave like $\overrightarrow{OP}' = -\overrightarrow{OP}$. The same results if we apply a mirror-symmetry with the sustaining plane as the mirror.

FACIT: since angular velocity is a prototype for a pseudo-vector, they are also called **axial vectors**[26].

24.5. **j**) What is the difference between axial and polar vectors under rotations?
HINT: a rotation is obtained by the succession of an *even* number of mirror-symmetries.
RESULT: nothing

(Solution:)

Each mirror-symmetry gives an additional factor of (-1) compared to a polar vector.

[26] The axis $\hat{\omega} = \frac{\vec{\omega}}{\omega}$ is also a pseudo-vector.

Since a rotation involves an even number of mirror-symmetries these signs drop out.

24.5. **k**) Write down equations analogous to (10), (11) and (12) for an axial vector, e.g. for $\vec{\omega}$.
HINT: compared to a polar vector an additional (-1) occurs for each mirror symmetry.

(Solution:)

$$\vec{\omega} \mapsto \vec{\omega}' = \vec{\omega} \qquad \text{point symmetry for an axial vector} \tag{10'}$$

$$\begin{aligned} \vec{\omega} &= (\omega_1,\, \omega_2,\, \omega_3) \\ \vec{\omega}'' &= M_{xy}\vec{\omega} = (-\omega_1,\, -\omega_2,\, \omega_3) \end{aligned} \tag{11'}$$

$$\begin{aligned} \vec{\omega}''' &= M_{xz}\vec{\omega}'' = M_{xz}M_{xy}\vec{\omega} = (\omega_1,\, -\omega_2,\, -\omega_3) \\ \vec{\omega}'''' &= M_{yz}\vec{\omega}''' = M_{yz}M_{xz}M_{xy}\vec{\omega} = (\omega_1,\, \omega_2,\, \omega_3) = \vec{\omega} = \vec{\omega}' \end{aligned} \tag{12'}$$

SUMMARY

$$\boxed{\text{A position vector is the prototype of an ordinary vector.}} \tag{21}$$

$$\boxed{\text{ordinary vector} \;\equiv\; \text{true vector} \;\equiv\; \text{polar vector}} \tag{22}$$

$$\boxed{\text{Angular velocity is the prototype of a pseudo-vector.}} \tag{23}$$

$$\boxed{\text{pseudo vector} \;\equiv\; \text{axial vector}} \tag{24}$$

$$\boxed{\text{The vector product of two ordinary vectors is a pseudo-vector.}} \tag{25}$$

$$\boxed{\text{Under rotation, vectors and pseudo-vectors behave identically.}} \tag{26}$$

$$\boxed{\begin{array}{l}\text{Under mirror-symmetries (and point symmetries),} \\ \text{a pseudo-vector assumes an additional } (-1) \\ \text{compared to an ordinary vector.}\end{array}} \tag{27}$$

REM 1: Pseudo-vectors come into play because of the notion of a right-screw, which was indeed used both in the definition of vector product and in the definition of angular velocity. While we did perform the mirror-symmetry, the right-screw remained unchanged. Had we applied the mirror-symmetry to both the top **and** the right-screw, it would have then been transformed into a left-screw, and if we had used the latter for the definition of $\vec{\omega}$, $\vec{\omega}$ would have behaved like an ordinary vector.

REM 2: Instead of ‘ordinary vector’ we say also ‘**true vector**’. The notations ‘true’ and ‘pseudo’ have historical origins, because the position vector was dicovered first and was considered to be true. From a mathematical point of view pseudo-vectors are as good as true vectors, they are only different.

25 Wedge product. Multiple vector products.

25.**Q 1: Wedge product**
What is the **wedge product** [$\overset{\mathbf{G}}{=}$ `Spatprodukt`]
25.1. **a**) expressed by a vector product.

(Solution:)

$$(\vec{a} \times \vec{b})\vec{c} \tag{1}$$

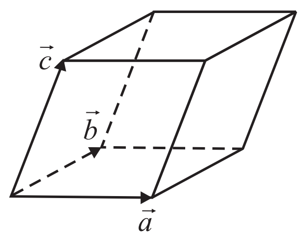
which is a scalar.

25.1. **b**) What is its geometrical meaning?

(Solution:)

Geometrically it represents the volume of the **parallelepipedon** [$\overset{\mathbf{G}}{=}$ `Spat`].

REM: **wedge** [$\overset{\mathbf{G}}{=}$ `Keil`]



`Fig`25.1. **1**: the wedge product $(\vec{a} \times \vec{b})\vec{c}$ is the volume of the parallelepipedon spanned by the three vectors $\vec{a}, \vec{b}, \vec{c}$.

$$V = (\vec{a} \times \vec{b})\vec{c} \tag{2}$$

REM: V is the oriented volume, i.e. it can be negative, indeed

$$(\vec{a} \times \vec{b})\vec{c} = -(\vec{b} \times \vec{a})\vec{c} \tag{3}$$

i.e. the oriented volume depends on the order of the vectors spanning the parallelepipedon.
The usual volume, which is always positive, is obtained by taking the absolute value.

$$V = |(\vec{a} \times \vec{b})\vec{c}| \tag{4}$$

instead of (1)

25.1. **c**) Give the **cyclic permutation rule** of the wedge product.

(Solution:)

$$(\vec{a} \times \vec{b})\vec{c} = (\vec{c} \times \vec{a})\vec{b} = (\vec{b} \times \vec{c})\vec{a} \tag{5}$$

where letters have been permutated cyclically:

$$\begin{array}{ccc} & b & \\ \nearrow & & \searrow \\ a & \leftarrow & c \end{array}$$

`Fig`25.1. **2**: Cyclic permutation of three symbols a, b, c

25.1. **d**) What is the connection with determinants?

(Solution:)

$$(\vec{a} \times \vec{b})\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{6}$$

REM: For the definition of determinants, see Ex 4 of the previous chapter.

25.**Q 2: Multiple vector products**

With the help of a formulary check the following **formula for multiple vector products**[$\overset{\mathbf{G}}{=}$ `Entwicklungssatz`]:

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b})} \tag{1}$$

25.**Ex 3: Other formulas for multiple vector products**

25.3. **a**) Verify

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\,\vec{c}) - \vec{c}(\vec{a}\,\vec{b}) \tag{1}$$

for

$$\begin{aligned} \vec{a} &= (0,\,0,\,1) \\ \vec{b} &= (1,\,0,\,-1) \\ \vec{c} &= (1,\,2,\,0) \end{aligned} \tag{2}$$

(Solution:)

Using the scheme

$$\begin{matrix} 1 & 0 & -1 \\ 1 & 2 & 0 \end{matrix} \tag{3}$$

we obtain

$$\vec{b} \times \vec{c} = (2,\ -1,\ 2). \tag{4}$$

Using the scheme

$$\begin{matrix} 0 & 0 & 1 \\ 2 & -1 & 2 \end{matrix} \tag{5}$$

we obtain

$$\vec{a} \times (\vec{b} \times \vec{c}) = (1,\ 2,\ 0). \tag{6}$$

On the other hand we have

$$\begin{aligned} \vec{a}\,\vec{c} &= 0 \\ \vec{a}\,\vec{b} &= -1 \end{aligned} \tag{7}$$

and the right-hand side of (1) gives $+\vec{c}$ which is identical to (6). q.e.d.

25.3. **b**) Look up a formula for $(\vec{a} \times \vec{b})(\vec{c} \times \vec{d})$ in a formulary.
RESULT:

$$\boxed{(\vec{a} \times \vec{b})(\vec{c} \times \vec{d}) = (\vec{a}\,\vec{c})(\vec{b}\,\vec{d}) - (\vec{a}\,\vec{d})(\vec{b}\,\vec{c})} \tag{8}$$

25.Ex 4: Purely vectorial treatment of a regular tetrahedron

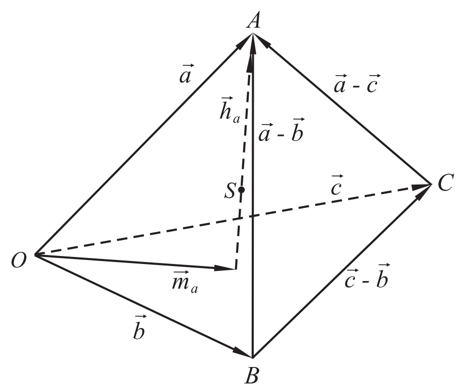


Fig25.4. **1**: A regular tetrahedron with corners O, A, B and C spanned by three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ having 6 edges of equal length taken to be unity: $a = b = c = 1$ etc. It has 4 faces which are equilateral triangles. $\vec{h}_a$ is the height from corner A to its opposing base. All four heights intersect at a single point S.

REM: In a previous exercise we dealt with a regular tetrahedron (*tetra* Greek = 4, *hedron* Greek = face) using special coordinates. Here, we do a purely vectorial treatment, i.e. the tetrahedron has a general orientation. Such a treatment is also called a **coordinate-independent treatment** or a **covariant**[27] **treatment**.

A regular tetrahedron with 6 **edges**[$\overset{\mathbf{G}}{=}$ `Kanten`] of length 1 is spanned by three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, see fig. 1.

25.4. **a**) Derive the conditions of 6 unit length edges in vertical form.

RESULT:

$$\begin{aligned} &\vec{a}^{\,2} = \vec{b}^{\,2} = \vec{c}^{\,2} = 1 \\ &\vec{a}\,\vec{b} = \vec{a}\,\vec{c} = \vec{b}\,\vec{c} = \frac{1}{2} \end{aligned} \tag{1}$$

(Solution:)

$$1 \overset{!}{=} (\vec{a} - \vec{c})(\vec{a} - \vec{c}) = \underbrace{\vec{a}^{\,2} + \vec{c}^{\,2}}_{2} - 2\vec{a}\,\vec{c} \quad \Rightarrow \tag{2}$$

$$\vec{a}\,\vec{c} = \frac{1}{2} \tag{3}$$

The two remaining edges are obtained by applying formal cyclic symmetry

$$\begin{array}{ccc} a & \longrightarrow & b \\ & \nwarrow \; c \; \swarrow & \end{array} \tag{4}$$

Thus we can apply this cyclic symmetry directly to (3) to obtain the remaining conditions in (1).

25.4. **b**)

$$\boxed{\text{In a regular tetrahedron opposite edges are perpendicular.}} \tag{5}$$

Prove this for sides $\vec{a}$ and $\vec{c} - \vec{b}$.

(Solution:)

$$\vec{a}(\vec{c} - \vec{b}) = \vec{a}\,\vec{c} - \vec{a}\,\vec{b} = \frac{1}{2} - \frac{1}{2} = 0 \qquad \text{q.e.d.}$$

25.4. **c**) The middle of face a (i.e. opposite to the corner A, also called a **face center**) is given by the position vector

$$\vec{m}_a = \frac{1}{3}(\vec{b} + \vec{c}) \tag{6}$$

[27] because everything can vary depending on the (choice of the) coordinates.

REM: because this face is a regular (i.e. unilateral) triangle, this point (face center) can be defined in several equivalent ways:
1) it is the center of mass if the corners O, B and C of the triangle are equal mass-points.
2) It is the center of mass if the triangle O, B, C is a plate with homogeneous mass distribution.
3) It is the (common) intersection of the three bisectors of the angle.
4) It is the (common) intersection of the bisectors of the (opposite) sides.
5) It is the (common) intersection of the (triangle's) heights.
6) Here we prove that it has the same distance from all corners O, B and C.

Calculate the length of $\vec{m}_a$ and prove that its tip (= face center) has the same distance from C and B.
RESULT:

$$|\vec{m}_a| = \frac{1}{\sqrt{3}} \tag{7}$$

(Solution:)

$$\vec{m}_a^2 = \frac{1}{9}(\vec{b}+\vec{c})(\vec{b}+\vec{c}) = \frac{1}{9}(\vec{b}^2 + \vec{c}^2 + 2\vec{b}\,\vec{c}) \overset{(1)}{=} \frac{1}{3} \tag{8}$$

Distance of the face center from B:

$$(\vec{b} - \vec{m}_a)^2 = \left(\frac{2}{3}\vec{b} - \frac{1}{3}\vec{c}\right)^2 = \frac{1}{9}(4\vec{b}^2 + \vec{c}^2 - 4\vec{b}\,\vec{c}) = \frac{1}{9}\left(4 + 1 - 4\cdot\frac{1}{2}\right) = \frac{1}{3} \tag{9}$$

Identical to (8).
The distance of the face center from C is obtained from (9) by the formal symmetry

$$B \leftrightarrow C \tag{10}$$

which does not affect the result from (9), i.e. $\frac{1}{3}$. q.e.d.

25.4. **c**) Calculate the height $\vec{h}_a$ as a vector and also its length (magnitude).
RESULT:

$$\vec{h}_a = \vec{a} - \vec{m}_a = \vec{a} - \frac{1}{3}\vec{b} - \frac{1}{3}\vec{c} \tag{11}$$

$$h_a = \sqrt{\frac{2}{3}} \tag{12}$$

$$\boxed{\text{The height } h_a \text{ of a regular tetrahedron with side length } a \text{ is } \sqrt{\frac{2}{3}}\,a.} \tag{13}$$

(Solution:)

$$\vec{h}_a^2 = \vec{a}^{\,2} + \frac{1}{9}\vec{b}^{\,2} + \frac{1}{9}\vec{c}^{\,2} - \frac{2}{3}\vec{a}\,\vec{b} - \frac{2}{3}\vec{a}\vec{c} + \frac{2}{9}\vec{b}\,\vec{c}$$
$$\overset{(1)}{=} 1 + \frac{1}{9} + \frac{1}{9} - \frac{1}{3} - \frac{1}{3} + \frac{1}{9} = \frac{9+1+1-3-3+1}{9} = \frac{6}{9} = \frac{2}{3} \tag{14}$$

25.4. **e**) Applying the formal cyclic symmetry (5) also find $\vec{h}_b$, $\vec{h}_c$.
Result:

$$\vec{h}_b = \vec{b} - \frac{1}{3}\vec{c} - \frac{1}{3}\vec{a} \tag{11'}$$

$$\vec{h}_c = \vec{c} - \frac{1}{3}\vec{a} - \frac{1}{3}\vec{b} \tag{11''}$$

25.4. **f**) Calculate the remaining fourth height h_o as the center of mass of the corners of face A, B, C, i.e. as the average of their position vectors.
Result:

$$\vec{h}_o = \frac{1}{3}(\vec{a} + \vec{b} + \vec{c}) \tag{15}$$

25.4. **g**) Check that $\vec{h}_o$ has length $\sqrt{\frac{2}{3}}$.

(Solution:)

$$\vec{h}_o^2 = \frac{1}{9}(\vec{a}^{\,2} + \vec{b}^{\,2} + \vec{c}^{\,2} + 2\vec{a}\,\vec{b} + 2\vec{a}\,\vec{c} + 2\vec{b}\,\vec{c})$$
$$= \frac{1}{9}(1+1+1+1+1+1) = \frac{6}{9} = \frac{2}{3}, \qquad h_0 = \sqrt{\frac{2}{3}} \tag{16}$$

25.4. **h**) Calculate the angle ϑ between neighboring faces.
Hint:

$$\boxed{\textbf{The angle between two planes is defined as the angle between their normals.}} \tag{17}$$

Result:

$$\vartheta = 70.5288^\circ \tag{18}$$

(Solution:)

$$\begin{aligned}\vec{h}_a\,\vec{h}_o &= \frac{1}{9}(3\vec{a}-\vec{b}-\vec{c})(\vec{a}+\vec{b}+\vec{c}) = \\ &= \frac{1}{9}(3\vec{a}^{\,2}+3\vec{a}\,\vec{b}+3\vec{a}\,\vec{c}-\vec{a}\vec{b}-\vec{b}^2-\vec{b}\,\vec{c}-\vec{a}\,\vec{c}-\vec{b}\,\vec{c}-\vec{c}^{\,2}\,) =\end{aligned} \tag{19}$$

$$\begin{aligned}&= \frac{1}{9}(3+\frac{3}{2}+\frac{3}{2}-\frac{1}{2}-1-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-1) = \\ &= \frac{1}{18}(6+3+3-1-2-1-1-1-2) = \frac{4}{18}\end{aligned} \tag{20}$$

$$= h_a\,h_o\cos\vartheta = \sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}}\cos\vartheta = \frac{2}{3}\cos\vartheta \tag{21}$$

$$\cos\vartheta = \frac{1}{3} \quad \Rightarrow \quad (18) \tag{22}$$

25.4. **i**) Let S be the center of the tetrahedron. S is found by averaging the position vectors of all the corners.

$$\overrightarrow{OS} = \vec{S} = \frac{1}{4}(\vec{a}+\vec{b}+\vec{c}) \tag{23}$$

Rem : 1) S is the center of mass when the tetrahedron is homogeneously (i.e. uniformly) filled with mass.
2) S is the center of mass when all four corners are equal mass points.
3) S is the center of mass when all faces are plates with homogeneous mass distribution.
4) S is the common intersection of all heights $\vec{m}_a$, $\vec{m}_b$, $\vec{m}_c$, $\vec{m}_o$.
Calculate the distance of S from all corners and show that they are equal.
Result:

$$|OS| = \frac{1}{2}\sqrt{\frac{3}{2}} \tag{24}$$

(Solution:)

$$\begin{aligned}|\vec{s}|^2 &= \frac{1}{16}(\vec{a}^{\,2}+\vec{b}^{\,2}+\vec{c}^{\,2}+2\vec{a}\vec{b}+2\vec{a}\vec{c}+2\vec{b}\vec{c}) = \\ &= \frac{1}{16}\left(1+1+1+\frac{2}{2}+\frac{2}{2}+\frac{2}{2}\right) = \frac{6}{16} = \frac{3}{8}\end{aligned} \tag{25}$$

$$\begin{aligned}
|AS| &= \left(\frac{3}{4}\vec{a} - \frac{1}{4}\vec{b} - \frac{1}{4}\vec{c}\right)^2 = \frac{1}{16}(3\vec{a} - \vec{b} - \vec{c})(3\vec{a} - \vec{b} - \vec{c}) = \\
&= \frac{1}{16}(9\vec{a}^{\,2} - 3\vec{a}\vec{b} - 3\vec{a}\vec{c} - 3\vec{a}\vec{b} + \vec{b}^{\,2} + \vec{b}\vec{c} - 3\vec{a}\vec{c} + \vec{b}\vec{c} + \vec{c}^{\,2}) = \\
&= \frac{1}{16}\left(9 - \frac{3}{2} - \frac{3}{2} - \frac{3}{2} + 1 + \frac{1}{2} - \frac{3}{2} + \frac{1}{2} + 1\right) = \frac{12}{32} = \frac{3}{8}
\end{aligned} \tag{26}$$

By formal cyclic symmetries we obtain the same value for the remaining corners.

25.4. **j**) Show that all heights intersect at a common point which is S.
HINT: By writing down the equation of a straight line bearing the height, show that each height passes through S.

(Solution:)

Equation for the straight line having height h_o:
($\vec{r}$ is an arbitrary point on that straight line.)

$$\vec{r} \overset{(15)}{=} \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})\lambda \tag{27}$$

For the parameter value $\lambda = \frac{3}{4}$ we obtain $\vec{r} = \vec{s}$ from (23).
For the height h_a:

$$\vec{r} \overset{(11)}{=} \vec{a} + (\vec{a} - \frac{1}{3}\vec{b} - \frac{1}{3}\vec{c})\lambda \tag{28}$$

$\vec{r} = \vec{s}$ for $\lambda = -\frac{3}{4}$.
By cyclic symmetry the same is true for the remaining heights. q.e.d.

25.4. **k**) Write down a parametric represention of the plane having face ABC.
RESULT:

$$\vec{r} = \vec{a} + (\vec{b} - \vec{a})\lambda + (\vec{c} - \vec{a})\mu \tag{29}$$

($\vec{r}$ is an arbitrary point on that plane.)

25.4. **l**) Check that the points A, B, C are obtained from (29) by suitable values of the parameters.

25.4. **m**)

Fig25.4. **2**: Inscribed and circumscribed sphere of a tetrahedron.

Give the equation of a sphere passing through all corners (circumscribed sphere), see fig. 2.
($\vec{r}$ is an arbitrary point on that sphere.)
RESULT:

$$(\vec{r} - \vec{s})^2 = R^2 \tag{30}$$

with

$$R = |OS| \overset{(24)}{=} \frac{1}{2}\sqrt{\frac{3}{2}} \tag{31}$$

25.4. **o**) Give the equation of a sphere touching the four faces.

HINT: The center of that sphere is the point S given by the position-vector $\overline{OS}$. One point on (the surface of) that sphere is $\vec{h}_0$, see f). $\vec{h}_0$ and $\overline{OS}$ are parallel.

RESULT: The same as (30) but

$$R = h_0 - |OS| \quad \overset{(16)(20)}{=} \quad \sqrt{\frac{2}{3}} - \frac{1}{2}\sqrt{\frac{3}{2}} = \frac{1}{4}\sqrt{\frac{2}{3}} \tag{32}$$

25.Ex 5: Volume of a cuboid calculated by wedge product and determinants

Consider three orthogonal vectors

$$\begin{aligned} \vec{a} &= (a,\ 0,\ 0) \\ \vec{b} &= (0,\ 0,\ b) \\ \vec{c} &= (0,\ c,\ 0) \end{aligned} \tag{1}$$

with

$$a > 0, \quad b > 0, \quad c > 0. \tag{2}$$

They span a cuboid with volume

$$V = abc \tag{3}$$

25.5. **a**) Verify this by calculating the wedge product.

(Solution:)

$\vec{a} \times \vec{b}$ is calculated by the scheme

$$\begin{matrix} a & 0 & 0 \\ 0 & 0 & b \end{matrix} \tag{4}$$

i.e.

$$\vec{a} \times \vec{b} = (0, \, -ab, \, 0) \tag{5}$$

Thus

$$(\vec{a} \times \vec{b})\vec{c} = 0 \cdot 0 - abc + 0 \cdot 0 = -abc \tag{6}$$

We have obtained the oriented volume which, in this case, turned out to be negative. The absolute value of (6) is (3).

25.5. **b**) Write the wedge product as a determinant.
RESULT:

$$(\vec{a} \times \vec{b})\vec{c} = \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{vmatrix} =: \ \det \tag{7}$$

25.5. **c**) A determinant is calculated by the following rule: take any row (or column) then multiply each element by the corresponding sign given by

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \tag{8}$$

and the corresponding **subdeterminant**[$\stackrel{\mathbf{G}}{=}$ `Unterdeterminante`] then add up these products.
For definiteness we take the uppermost row in (7) (and also in (8))

$$\begin{vmatrix} \boxed{a} & \boxed{0} & \boxed{0} \\ 0 & 0 & b \\ 0 & c & 0 \end{vmatrix} \tag{7'}$$

$$\begin{pmatrix} \boxed{+} & \boxed{-} & \boxed{+} \\ - & + & - \\ + & - & + \end{pmatrix} \tag{8'}$$

and we need the subdeterminants

$$\begin{pmatrix} \boxed{S_1} & \boxed{S_2} & \boxed{S_3} \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \tag{9}$$

Calculate S_1, S_2 and S_3.
HINT 1: A subdeterminant to an element is found by crossing out the columns and the rows to which that element belongs.
HINT 2: A 2×2 determinant is defined as:

$$\boxed{\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \beta\gamma} \tag{10}$$

$$S_1 = \begin{vmatrix} \cancel{a} & \cancel{0} & \cancel{0} \\ \cancel{0} & 0 & b \\ \cancel{0} & c & 0 \end{vmatrix} = \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = 0 - bc = -bc \tag{11}$$

$$S_2 = \begin{vmatrix} \cancel{a} & \cancel{0} & \cancel{0} \\ 0 & \cancel{0} & b \\ 0 & \cancel{c} & 0 \end{vmatrix} = \begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0 \tag{12}$$

$$S_2 = \begin{vmatrix} \cancel{a} & \cancel{0} & \cancel{0} \\ 0 & 0 & \cancel{b} \\ 0 & c & \cancel{0} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix} = 0 - 0 = 0 \tag{13}$$

25.5. **d**) According to the rule in c) the determinant is

$$\det \; = \boxed{a}\,\boxed{(+1)}\,\boxed{S_1} + \boxed{0}\,\boxed{(-1)}\,\boxed{S_2} + \boxed{0}\,\boxed{(+1)}\,\boxed{S_3} \tag{14}$$

Calculate det and show that it is identical to (6).

(Solution:)

$$a(-bc) + 0 + 0 = -abc \tag{15}$$

25.5. **e**) We would like to calculate the determinant in (10) by a similar rule with the signs

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix} \tag{16}$$

and the rule

$$|\alpha| = \alpha \tag{17}$$

Rem: The vertical lines in (17) denote the determinant, not an absolute value. Normally this does not lead to confusion since the determinant of a 1×1 matrix, as in (17), is rarely used.
Hint: For definiteness take the uppermost row

$$\begin{vmatrix} \boxed{\alpha} & \boxed{\beta} \\ \gamma & \delta \end{vmatrix} \tag{10'}$$

$$\begin{pmatrix} \boxed{+} & \boxed{-} \\ - & + \end{pmatrix} \tag{16'}$$

$$\begin{pmatrix} \boxed{s_1} & \boxed{s_2} \\ ? & ? \end{pmatrix} \tag{18}$$

Calculate the subdeterminates s_1 and s_2.

(Solution:)

$$s_1 = \begin{vmatrix} \cancel{\alpha} & \cancel{\beta} \\ \cancel{\gamma} & \delta \end{vmatrix} = |\delta| = \delta \tag{19}$$

$$s_2 = \begin{vmatrix} \cancel{\alpha} & \cancel{\beta} \\ \gamma & \cancel{\delta} \end{vmatrix} = |\gamma| = \gamma \tag{20}$$

25.5. **f**) Verify (10) by calculating the determinant in (10).

(Solution:)

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \boxed{\alpha}\ \boxed{(+1)}\ \boxed{s_1} + \boxed{\beta}\ \boxed{(-1)}\ \boxed{s_2} = \alpha\delta + \beta(-1)\gamma = \alpha\delta - \beta\gamma \qquad \text{q.e.d.}$$

26 Leibniz's product rule for vectors

26.**Q 1: Leibniz's product rule for vectors**

Give the product rule for the following quantities ($\vec{a} = \vec{a}(t),\ \vec{b} = \vec{b}(t),\ \ \lambda = \lambda(t)$):

26.1. **a**) $\frac{d}{dt}\,(\vec{a}\vec{b}) = ?$

(Solution:)

$$\boxed{\frac{d}{dt}\,(\vec{a}\vec{b}) = \frac{d\vec{a}}{dt}\,\vec{b} + \vec{a}\,\frac{d\vec{b}}{dt}} \quad \text{product rule for scalar product} \tag{1}$$

26.1. **b**) $\frac{d}{dt}\,(\vec{a} \times \vec{b}) = ?$

(Solution:)

$$\boxed{\frac{d}{dt}\,(\vec{a} \times \vec{b}) = \frac{d\vec{a}}{dt}\ \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}} \quad \text{product rule for vector product} \tag{2}$$

(Mind the order!)

26.1. **c**) $\frac{d}{dt}\,(\lambda\vec{b}) = ?$

(Solution:)

$$\boxed{\frac{d}{dt}\,(\lambda\vec{b}) = \frac{d\lambda}{dt}\,\vec{b} + \lambda\,\frac{d\vec{b}}{dt}} \quad \text{product rule for scalar multiplication} \tag{3}$$

26.**Ex 2: Proof of Leibniz's product rule for vectors**

26.2. **a**) Prove the Leibniz product rule for scalar product for 2-dimensional vectors ($n = 2$).

(Solution:)

$$\begin{aligned} \vec{a} &= \vec{a}(t) = (a_1(t),\, a_2(t)) \equiv (a_1,\, a_2) \\ \vec{b} &= \vec{b}(t) = (b_1(t),\, b_2(t)) \equiv (b_1,\, b_2) \end{aligned} \tag{1}$$

$$\frac{d}{dt}(\vec{a}\,\vec{b}) \equiv (\vec{a}\,\vec{b})^{\cdot} = (a_1\,b_1 + a_2\,b_2)^{\cdot} \overset{\clubsuit}{=} a_1\dot{b}_1 + \dot{a}_1 b_1 + a_2\dot{b}_2 + \dot{a}_2 b_2 = \vec{a}\,\dot{\vec{b}} + \dot{\vec{a}}\,\vec{b} \tag{2}$$

♣ derivative of a sum = sum of the derivatives Leibniz's product rule for scalar function.

26.2. **b**) Prove Leibniz's product rule for vector product (first component only, for a 3-dimensional vector ($n = 3$)).

(Solution:)

According to the scheme

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \tag{3}$$

the first component of $\vec{a} \times \vec{b}$ is given by

$$(\vec{a} \times \vec{b})_1 = a_2 b_3 - a_3 b_2 \tag{4}$$

Since differentiation of a vector is componentwise, i.e. the first component of the derivative of a vector is the derivative of the first component, we have to prove

$$\begin{aligned} \left[\frac{d}{dt}(\vec{a} \times \vec{b})\right]_1 &\equiv \left[(\vec{a} \times \vec{b})^{\cdot}\right]_1 = \left[(\vec{a} \times \vec{b})_1\right]^{\cdot} \\ &\overset{(4)}{=} (a_2 b_3 - a_3 b_2)^{\cdot} = (\dot{a}_2 b_3 - \dot{a}_3 b_2) + (a_2 \dot{b}_3 - a_3 \dot{b}_2) \end{aligned} \tag{5}$$

On the other hand, the right-hand side of Leibniz's product rule states (for the first component)

$$\left[\frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}\right]_1 = \dot{a}_2 b_3 - \dot{a}_3 b_2 + a_2 \dot{b}_3 - a_3 \dot{b}_2$$

where we have used a formula analogous to (4) and

$$\left[\frac{d\vec{a}}{dt}\right]_2 \equiv (\dot{\vec{a}})_2 = (a_2)^{\cdot} \equiv \dot{a}_2$$

q.e.d.

26.**Ex 3: Velocity and acceleration of circular motion**

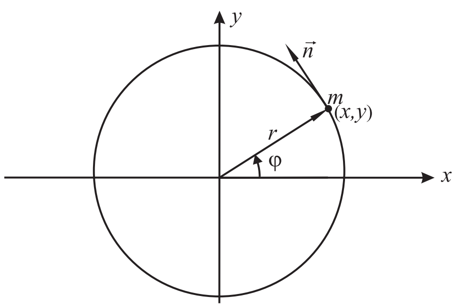


Fig26.3. **1**: Calculation of velocity and acceleration of a mass point m moving along a circle with radius r and with arbitrary angle $\varphi = \varphi(t)$. $\vec{n}$ is a unit-tangential vector to the circle.

A mass point (e.g. a car on a road, or the earth around the sun) moves along a circle with radius r (see fig. 1) in an arbitrary (e.g. non-uniform) way:

$$\varphi = \varphi(t) \tag{1}$$

26.3. **a**) Calculate the position vector

$$\vec{r} = \vec{r}(t) = (x,\, y) = (x(t),\, y(t)) \tag{2}$$

RESULT:

$$\vec{r} = r(\cos\varphi(t),\, \sin\varphi(t)) \tag{3}$$

(Solution:)

$$\begin{aligned} &\text{projection } x = r\cos\varphi \\ &\text{side-projection } y = r\sin\varphi \end{aligned} \tag{4}$$

26.3. **b**) Calculate the velocity

$$\vec{v} = \vec{v}(t) = (v_x(t),\, v_y(t)) = \dot{\vec{r}} \tag{5}$$

HINTS:

$$v_x = \dot{x} = \frac{dx}{dt} = \frac{dx}{d\varphi}\frac{d\varphi}{dt} \tag{6}$$

according to the chain rule, and we can write:

$$\frac{d\varphi}{dt} = \dot{\varphi}(t) \tag{7}$$

RESULT:

$$\vec{v} = r\dot{\varphi}(-\sin\varphi,\, \cos\varphi) \tag{8}$$

(Solution:)

$$x = r\cos\varphi, \quad \frac{dx}{d\varphi} = -r\sin\varphi, \tag{9}$$

$$\frac{dx}{dt} = \frac{dx}{d\varphi}\frac{d\varphi}{dt} = -r\sin\varphi \cdot \dot{\varphi} \tag{10}$$

Similarly:

$$v_y = \frac{dy}{dt} = r\cos\varphi \cdot \dot{\varphi} \tag{11}$$

26.3. **c**) Prove that at any momemt $\vec{r}$ and $\vec{v}$ are orthogonal.
HINT: Calculate their scalar product using (3) and (8).

(Solution:)

$$\vec{r}\vec{v} = r^2\dot{\varphi}(-\cos\varphi\sin\varphi + \sin\varphi\cos\varphi) = 0 \quad \Rightarrow \quad \vec{r} \perp \vec{v} \tag{12}$$

26.3. **d**) Alternatively, prove that directly from

$$\vec{r}^{\,2} = r^2 \tag{3}$$

HINT: $\vec{r}^{\,2} = \vec{r}\vec{r}$. Use the rule for differentiation of a scalar product. Use $r =$ const, the symmetry of the scalar product and $\dot{\vec{r}} = \vec{v}$.

(Solution:)

$$\frac{d}{dt}\vec{r}^{\,2} = \frac{d}{dt}r^2 = 0 \tag{14}$$

since r^2 is constant

$$\vec{r}\dot{\vec{r}} + \dot{\vec{r}}\vec{r} = 0 \tag{15}$$

$$2\vec{r}\dot{\vec{r}} = 0 \tag{16}$$

$$\vec{r}\dot{\vec{r}} = 0, \text{ i.e. } \vec{r}\vec{v} = 0 \quad \Rightarrow \quad \vec{r} \perp \vec{v} \tag{17}$$

26.3. **e**) $\vec{v}$ is a tangential vector along the path (i.e. the circle). Introduce the corresponding unit-vector,

$$\vec{n} = \hat{v} \tag{18}$$

see fig. 1.
Calculate $\vec{n}$ and express $\vec{v}$ in terms of $\vec{n}$.
HINT: first calculate $v = |\vec{v}|$ using (8) and $\sin^2 + \cos^2 = 1$. Assume $\dot{\varphi} > 0$ for reasons of simplicity.
RESULT:

$$v = r\dot{\varphi} \tag{19}$$

$$\vec{n} = (-\sin\varphi,\, \cos\varphi) \tag{20}$$

$$\vec{v} = r\dot{\varphi}\,\vec{n} \tag{21}$$

(Solution:)

$$v = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{r^2\dot{\varphi}^2\,\underbrace{(\sin^2\varphi + \cos^2\varphi)}_{1}} = r|\dot{\varphi}| = r\dot{\varphi} \tag{22}$$

$\dot{\varphi}$ is called the **angular velocity** [$\stackrel{\mathbf{G}}{=}$ `Winkelgeschwindigkeit`], mostly denoted by ω,

$$\boxed{\omega \equiv \dot{\varphi} \qquad \text{angular velocity}} \tag{23}$$

$$\boxed{\begin{array}{l} v = r\dot{\varphi} \\ \text{tangential velocity } v = \text{ radius } \cdot \text{ angular velocity } \dot{\varphi} \end{array}} \tag{24}$$

$$\vec{n} = \hat{v} = \frac{\vec{v}}{v} = (-\sin\varphi,\, \cos\varphi) \tag{25}$$

solving (25) for $\vec{v}$ gives (21).

26.3. **f**) From (8) calculate the acceleration of the mass-point

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = (\ddot{x},\, \ddot{y}) = (a_x,\, a_y) = (\dot{v_x},\, \dot{v_y}) \tag{26}$$

and express the result in terms of $\hat{r}$ and $\vec{n}$.
HINT: use the product rule for differentiation.
RESULT:

$$\boxed{\vec{a} = r\ddot{\varphi}\,\vec{n} - r\dot{\varphi}^2\hat{r}} \tag{27}$$

In words:

$$\boxed{\text{The centripetal acceleration is radius } \cdot \text{ velocity squared.}} \tag{28}$$

REM:
centripetal = to strive in the dirction of the center, from Latin *petere* = **to strive for** [$\stackrel{\mathbf{G}}{=}$ `streben nach`].

$$\boxed{\text{The tangential acceleration is radius } \cdot \text{ angular acceleration } (\ddot{\varphi}).} \tag{29}$$

(Solution:)

$$v_x \overset{(8)}{=} -r\dot{\varphi}\sin\varphi \tag{30}$$

$$a_x = \dot{v}_x = -r\ddot{\varphi}\sin\varphi - r\dot{\varphi}\cos\varphi \cdot \dot{\varphi} \tag{31}$$

$$v_y \overset{(9)}{=} r\dot{\varphi}\cos\varphi \tag{32}$$

$$a_y = \dot{v}_y = r\ddot{\varphi}\cos\varphi - r\dot{\varphi}\sin\varphi \cdot \dot{\varphi} \tag{33}$$

$$\vec{a} = r\ddot{\varphi}\underbrace{(-\sin\varphi,\ \cos\varphi)}_{\vec{n}} - r\dot{\varphi}^2\underbrace{(\cos\varphi,\ \sin\varphi)}_{\hat{r}} \tag{34}$$

26.3. **g**) Specialize this for constant angular velocity

$$\dot{\varphi} = \omega = \text{ const.} \tag{35}$$

RESULT:

$$\vec{a} = -r\omega^2\hat{r} \tag{36}$$

(Solution:)

$$\ddot{\varphi} = 0 \tag{37}$$

26.3. **h**) According to Newton's second law, the force $\vec{F}$ necessary to produce the acceleration $\vec{a}$ of a mass m is

$$\boxed{\vec{F} = m\vec{a}} \qquad \text{(Newton's second law)} \tag{38}$$

in words:

$$\boxed{\text{force} = \text{mass} \ \cdot \ \text{acceleration}} \tag{38'}$$

and Newton's law of gravitation states (see fig. 2),

$$\boxed{\vec{F}_2 = \frac{\gamma\, m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}\,(\vec{r}_1 - \vec{r}_2)} \qquad \text{(Newton's law of gravitation)} \tag{39}$$

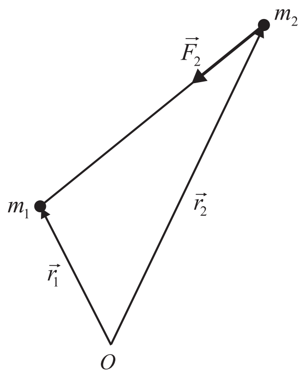


Fig26.3. **2**: Newton's law of gravitation gives the force $\vec{F}_2$ on a mass m_2 at position $\vec{r}_2$ produced by the gravitational attraction due to mass m_1 at position $\vec{r}_1$.

where

$$\boxed{\gamma = 6.7 \cdot 10^{-8} \text{ cm gr}^{-1}\text{sec}^{-2}} \qquad \text{(gravitational constant)} \tag{40}$$

Specialize this for $\vec{r}_1 \equiv 0$, $m_1 = M =$ mass of the sun, $m_2 = m =$ mass of the earth. Write $\vec{r}_2 = \vec{r}$ and use (36), (38) and (39) to calculate the distance r between the earth and the sun (assume that $r =$ const. and $\omega =$ const.) .

RESULT:

$$r = \left(\frac{\gamma M}{\omega^2}\right)^{\frac{1}{3}}, \qquad \omega = \frac{2\pi}{1 \text{ year}} \tag{41}$$

(Solution:)

$$\vec{F} = \vec{F}_2 \stackrel{(39)}{=} -\frac{\gamma m M}{r^3}\vec{r} \stackrel{(38)}{=} m\vec{a} \stackrel{(36)}{=} -mr\omega^2\hat{r} \tag{42}$$

since $\vec{r} = r\hat{r}$ (43)

$$\frac{\gamma M}{r^2} = r\omega^2 \tag{44}$$

26.Ex 4: Mathematical pendulum

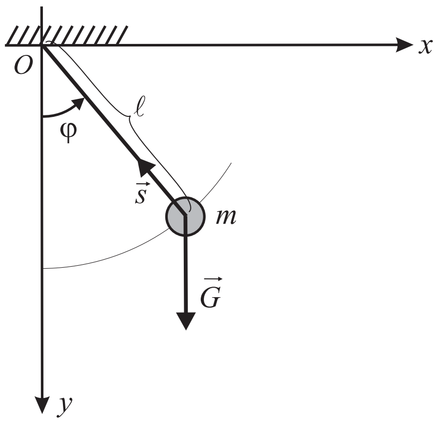


Fig26.4. **1**: Mathematical pendulum with mass m and thread length ℓ. The mass is being acted upon by vertical gravitational force $\vec{G}$ and by a strain force $\vec{S}$ along the thread.

A mathematical pendulum is a point-mass m suspended by a thread of fixed length ℓ, see fig. 1. We consider the simple case of the pendulum moving in a plane - the x-y-plane.

26.4. **a**) Calculate the position vector $\vec{r}$ of mass m (relative to the origin O) in terms of the elongation angle φ.

RESULT:

$$\vec{r} = (x,\, y) = \ell(\sin\varphi,\, \cos\varphi) \tag{1}$$

(Solution:)

y is the projection of length ℓ, x is the side projection, thus

$$\begin{aligned} y &= \ell\cos\varphi \\ x &= \ell\sin\varphi \end{aligned} \tag{2}$$

26.4. **b**) Formally calculate the length of vector $\vec{r}$ in (1) and check that it is ℓ.

(Solution:)

$$r = |\vec{r}\,| = \sqrt{x^2+y^2} = \sqrt{\ell^2\sin^2\varphi + \ell^2\cos^2\varphi} = \sqrt{\ell^2\underbrace{(\sin^2\varphi+\cos^2\varphi)}_{1}} = \sqrt{\ell^2} = \ell \tag{3}$$

26.4. **c**) Calculate the unit vector $\hat{r}$ in the direction of $\vec{r}$.

RESULT:

$$\hat{r} = (\sin\varphi,\, \cos\varphi) \tag{4}$$

(Solution:)

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\ell}{\ell}(\sin\varphi,\, \cos\varphi) \tag{5}$$

26.4. **d**) Calculate the unit-vectors $\vec{e}_x$ and $\vec{e}_y$ in the direction of the x-axis and the y-axis, respectively.
RESULT:

$$\vec{e}_x = (1,\, 0), \quad \vec{e}_y = (0,\, 1) \tag{6}$$

26.4. **e**) Formally check that they are perpendicular.
HINT: Calculate their scalar product.

(Solution:)

$$\vec{e}_x \vec{e}_y = 1 \cdot 0 + 0 \cdot 1 = 0 \tag{7}$$

26.4. **f**) The gravitational force $\vec{G}$ is vertical, i.e. has direction $\vec{e}_y$ and has magnitude

$$G = |\vec{G}| = mg \tag{8}$$

where g is the (local) **gravitational acceleration due to the earth**[$\stackrel{\mathbf{G}}{=}$ `Erdbeschleunigung`], which is approximately

$$g = 9.81 \text{ m sec}^{-2} \tag{9}$$

Calculate $\vec{G}$ and also write it as a mutiple of $\vec{e}_y$.
RESULT:

$$\vec{G} = G\vec{e}_y = mg\vec{e}_y = mg(0,\, 1) \tag{10}$$

26.4. **g**)

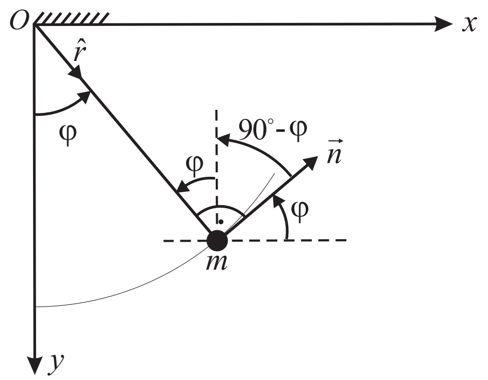


Fig26.4. **2**: The path of mass m is a circle with radius ℓ. We calculate the tangential vector $\vec{n}$ to the path which is perpendicular to the position vector $\vec{r}$.
Note that pairwise orthogonal legs lead to equal angles: $\varphi = \varphi$.

We would like to calculate a unit vector $\vec{n}$ which is a tangential vector to the path of the mass. Since the path is a circle, $\vec{n}$ is perpendicular to $\vec{r}$.
In fig. 2, besides the original elongation angle φ, you see two additional angles φ and the angle $90° - \varphi$. Prove these relationships using the following well-known rules from plane trigonometry which, in condensed form, read:

$$\boxed{\textbf{alternating angles are equal}} \tag{11}$$

$$\boxed{\textbf{pair-wise orthogonal legs} \quad \Rightarrow \quad \textbf{equal angles}} \tag{12}$$

(Solution:)

The y-axis and the dotted vertical line are parallel and are both intersected by the line of position vector $\vec{r}$. The angles on both sides of that line are called alternating angles which, by (11), are equal.
Since $\vec{r}$ and $\vec{n}$ are orthogonal, $90° - \varphi$ is a complementary angle.
The original elongation angle has **legs**`[`$\stackrel{\mathbf{G}}{=}$ `Schenkel]` y-axis and position vector $\vec{r}$. The third angle φ has legs $\vec{n}$ and the horizontal dotted line. These legs are pair-wise orthogonal thus, by (12) both angles are equal ($= \varphi$).

26.4. **h**) From the angles indicated in fig. 2 calculate the tangential vector $\vec{n}$.
RESULT:

$$\vec{n} = (\cos\varphi, \ -\sin\varphi) \tag{13}$$

(Solution:)

$$\text{projection} \quad \Rightarrow \quad n_x = 1 \cdot \cos\varphi \tag{14}$$

$$\text{side-projection} \quad \Rightarrow \quad -n_y = 1 \cdot \sin\varphi \tag{15}$$

(Alternatively:

$$\text{projection} \quad \Rightarrow \quad -n_y = 1 \cdot \cos(90^\circ - \varphi) = 1 \cdot \sin\varphi) \tag{16}$$

26.4. **i)** Check that $\vec{n}$ from (13) and $\hat{r}$ from (4) are orthogonal.

(Solution:)

$$\vec{n}\hat{r} = \sin\varphi\cos\varphi - \cos\varphi\sin\varphi = 0 \tag{17}$$

26.4. **j)**☻ We have derived $\vec{n}$ geometrically. Alternatively derive (13) analytically (i.e. formally)

$$\vec{n} = (n_x,\, n_y) \tag{18}$$

by using the fact that $\vec{n}$ is a unit-vector orthogonal to $\hat{r}$.
Hint: Solve $\vec{n}\hat{r} = 0$ for n_x. Find a common denominator. Use $\sin^2 + \cos^2 = 1$. Use fig. 2 to choose the sign of $\vec{n}$.

(Solution:)

$$1 \stackrel{!}{=} \vec{n}^2 = {n_x}^2 + {n_y}^2 \tag{19}$$

$$0 \stackrel{!}{=} \vec{n}\hat{r} \stackrel{(4)}{=} n_x \sin\varphi + n_y \cos\varphi \quad \Rightarrow \tag{20}$$

$$n_x = -\frac{\cos\varphi}{\sin\varphi}\, n_y \tag{21}$$

Thus (19) reads

$$1 = {n_y}^2 \left(\left(\frac{\cos\varphi}{\sin\varphi} \right)^2 + 1 \right) \quad \Rightarrow \quad n_y = \frac{1}{\sqrt{1 + (\frac{\cos\varphi}{\sin\varphi})^2}} \tag{22}$$

with a common denominator

$$n_y = \pm \frac{\sin\varphi}{\sqrt{\sin^2\varphi + \cos^2\varphi}} = \pm \sin\varphi \tag{23}$$

We choose the lower sign to conform to fig. 2. (The conditions of unit length and orthogonality of the tangential vector $\vec{n}$ is still unspecified by a sign.)

$$n_y = -\sin\varphi \tag{24}$$

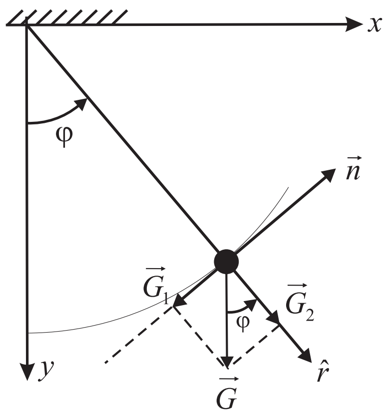
(21) now gives

$$n_x = \cos\varphi \tag{25}$$

26.4. **k**) Besides the gravity force $\vec{G}$, a second force $\vec{S}$ along the thread is acting on mass m. Up to now we only know its direction - along the thread, i.e.

$$\vec{S} = -S\hat{r} \tag{26}$$

but we do not know its magnitude S. S is just big enough (or small enough) to enforce the constant length of the thread. Therefore, $\vec{S}$ is called a **coercion force**[$\stackrel{\mathbf{G}}{=}$ `Zwangskraft`].[28] Since we can find out its strength S only after having solved the problem, we must now express the **equation of motion** of m not in the **fixed reference frame**[$\stackrel{\mathbf{G}}{=}$ `Bezugssystem`] $\vec{e}_x$, $\vec{e}_y$, but in the **moving reference frame** $\hat{r}$, $\vec{n}$.
Thus we pose the following task: break $\vec{G}$ down into a component in the direction of $\hat{r}$ and a component in the direction of $\vec{n}$.



`Fig`26.4. **3**: The gravitational force $\vec{G}$ is broken down into a component $\vec{G}_1$ in the (opposite) direction of the (instantaneous) tangent $\vec{n}$ of the path and into a component $\vec{G}_2$ in the direction of the (instantaneous) position unit-vector $\hat{r}$: $\vec{G} = \vec{G}_1 + \vec{G}_2$.

RESULT:

$$\vec{G} = mg\cos\varphi\,\hat{r} - mg\sin\varphi\,\vec{n} \tag{27}$$

---

[28]Physically, it is an ordinary force, e.g. an electromagnetic force in this case, called an intermolecular (or interatomic) force which becomes active when we try to enlarge the distance between the molecules (or atoms) of the thread. In reality, the thread will be slightly streched. Treating the intermolecular force as a coercion force is an approximation which means that the intermolecular force is always so that the length of the thread becomes exactly constant, i.e. fullfils the coercion condition $r = \ell =$ const.

26.4. **l**) In a previous exercise we calculated the acceleration corresponding to the circular motion of the pendulum mass m. The result was

$$\vec{a} = r\ddot{\varphi}\,\vec{n} - r\dot{\varphi}^2\hat{r}, \quad r \equiv \ell \tag{28}$$

Express Newton's second law which states that mass $\times$ acceleration is the (total) force, i.e. $\vec{G} + \vec{S}$.
RESULT:

$$\underbrace{m\ell\ddot{\varphi}\vec{n} - m\ell\dot{\varphi}^2\hat{r}}_{m\vec{a}} = \underbrace{mg\cos\varphi\,\hat{r} - mg\sin\varphi\,\vec{n}}_{\vec{G}}\underbrace{-S\hat{r}}_{\vec{S}} \tag{29}$$

26.4. **m**) Write this equation in components (using $\vec{n}$, $\hat{r}$ as the reference frame).
RESULT:

$$m\ell\ddot{\varphi} = -mg\sin\varphi \tag{30}$$

$$-m\ell\dot{\varphi}^2 = mg\cos\varphi - S \tag{31}$$

Equation (31) is useful for calculating the force S in the thread.

REM: The method of decomposing into the moving reference frame $\hat{r}, \hat{n}$ has now turned out to be successful: (30) is a differential equation for $\varphi = \varphi(t)$ independent of the unknown coercion force S.

26.4. **n**) The differential equation (30) cannot be solved exactly. Instead make a linear approximation for small angles φ.
RESULT:

$$\ddot{\varphi} = -\frac{g}{\ell}\varphi \tag{30$'$}$$

26.4. **o**) Show that a solution is given by

$$\varphi = \varphi_0 \sin[\omega(t - t_0)] \tag{32}$$

with

$$\omega = \sqrt{\frac{g}{\ell}} \tag{33}$$

with two integration constants φ_0 and t_0.

(Solution:)

$$\begin{aligned} \dot{\varphi} &= \omega\varphi_0\cos[\omega(t - t_0)] \\ \ddot{\varphi} &= -\omega^2\varphi_0\sin[\omega(t - t_0)] \end{aligned} \tag{34}$$

q.e.d.

26.**Ex 5: Conservation of angular momentum**

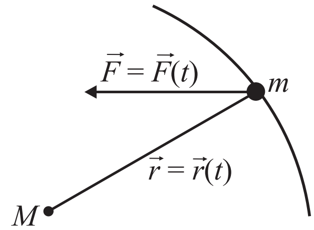


Fig26.5. **1**: A body m is moving ($\vec{r} = \vec{r}(t)$) under the influence of an arbitrary force $\vec{F} = \vec{F}(t)$. When the force $\vec{F}$ is a central force, i.e. is always directed to a fixed center M, the angular momentum of the body is conserved and it moves in a plane through M. In the gravitational case, m could be the mass of the earth and M the mass of the sun.

Relative to a fixed center M, taken as the origin of the position vector $\vec{r}$, **angular momentum**[$\stackrel{\mathbf{G}}{=}$ `Drehimpuls`] is defined as

$$\boxed{\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}}} \qquad \text{(angular momentum)} \tag{1}$$

where $\vec{p}$ is the ordinary or **linear momentum**[$\stackrel{\mathbf{G}}{=}$ `Linearimpuls`]

$$\boxed{\vec{p} = m\vec{v} = m\dot{\vec{r}}} \qquad \text{(linear momentum)} \tag{2}$$

26.5. **a**) From Newton's second law

$$\vec{F} = m\ddot{\vec{r}} \tag{3}$$

deduce the following **angular momentum law**[$\stackrel{\mathbf{G}}{=}$ `Drehimpulssatz`]

$$\boxed{\dot{\vec{L}} = \vec{N}} \qquad \text{(angular momentum law)} \tag{4}$$

where $\vec{N}$ is the **torque**[$\stackrel{\mathbf{G}}{=}$ `Drehmoment`]

$$\boxed{\vec{N} = \vec{r} \times \vec{F}} \qquad \text{(torque)} \tag{5}$$

HINTS: m is a constant.

$$\vec{a} \times \vec{a} \equiv 0 \tag{6}$$

(Solution:)

$$\dot{\vec{L}} \overset{(1)}{=} (m\vec{r} \times \dot{\vec{r}})^{\cdot} \overset{\clubsuit}{=} \underbrace{\dot{m}}_{0\,\bullet}(\vec{r} \times \dot{\vec{r}}) + m(\vec{r} \times \dot{\vec{r}})^{\cdot} \overset{\spadesuit}{=} m\underbrace{(\dot{\vec{r}} \times \dot{\vec{r}})}_{0\,(6)} + m(\vec{r} \times \ddot{\vec{r}})$$
$$= \vec{r} \times m\ddot{\vec{r}} \overset{(3)}{=} \vec{r} \times \vec{F} \overset{(5)}{=} \vec{N} \qquad \text{q.e.d.} \tag{7}$$

♣	Leibniz's product rule for scalar multiplication
•	m is a constant
♠	Leibniz's product rule for vector product

26.5. **b**) A force is called a **central force** [$\overset{\mathrm{G}}{=}$ `Zentralkraft`] when it is always directed toward (or against) a *fixed* center. Taking that center as the origin of the position vetcors (M in fig. 1), we have

$$\boxed{\vec{F} = \lambda(\vec{r}, t)\vec{r}} \qquad \text{(definition of a central force)} \tag{8}$$

i.e. λ is arbitrary.
Prove:

$$\boxed{\text{central force} \quad \Rightarrow \quad \text{conservation of angular momentum}} \tag{9}$$

HINT: Calculate the torque.

(Solution:)

$$\vec{N} \overset{(5)}{=} \vec{r} \times \vec{F} = \lambda\vec{r} \times \vec{r} \overset{(6)}{=} 0 \quad \overset{(4)}{\Rightarrow} \quad \dot{\vec{L}} = 0 \quad \Rightarrow \quad \vec{L} = \text{ const.} \tag{10}$$

26.5. **c**) ☻ The following statement is true:

$$\boxed{\text{conservation of angular momentum} \quad \Rightarrow \quad \text{plane motion}} \tag{11}$$

Prove this for $\vec{L} \neq 0$.
HINT: Use linear independence expressed by vector product. Consider the plane spanned by $\vec{r}$ and $\vec{v}$. Is that plane dependent on t? What is the normal vector of that plane (i.e. a vector perpendicular to that plane)?

(Solution:)

$\vec{L} \neq 0 \;\Rightarrow\; \vec{r}, \vec{v}$ are linearly independent, i.e. they determine (span) a unique plane (at any t). That plane is independent of t, i.e. it is a *fixed plane* since it is always perpendicular to $\vec{L}$, which is constant, and since it must go through the fixed point M. Since m is always on that fixed plane, m performs a plane motion.

27 ☻ Complex numbers

27.Q 1: Complex numbers

27.1. **a**) What are **complex numbers** (historical introduction)? Why the word 'complex'? What is an **imaginary number**? What is i ?

(Solution:)

Complex numbers are "numbers" of the form:

$$\boxed{z = x + iy} \quad x, y \,\epsilon\, \mathbb{R}, \;\; z \,\epsilon\, \mathbb{C} \tag{1}$$

($\mathbb{C}$ = set of all complex numbers),
where

$$\boxed{i = \sqrt{-1}} \tag{2}$$

('complex' because they are composed of two real numbers x, y and i)
There is no real number $i \,\epsilon\, \mathbb{R}$. Therefore, i was called the **imaginary unit** (**imaginary**[$\stackrel{\mathbf{G}}{=}$ `imaginär, frei erfunden`]) and ib with $b \,\epsilon\, \mathbb{R}$ are called imaginary numbers.

Historically one had observed very early that by assuming all calculation rules known from real numbers together with

$$\boxed{i^2 = -1} \tag{2'}$$

one obtains a consistent, beautiful and very useful mathematical theory.

27.1. **b**) What is the **real part**[$\stackrel{\mathbf{G}}{=}$ `Realteil`], what is the **imaginary part**[$\stackrel{\mathbf{G}}{=}$ `Imaginärteil`] of a complex number z, and how are they denoted?

(Solution:)

$$z = x + iy, \;\; x, y \,\epsilon\, \mathbb{R} \quad \Rightarrow \quad \Re z = x, \;\; \Im z = y \tag{3}$$

$\Re$ = real part, $\Im$ = imaginary part.
Alternative notation:
Re $z = x$, Im $z = y$

Rem 1: Measured physical quantities are always real. Sometimes two related physical quantities can be combined in an elegant way as the real- and imaginary part of a **complex physical quantity**.

Rem 2: Sometimes iy (instead of y) is called the imaginary part of z.

27.Ex 2: Hieronimo Cardano's problem from the year 1545

27.2. **a**) Split the number 10 into two parts x and y so that their product is 40.

HINTS: Formulate the problem as two equations. Eliminate y to obtain a quadratic equation, which is solved formally.
RESULTS:

$$\begin{aligned} x &= 5 + i\sqrt{15} \\ y &= 5 - i\sqrt{15} \end{aligned} \tag{1}$$

(Solution:)

$$\left| \begin{array}{ll} x + y & = 10 \\ xy & = 40 \end{array} \right. \tag{2}$$

$$y = \frac{40}{x} \tag{3}$$

$$x + \frac{40}{x} = 10 \tag{4}$$

$$x^2 + 40 - 10x = 0 \tag{5}$$

$$x = \frac{10 \pm \sqrt{(10)^2 - 4 \cdot 1 \cdot 40}}{2} = \frac{10 \pm \sqrt{-60}}{2} = \tag{6}$$

$$= 5 \pm \sqrt{-15} = 5 \pm i\sqrt{15} \tag{7}$$

$$y = 10 - x = 5 \mp i\sqrt{15} \tag{8}$$

Since it is irrelevant what is x and what is y, we take the upper sign in (7)(8) to obtain (1).

27.2. **b**) Using the formal rule for i, check that the product of x and y is indeed 40.

(Solution:)

$$xy = (5 + i\sqrt{15})(5 - i\sqrt{15}) \stackrel{\spadesuit}{=} 25 - \underbrace{i^2}_{-1} \sqrt{15}^2 = 25 + 15 = 40 \tag{9}$$

$\boxed{\spadesuit \text{ third binomic formula}}$ q.e.d.

27.**Q 3: Addition and multiplication in components**

For two complex numbers $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2, (x_i, y_i \in \mathbb{R})$ compute the following expression by decomposing the result into real- and imaginary parts.

27.3. **a**) $z_1 + z_2 = ?$

(Solution:)

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = \tag{1}$$

$$\boxed{z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)} \tag{2}$$

Addition of complex numbers is done componentwise

27.3. **b)** $z_1 z_2 = ?$

(Solution:)

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + x_1iy_2 + iy_1x_2 + iy_1iy_2 = \\ &= x_1x_2 + i(x_1y_2 + y_1x_2) + i^2y_1y_2) = \end{aligned} \tag{3}$$

$$\boxed{z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)} \quad \text{multiplication of complex numbers} \tag{4}$$

27.T 4: Fundamental theorem of algebra

In the real domain not all quadratic equations have a solution. E.g.

$$a_0 + z^2 = 0 \tag{1}$$

has only a solution for $a_0 \leq 0$. For mathematicians the theory of complex numbers and functions is a favourite topic since in the complex domain a lot of beautiful theorems are valid, whose analog in the real domain are plagued with ugly exceptions.

The first example is the so called **fundamental theorem of algebra**[$\stackrel{\mathbf{G}}{=}$ `Fundamentalsatz der Algebra`]:

Every algebraic equation (of n^{th}-order), i.e.

$$\sum_{k=0}^{n} a_k z^k = 0, \quad (z, a_k \,\epsilon\, \mathbb{C}, a_n \neq 0) \tag{2}$$

has at least one solution z.

REM: In general it has n solutions (e.g. $n = 2$ for quadratic equations). In special cases it has fewer solution, but at least one.
E.g. $a_0 = 0$ in (1) has one solution only. In these cases we say that two (or more) solutions have **coalesced**[$\stackrel{\mathbf{G}}{=}$ `zusammengefallen`], or we speak of **multiple solutions**[$\stackrel{\mathbf{G}}{=}$ `Mehrfachlösungen`].

27.**Ex 5: Example: square root of i**
As a special case show that

$$z^2 = i \tag{1}$$

has two solutions. Calculate them, i.e. $\sqrt{i}$, giving the answer decomposed in real and imaginary parts.

(Solution:)

Let be

$$z = \sqrt{i} =: x + iy \quad x, y \,\epsilon\, \mathbb{R} \tag{2}$$

$$z^2 = x^2 - y^2 + 2ixy \stackrel{!}{=} i \tag{3}$$

$$\boxed{\begin{array}{c}\textbf{every complex equation is equivalent to 2 real equations:}\\ \textbf{the real and the imaginary part of that complex equation.}\end{array}} \tag{4}$$

$$\Re z^2 = x^2 - y^2 = \Re i = 0 \tag{5}$$

$$\Im z^2 = 2xy = \Im i = 1 \tag{6}$$

$$(5) \Rightarrow x = \pm y, \qquad (6) \Rightarrow \pm 2y^2 = 1 \tag{7}$$

Since $y \,\epsilon\, \mathbb{R}$, only the upper sign in (7) is possible:

$$x = y, \quad 2y^2 = 1 \tag{7'}$$

i.e.

$$x = y = \pm\frac{\sqrt{2}}{2} \tag{8}$$

$$\boxed{\sqrt{i} = \pm\frac{\sqrt{2}}{2}\,(1+i)} \tag{9}$$

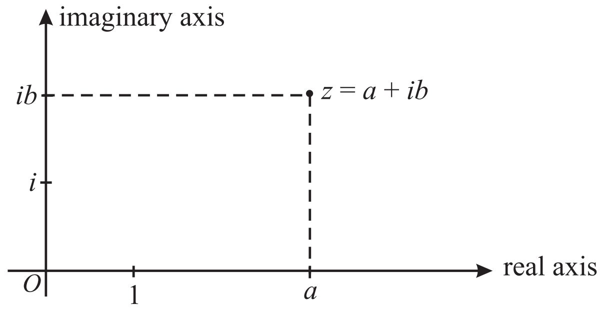
27.**Q 6: Real models of $\mathbb{C}$**

27.6. **a**) Give a real representation (i.e. a **real model**) for complex numbers.

(Solution:)

Complex numbers are not as “imaginary” as one had believed in former times. Indeed, they have a **real model** as points on a plane (the so called **complex plane**[$\stackrel{\mathbf{G}}{=}$ `komplexe Zahlenebene`], also called **Gaussian plane**[$\stackrel{\mathbf{G}}{=}$ `Gaußsche`

`Zahlenebene]`).



`Fig`27.6. **1**: Complex numbers can be viewed as points on a plane (the Gaussian plane)

The complex number $z = a + ib$ is the point with cartesian coordinates (a, b). The x-axis is called the **real axis** [$\stackrel{\mathbf{G}}{=}$ `reelle Achse`], the y-axis is called the **imaginary axis**.

REM 1: Alternatively, we could say that complex numbers form a (real) 2-vector space (with a Euclidean metric) with two distinguished orthogonal unit vectors, denoted by 1 and i. Addition of complex numbers corresponds to vector addition.

REM 2: Alternatively, we could say: Complex numbers are **pairs of real numbers**, e.g. $z_1 = (a_1, b_1)$, $z_2 = (a_2, b_2)$ with the following **definition** for addition and multiplication of pairs (see Q3 (3) and (4)):

$$z_1 + z_2 = (a_1 + a_2,\ b_1 + b_2) \tag{1}$$

$$z_1 z_2 = (a_1 a_2 - b_1 b_2,\ a_1 b_2 + b_1 a_2) \tag{2}$$

The **marvelous** [$\stackrel{\mathbf{G}}{=}$ `fabelhaft`] fact is that with these curious calculation laws for these pairs (almost) all properties and theorems known from real numbers still hold - and a lot more beautiful theorems like the fundamental theorem of algebra.

REM 3: Already negative integers $-n$ and rational numbers
m/n $(\notin \mathbb{N})$ are "imaginary" as long as numbers are conceived as the result of counting (natural numbers). But they have found a "real" interpretation and a useful application as points on a straight line, the so called **real axis** $\mathbb{R}$.

27.6. **b**) What is the geometrical interpretation of addition in $\mathbb{C}$?

(Solution:)

vector addition (when the z are regarded as position vectors in the complex plane $\mathbb{C}$).

27.6. **c**) What is the geometric interpretation of multiplication by a real number $\lambda \in \mathbb{R}$ (with proof).

(Solution:)

For $z_1 = \lambda \in \mathbb{R}$, $z = a + ib$ we have

$$\lambda z = \lambda(a + ib) = (a\lambda) + i(\lambda b) \tag{3}$$

i.e. it corresponds to multiplication by a scalar for the 2-vectors.

27.6. **d**) What is the absolute value, what is the **arcus** of a complex number (geometrical interpretation and notation).

(Solution:)

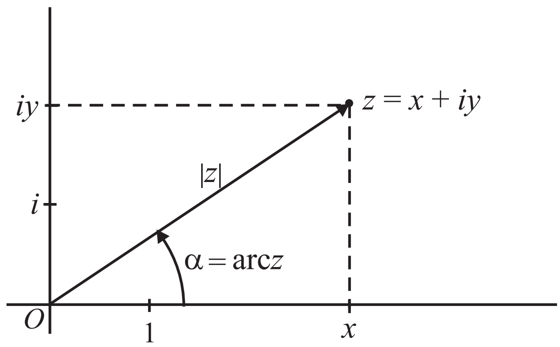


Fig27.6. **2**: Absolute value ($|z|$) and arcus (arc z) of a complex number z in the Gaussian plane

$|z|$ = absolute value is the length of the vector z.

$$\boxed{|z| =_+ \sqrt{x^2 + y^2} \quad \text{for } z = x + iy} \quad \text{absolute value of a complex number} \tag{4}$$

arcus z = angle α (measured in the mathematically positive sense) of the vector z and the real axis:

$$\boxed{\text{arcus } z = \text{arc } z = \arctan \frac{y}{x}} \tag{5}$$

27.6. **e**) Give the representation of a complex number by its absolute value and its arcus.

(Solution:)

$$\boxed{z = |z|(\cos\alpha + i \sin\alpha)} \quad \text{with } \alpha = \text{ arc } z \tag{6}$$

polar representation of a complex number

27.6. **f)** What is the geometrical interpretation of multiplication of two complex numbers.

(Solution:)

$$\boxed{z = z_1 z_2 \quad \Rightarrow \quad |z| = |z_1|\,|z_2|} \tag{7}$$

REM 1: Thus we see that the fundamental law for the absolute value: 'the absolute value of a product is the product of the absolute values of the factors' is valid also in the complex domain.

$$\boxed{z = z_1 z_2 \quad \Rightarrow \quad \text{arc } z = \text{arc } z_1 + \text{arc } z_2} \tag{8}$$

Multiplication of z_1 by z_2 has the following geometrical interpretation:
The vector z_2 is rotated by arc z_1 and its length ($|z_2|$) is multiplied by the length $|z_1|$ of z_1. (9)

REM 2: For $z_2 = \lambda \in \mathbb{R}$ the rotation is zero and complex multiplication is multiplication by a scalar.

REM 3: Multiplication of complex numbers is **not** the scalar product $\vec{z}_1\vec{z}_2$ of the corresponding vectors. Scalar product is present in $\mathbb{C}$ (see Ex 14) but rarely used. The multiplication when writing z_1z_2 is not a scalar product, but complex multiplication.

REM 4: Because of that difference, it is sometimes erroneously argued that complex numbers are not vectors and in German a new word: **Zeiger** is used instead of vectors. However, that view is incorrect: $\mathbb{C}$ *is* a vector space. But it has additional structures, e.g. **complex multiplication** and the selection of two unit vectors 1 and i.

REM 5: That complex multiplication has nothing to do with the vector product is obvious because the latter cannot be defined in a two dimensional vector space.

27.**Ex 7: Proof of the multiplication law**
Prove Q6 (7) and (8).

HINT: Use the polar representation (Ex 6 (6)) and the addition theorem for trigonometric functions.

(Solution:)

$$z_1 = |z_1|(\cos\alpha_1 + i\sin\alpha_1) \tag{1}$$

$$z_2 = |z_2|(\cos\alpha_2 + i\sin\alpha_2) \tag{2}$$

$$\begin{aligned} z_1 z_2 &= |z_1|\,|z_2|[(\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2) + i(\cos\alpha_1\sin\alpha_2 + \sin\alpha_1\cos\alpha_2)] = \\ &= |z_1|\,|z_2|[\cos(\alpha_1+\alpha_2) + i\sin(\alpha_1+\alpha_2)] \end{aligned} \tag{3}$$

q.e.d.

REM: The last expression is the polar representation of $z = z_1 z_2$. Thus arc $z = \alpha_1 + \alpha_2 =$ arc $z_1 +$ arc z_2 and $|z| = |z_1|\,|z_2|$.

27.Q 8: Complex conjugation

27.8. **a**) What is the **complex conjugate**[$\stackrel{\mathbf{G}}{=}$ `komplex-konjugiertes`] (algebraically, geometrically, 2 notations) of a complex number.

(Solution:)

$$\boxed{z^* \equiv \overline{z} = a - ib} \quad \text{for } z = a + ib, \quad a, b \,\epsilon\, \mathbb{R} \tag{1}$$

(complex conjugation)
Geometrically it is (mirror-) reflexion at the real axis.

REM: In mathematical literature overlining the number ($\overline{z}$), in physical literature starring (z^*) is more usual to denote the operation of complex conjugation.

27.8. **b**) Express **reality** by complex conjugation.

(Solution:)

$$\boxed{z = z^* \quad \Leftrightarrow \quad z \,\epsilon\, \mathbb{R}} \tag{2}$$

or in words:

A complex number is real if and only if it is identical to its complex conjugate. (2')

PROOF:

$$z = a + ib = z^* = a - ib \quad \Leftrightarrow b = 0 \tag{3}$$

27.8. **c)** What is the **square of the operation** of complex conjugation?

(Solution:)

The square of an operator is the operator applied twice in succession: $z^* = a - ib$

$$(z^*)^* = a + ib = z \tag{4}$$

or

$$\boxed{z^{**} = z} \tag{5}$$

i.e. it is the **identical operation** id, which does nothing:

$$\boxed{\mathrm{id}(z) = z} \quad \text{identical operation} \tag{6}$$

symbolically

$$\boxed{** = id} \tag{7}$$

27.8. **d)** What is the **inverse of the operation of complex conjugation**?

(Solution:)

Complex-conjugation is a **bijective mapping** (also called: a **1-1-mapping**) of $\mathbb{C}$ **unto itself**

$$\boxed{\mathbb{C}^* = \mathbb{C}} \tag{8}$$

i.e. there is a unique **inverse mapping** [$\overset{\mathbf{G}}{=}$ `Umkehrabbildung`], which from z^* leads back to z from which we have started. Because of (5) that is again *, symbolically:

$$\boxed{*^{-1} = *} \tag{9}$$

27.8. **e)** What does it mean that complex conjuagtion is an **automorphism** [$\overset{\mathbf{G}}{=}$ `strukturerhaltende Abbildung`]?

(Solution:)

$$\boxed{(z_1 + z_2)^* = z_1^* + z_2^*} \tag{10}$$

$$\boxed{(z_1 z_2)^* = z_1^* z_2^*} \tag{11}$$

(complex conjugation as an automorphism)

Rem 1: In words: the operation of addition and the operation of complex conjugation can be interchanged: the complex conjugate of a sum is the sum of the complex conjugates of the summands (and similarly for multiplication).

REM 2: In still other words: the mapping of complex conjugation is **structure preserving** (where structure means here the additive and multiplicative structure; Greak: morphos = form, structure): The **images**[$\stackrel{\mathbf{G}}{=}$ `Bilder`], i.e. the results of the mapping $(z_1^*, z_2^*, (z_1+z_2)^*, (z_1 z_2)^*)$ have the same (additive and multiplicative) relations to each other as the **originals**[$\stackrel{\mathbf{G}}{=}$ `Urbilder`] $(z_1, z_2, z_1+z_2, z_1 z_2)$.

REM 3 Only because * is an automorphism, it is a relevant operation in $\mathbb{C}$. The automorphism means that $\mathbb{C}$ has a (mirror-)symmetry.

REM 4: The symmetry means that $i = \sqrt{-1}$ is as good as $i = -\sqrt{-1}$ to define complex numbers.

REM 5: * and id are the only automorphisms of $\mathbb{C}$.

REM 6: $\mathbb{R}$ has no non-trivial automorhism (i.e. none except id).

27.**Ex 9: Reflexion at the imaginary axis is not an automorphism**
Show that

$$\tilde{z} := -a + ib \tag{1}$$

is *not* an automorphism.

(Solution:)

$$\begin{aligned}(z_1 z_2)^\sim &= [(a_1+ib_1)(a_2+ib_2)]^\sim = \\ &= [(a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)]^\sim = \\ &= (b_1b_2 - a_1a_2) + i(a_1b_2 + b_1a_2)\end{aligned} \tag{2}$$

On the other hand we have:

$$\begin{aligned}z_1^\sim z_2^\sim &= (-a_1+ib_1)(-a_2+ib_2) = \\ &= (a_1a_2 - b_1b_2) + i(-a_1b_2 - a_2b_1)\end{aligned} \tag{3}$$

which is not identical with (2)

REM:

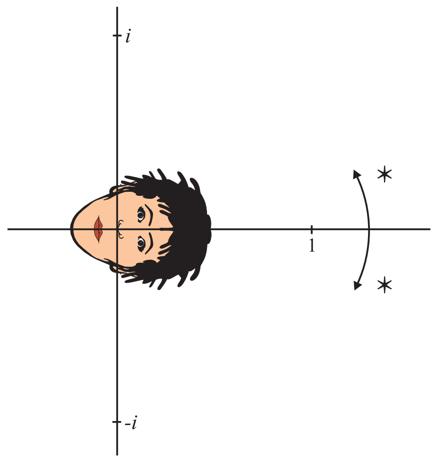


Fig27.9. **1**: Complex numbers have a mirror symmetry at the real axis, very like that face. The symmetry operation is given by complex conjugation (*).

$\mathbb{C}$ has a symmetry as the above face. * is the corresponding symmetry-operation (symmetry-mapping).

27.**Q 10: $\Re$ and $\Im$ expressed by $*$**

Express the real- and imaginary part with the help of complex conjugation.

(Solution:)

$$\boxed{\Re\, z = \frac{1}{2}(z + z^*)} \tag{1}$$

$$\boxed{\Im\, z = -\frac{1}{2}\, i\, (z - z^*)} \tag{2}$$

Proof:

$$\frac{1}{2}(z + z^*) = \frac{1}{2}(a + ib + a - ib) = a \tag{3}$$

$$-\frac{1}{2}\, i\, (z - z^*) = -\frac{1}{2}\, i\, (a + ib - a + ib) = b \tag{4}$$

27.**Q 11: $|z|$ expressed by $*$**

Express the absolute value of a complex number with the help of complex conjugation.

(Solution:)

$$|z| =_{+}\sqrt{zz^*} \tag{1}$$

$$\boxed{|z|^2 = zz^*} \tag{2}$$

PROOF:

$$zz^* = (a+ib)(a-ib) = a^2 - i^2b^2 = a^2 + b^2 = |z|^2 \tag{3}$$

27.Q 12: No < relation in $\mathbb{C}$

We had stated previously in a loose way that “almost all” properties known in $\mathbb{R}$ is also valid in $\mathbb{C}$. What is the essential structure which is lost in $\mathbb{C}$?

(Solution:)

The greater-than-relation (>) is lost, i.e. the statement $z_1 < z_2$ is meaningless, if not both z_1 and z_2 are in $\mathbb{R}$.

REM 1: Of course, $|z_1| < |\, z_2|$ is a meaningful statement.
But for real numbers $r_1,\ r_2$ we have the property that

$$\boxed{\text{either } r_1 < r_2, \text{ or } r_1 = r_2 \text{ or } r_1 > r_2} \tag{1}$$

That property is lost, if we tried to interpret z_1, z_2 as meaning $|z_1| < |z_2|$.

REM 2: It is possible to regard still higher dimensional vector spaces $V_n,\ n \geq 3$ as number systems. But then, at least, the commutative property of multiplication is also lost. The resulting theory (theory of matrices) is much less elegant and powerful than the theory of $\mathbb{C}$. Thus the generalizations from $\mathbb{N}$ to $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ to $\mathbb{C}$ has found a natural conclusion with $\mathbb{C}$.

27.Q 13: Quotients decomposed in real- and imaginary parts

27.13. **a**) Calculate z_1/z_2 by decomposing it into real- and imaginary parts. (Express the method also in words.)

(Solution:)

The trick is to enlarge the fraction by the conjugate complex of the denominator:

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{(a_1+ib_1)(a_2-ib_2)}{(a_2+ib_2)(a_2-ib_2)} = \tag{1}$$
$$= \frac{1}{a_2^2+b_2^2}\left[(a_1a_2+b_1b_2) + i(b_1a_2 - a_1b_2)\right]$$

27.13. **b**) In particular calculate $\frac{1}{i} = ?$

(Solution:)

$$\boxed{\frac{1}{i} = -i} \tag{2}$$

27.13. **c**) $\frac{1}{-i} = ?$

(Solution:)

$$\frac{1}{-i} = -\frac{1}{i} = i \tag{3}$$

27.**Ex 14: Scalar product expressed by complex multiplication and** $*$

Prove

$$\vec{z_1}\vec{z_2} = \frac{1}{2}\left(z_1 z_2^* + z_1^* z_2\right) \tag{1}$$

Rem: We have put arrow symbols above the symbols for complex numbers to indicate that their product is meant to be a scalar product (dot product), *not* complex multiplication.

(Solution:)

the right hand side of (1) gives

$$(a_1 a_2 + b_1 b_2) = \vec{z_1}\vec{z_2} \tag{2}$$

28 ☻ Complex functions

28.Q 1: Complex functions

28.1. **a**) Explain in words what is a **complex valued function**[$\stackrel{\mathbf{G}}{=}$ `komplexwertige Funktion`]

$$w = w(z) \tag{1}$$

of a complex variable z, and express it by real valued functions.

(Solution:)

To each complex number $z \in \mathbb{C}$ (or at least of a subset of $\mathbb{C}$ = range of definition of the function w) is uniquely attributed a function value $w = w(z)$ with $w \in \mathbb{C}$.

Splitting w in real- and imaginary parts, the complex function is equivalent to two real valued functions of two real variables x, y (with $z = x + iy$):

$$w = w(z) = u + iv = u(z) + iv(z) = u(x, y) + iv(x, y) \tag{2}$$

REM: So, $w = w(z)$ is a 2-dimensional vector field.

28.1. **b**) What is the meaning of functions known for real variables, e.g. $w = e^z$, $w = \sin z$, when generalized to complex arguments z ?

(Solution:)

When these functions have power series representations (given by Taylor's formula), e.g.

$$e^z = \sum_{k=1}^{\infty} \frac{1}{k!}\, z^k \tag{3}$$

it can be used as the definition of that function for $z \in \mathbb{C}$ (or at least for those z the series is convergent).

REM: That procedure is possible, because of the theorem of **uniqueness of power series**[$\stackrel{\mathbf{G}}{=}$ `Eindeutigkeit der Potenzreihenentwicklung`]: When a function is known on an infinite number of points $z \in \mathbb{C}$ (which have a limit point), e.g. on an interval of the real axis, the function can have at most one power series representation.

28.Ex 2: $w = z^2$ decomposed as two real valued functions

For the complex valued function

$$w = z^2 =: u + iv \tag{1}$$

give its real and imaginary parts u and v.

RESULTS:

$$u = x^2 - y^2 \qquad (2)$$
$$v = 2xy$$

(Solution:)

$$w = z^2 = (x+iy)^2 = x^2 + 2ixy + \underbrace{i^2}_{-1} y^2 \qquad (3)$$

28.**Q 3: Euler's formula**
Give Euler's formula connecting the exponential function with trigonomteric functions.

(Solution:)

$$\boxed{e^{iz} = \cos z + i \sin z} \qquad (1)$$

28.**Ex 4: Proof of Euler's formula**
Prove Euler's formula using the power series of all three functions (and assuming the terms in the infinite series can be rearranged).

HINT: In the power series for e^z replace $z \mapsto iz$.
Simplify i^k to $\pm 1, \ \pm i$.
Rearrange the series so that even powers of z come first. In the series of the odd powers pull an i before the series. Finally, compare with the known power series of sin and cos.

(Solution:)

$$e^{iz} = 1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \frac{(iz)^6}{6!} + \cdots \qquad (1)$$

Using

$$i^2 = -1, \quad , i^3 = -i, \quad , i^4 = 1, \quad , i^5 = i, \quad , i^6 = -1, \quad \cdots \qquad (2)$$

that reads

$$e^{iz} = (1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \pm \cdots) + \qquad (3)$$
$$+ i(\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} \mp \cdots) = \cos z + i \sin z$$

REM: Here we have assumed **tacitly**[$\stackrel{G}{=}$ `stillschweigend`] that it is allowed to rearrange the terms in an infinite series. That is a non-tivial statement which can be proved in rigorous mathematics.

28.Q 5: Phase: complex number on the unit circle

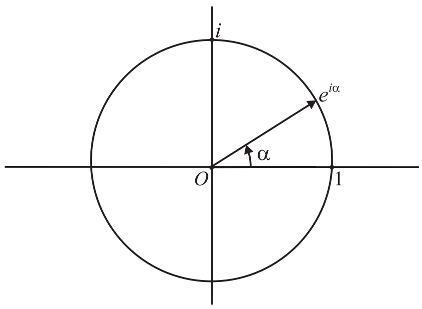
A complex number

$$\boxed{z = e^{i\alpha}, \quad (\alpha \in \mathbb{R})} \qquad (\textbf{phase}) \tag{1}$$

is called a 'phase'. (This is one of several meanings of the word 'phase' in physics and mathematics.)

28.5. **a**) Give a geometric interpretation of a phase.

(Solution:)



`Fig`28.5. **1**: A phase $e^{i\alpha}$ is a complex number on the unit circle.

According to Euler's formula

$$\text{arc } e^{i\alpha} = \alpha \tag{2}$$

$$|e^{i\alpha}| = \cos^2\alpha + \sin^2\alpha = 1 \tag{3}$$

i.e. phases are points on the **unit circle**[$\stackrel{\mathbf{G}}{=}$ `Einheitskreis`] (about 0).

28.5. **b**) Give the representation of an arbitrary complex number z as its absolute value times its phase (**exponential representation**).

(Solution:)

$$\boxed{z = |z|e^{i\alpha}, \quad \alpha = \text{arc } z} \tag{4}$$

(exponential representation of z)
in words: A complex number z is its absolute value $|z|$ times its phase $e^{i\alpha}$.

REM: Sometimes, instead of $e^{i\alpha}$, $\alpha = \text{arc}\ \alpha$ itself is called the phase of z.

28.Ex 6: Plane rotation

Derive the formula for rotation (about 0 with angle α) of a plane in cartesian coordinates, using complex methods.

HINTS: Let $z = x + iy$ be the original point, let $z' = x' + iy'$ be the rotated point (image point), so that the rotation reads

$$z' = e^{i\alpha}\, z \tag{1}$$

Give (x', y') as a function of (x, y).
Use Euler's formula.
Decompose the equation in real- and imaginary parts.

(Solution:)

$$x' + iy' = (\cos\alpha + i\sin\alpha)(x + iy) = (\cos\alpha\ x - \sin\alpha\ y) + i(\sin\alpha\ x + \cos\alpha\ y) \tag{2}$$

$$\boxed{\begin{aligned} x' &= \cos\alpha\ x - \sin\alpha\ y \\ y' &= \sin\alpha\ x + \cos\alpha\ y \end{aligned}} \quad \text{rotation in a plane by } \alpha \tag{3}$$

REM 1: This is the first example for using complex numbers to derive real results in an elegant way.

We take the occasion to collect some bracket conventions (or better: bracket habits used by physicist and mathematicians):

REM 2: **functional binding has highest priority**
Therefore,

$$\cos\alpha x := (\cos\alpha)x \tag{4}$$

REM 3: In physics very often, contradicting (4) and in a sloppy way:

$$\sin\omega t := \sin(\omega t) \tag{5}$$

since from context it is known what is meant.
To avoid such ambiguities, one writes:

$$\cos\alpha x \mapsto \cos\alpha\ x = x\cos\alpha \tag{6}$$

i.e. with an extra space before x or with x written before cos.

REM 4: The rule in Rem 2 is overridden when a typographical compact symbolism, e.g. a root symbol ($\sqrt{a}$), a fraction ($\frac{a}{b}$) or a upperset construction with an exponent (a^b) produces a compact block

$$\sin e^x := \sin(e^x) \tag{7}$$

and not $= (\sin e)^x$

REM 5:

$$\sin^2 x := (\sin x)^2 \tag{8}$$

The left hand side is a widely used, but a sloppy if not an incorrect notation. It is believed that sin alone has no meaning, so it is clear that at first $\sin x$ should be calculated and then the square of that expression has to be taken. However, $\sin^2$ does have a meaning, namely the square of the mapping sin, i.e. sin applied twice. So the left hand side of (8) could mean: $\sin(\sin x)$.

REM: The latter interpretation suggests itself when the exponent is -1, denoting the inverse function: When f is a function, the inverse function is denoted by f^{-1}

$$\boxed{f(f^{-1}(x)) = x = f^{-1}(f(x))} \quad f^{-1} \text{ is the inverse function of } f \tag{9}$$

or symbolically

$$\boxed{f \circ f^{-1} = id = f^{-1} \circ f} \tag{10}$$

In particular,

$$\boxed{\sin^{-1} x = \arcsin x} \tag{11}$$

(alternative notation for the inverse trigonometric functions)

Facit: A sloppy notation has to be interpreted according to context.

28.T 7: Properties of complex functions

Most formulae valid for real functions are again valid for their complex generalizations (because these properties are derivable formally from their power series), e.g.

$$e^{z_1+z_2} = e^{z_1} e^{z_2} \tag{1}$$

$$\sin(-z) = -\sin z \tag{2}$$

$$\cos(-z) = \cos z \tag{3}$$

$$\sin(z + 2\pi) = \sin z \tag{4}$$

$$\cos(z + 2\pi) = \cos z \tag{5}$$

$$(e^{z_1})^{z_2} = e^{z_1 z_2} \tag{6}$$

28.Ex 8: Functional properties derived from power series

From the corresponding power series derive that $\cos z$ is an even, $\sin z$ is an odd function

(Solution:)

$$\sin(-z) = \sum_{k=0}^{\infty}(-1)^k \frac{(-z)^{2k+1}}{(2k+1)!} \tag{1}$$

Since $2k+1$ is always an odd number, we have

$$(-z)^{2k+1} = (-1)^{2k+1} z^{2k+1} = -z^{2k+1} \tag{2}$$

$$\sin(-z) = -\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!} = -\sin z \tag{3}$$

(Similarly you can prove $\cos(-z) = \cos z$.)

28.Q 9: Trigonometric functions as exponentials

Give $\sin z$ and $\cos z$ in terms of e-functions.

(Solution:)

$$\boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}} \tag{1}$$

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}} \tag{2}$$

28.Ex 10: Parity of sine and cosine derived from Euler's formula

28.10. **a**) From the above definitions of sin and cos in terms of e-functions, derive again their **parity**[$\stackrel{\mathbf{G}}{=}$ `Parität`], i.e. if they are even or odd functions.

REM: Very often in mathematics it is a matter of taste what is a definition and what is a theorem. We had defined $\sin z$ by its power series known from the real domain, then

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{1}$$

is a theorem.
Alternatively, we could also regard (1) as a definition. Then its power series representation is a theorem.

(Solution:)

trivial

28.10. **b**) Derive (2) from Euler's formula and from the known parity of the trigonometric functions.

HINT: Write Eulers's formula also with $z \mapsto -z$

(Solution:)

$$e^{iz} + e^{-iz} = \cos z + i \sin z + \cos(-z) + i \sin(-z) = \\ = \cos z + i \sin z + \cos z - i \sin z = 2 \cos z \tag{2}$$

28.Q 11: e-function has an imaginary period

Give the periodicity property of the e-function.

(Solution:)

$$\boxed{e^{2\pi n i} = 1, \quad (n \,\epsilon\, \mathbb{Z})} \qquad \boxed{\textbf{e-function has period } 2\pi i} \tag{1}$$

or

$$e^{z+2\pi n i} = e^z \tag{2}$$

PROOF:

$$e^{2\pi n i} = \cos(2\pi n) + i \sin(2\pi n) = 1 + i0 = 1 \tag{3}$$

28.Ex 12: $*$ of an exponential

From the power series of the e-function and from * as an automorphism (applied to infinite series) prove

$$\boxed{(e^z)^* = e^{(z^*)}} \tag{1}$$

and in particular

$$\boxed{(e^{i\alpha})^* = e^{-i\alpha}, \quad \alpha \,\epsilon\, \mathbb{R}} \tag{2}$$

(Solution:)

$$(e^z)^* = \left(\sum_{k=0}^{\infty} \frac{1}{k!} z^k \right)^* = \sum_{k=0}^{\infty} (\frac{1}{k!} z^k)^* = \sum_{k=0}^{\infty} (\frac{1}{k!} (z^*)^k) = e^{(z^*)} \tag{3}$$

$$z = i\alpha \quad \Rightarrow \quad z^* = (i\alpha)^* = i^* \alpha = -i\alpha \tag{4}$$

28.Ex 13: Moivre's formula

Using Euler's formula and the formula for the exponential of a sum derive

$$\boxed{(\cos z + i \sin z)^n = \cos(nz) + i \sin(nz)} \quad \text{Moivre's formula} \tag{1}$$

(Prove it only for $n \in \mathbb{N}$)

(Solution:)

$$(\cos z + i \sin z)^n = \underbrace{e^{iz} e^{iz} \cdots e^{iz}}_{n-\text{times}} = e^{inz} = \cos nz + i \sin nz \tag{2}$$

REM: In the last expression we have used a sloppy notation. From context, it is clear that $\cos nz$ means $\cos(nz)$ and not $(\cos n)z$.

28.Ex 14: Addition theorem for trigonometric functions derived via $\mathbb{C}$

As a last example of elegant derivations of real results by complex methods, derive the addition theorem for trigonometric functions from

$$e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta} \tag{1}$$

HINT: Euler's theorem. Decomposition into real- and imaginary parts.

(Solution:)

$$\begin{aligned} e^{i(\alpha+\beta)} &= \cos(\alpha+\beta) + i \sin(\alpha+\beta) = \\ = e^{i\alpha} e^{i\beta} &= (\cos\alpha + i \sin\alpha)(\cos\beta + i \sin\beta) = \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \end{aligned} \tag{2}$$

Decomposing this complex equation into 2 real equations:

$$\boxed{\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}} \tag{3}$$

Index

:=, 278
=:, 278
0-vector, 275
1-1-mapping, 366

!, *see* factorials
$^\circ$, *see* degree
$'$, *see* derivative
$*$, *see* complex conjugation
0!, 270
Δ, *see* increment
$\Im$, *see* imaginary part
$\Longleftrightarrow$ (logical equivalence), 324
$\Re$, *see* real part
$\cdot$, derivative with respect to time, 150
$\circ$ (composition of operators), 174
ϵ-environment, 117
$\hbar$, 102
∞ (infinity), 116
$\int$ integral, 172
[ ] (boundaries of integrals), 177
$\neg$ (logical negation), 324
$\nparallel$ = non parallel, 301
ν, *see* frequency and period
ω, *see* angular frequency
∂ , *see* derivative, partial
$\perp$ = perpendicular = orthogonal, 305
π, 1
π with 50 decimals, 88
$\sim$ (equivalent), 175
θ-function, 128
$\times$, *see* vector product
$\vec{a}$ = vector, 272
$\mathcal{D}$, 12
$\mathcal{D}$ differential operator, 174
$\mathcal{J}$ integral operator, 174
$\mathbb{C}$, *see* complex numbers
$\mathbb{N}$, *see* numbers, natural
$\mathbb{Q}$, *see* numbers, rational
$\mathbb{R}$, *see* numbers, real
$\mathbb{Z}$ integers, 98
Å= angstrom, 102

Å= angstrom, 102
Abbildung $\stackrel{\mathbf{E}}{=}$ mapping, 280, 326
abbreviation $\stackrel{\mathbf{G}}{=}$ Abkürzung, 3
abgeschnitten $\stackrel{\mathbf{E}}{=}$ truncated, 77, 164
abhängige Variable $\stackrel{\mathbf{E}}{=}$ dependent-variable, 25
Abkürzung $\stackrel{\mathbf{E}}{=}$ abbreviation, 3
Ableitung, partielle, 216
abschätzen $\stackrel{\mathbf{E}}{=}$ estimate, 160
abschneiden $\stackrel{\mathbf{E}}{=}$ Truncate, 166
absolute error $\stackrel{\mathbf{G}}{=}$ absoluter Fehler, 77
absolute minimum or maximum, 154
absolute value $\stackrel{\mathbf{G}}{=}$ absoluter Betrag, 40
absolute value of a complex number, 363
absolute value of complex numbers expressed by complex conjugation, 368
absolute value of vector, 273
absolute value of vector expressed by scalar product, 304
absoluter Betrag $\stackrel{\mathbf{E}}{=}$ absolute value, 40
absoluter Fehler $\stackrel{\mathbf{E}}{=}$ absolute error, 77
Abweichungen $\stackrel{\mathbf{E}}{=}$ discrepancies, 105
abwickeln $\stackrel{\mathbf{E}}{=}$ unbending, 203
acceleration
    centripetal, 347
    circular motion, for, 347
    tangential, 347
acceleration $\stackrel{\mathbf{G}}{=}$ Beschleunigung, 267
acceleration as force divided by mass, 348
acceleration of circular motion, 344
addition of vectors, 277
addition theorem for trigonometric functions (sin, cos), 42
addition theorem for trigonometric functions (tan, cot), 52
adjacent $\stackrel{\mathbf{G}}{=}$ anliegend, 33
affect $\stackrel{\mathbf{G}}{=}$ beeinflussen, 119
algebra, 222
    fundamental theorem of, 360
algebraic equation, 250

allocation $\stackrel{\mathbf{G}}{=}$ Zuordnung, 14
almost all $\stackrel{\mathbf{G}}{=}$ fast alle, 117
alt $\stackrel{\mathbf{E}}{=}$ ancient, 115
alternating angles are equal, 352
ambiguity $\stackrel{\mathbf{G}}{=}$ Zweideutigkeit, 276
ambiguous $\stackrel{\mathbf{G}}{=}$ vieldeutig, 176
ambiguous $\stackrel{\mathbf{G}}{=}$ zweideutig, 91
amount $\stackrel{\mathbf{G}}{=}$ Menge, 155
amplitude, 24
An-Kathete $\stackrel{\mathbf{E}}{=}$ base, 10, 25, 26
analytic geometry, 279
analytical $\stackrel{\mathbf{G}}{=}$ formelmäßig, 38
ancient $\stackrel{\mathbf{G}}{=}$ alt, 115
Anfangs- $\stackrel{\mathbf{E}}{=}$ initial, 94
Anfangsbedingung $\stackrel{\mathbf{E}}{=}$ initial condition, 253, 265
angenähert $\stackrel{\mathbf{E}}{=}$ approximately, 1
angle's sum in a triangle, 32
angle, positive, 4
angstrom, *see* Å
angular frequency, 29, 53
angular frequency $\stackrel{\mathbf{G}}{=}$ Kreisfrequenz, 264
angular momentum $\stackrel{\mathbf{G}}{=}$ Drehimpuls, 356
angular momentum conservation, 356
angular momentum conservation yields plane motion, 357
angular momentum law $\stackrel{\mathbf{G}}{=}$ Drehimpulssatz, 356
angular velocity $\stackrel{\mathbf{G}}{=}$ Winkelgeschwindigkeit, 29, 347
angular velocity vector $\stackrel{\mathbf{G}}{=}$ Winkelgeschwindigkeitsvektor, 328
angular-frequency $\stackrel{\mathbf{G}}{=}$ Kreisfrequenz, 25
anliegend $\stackrel{\mathbf{E}}{=}$ adjacent, 33
Annäherung $\stackrel{\mathbf{E}}{=}$ approach, 120
Anordnungen $\stackrel{\mathbf{E}}{=}$ arrangements, 73
anticommutative law for vector product, 316
antiderivative $\stackrel{\mathbf{G}}{=}$ Stammfunktion, Aufleitung, 176
approach $\stackrel{\mathbf{G}}{=}$ Annäherung, 120
approximately $\stackrel{\mathbf{G}}{=}$ angenähert, 1
approximately $\stackrel{\mathbf{G}}{=}$ näherungsweise, 77
approximation
    first order, 77
    linear, 77
    lowest (non-vanishing) order, 78
approximation (linear, quadratic, zeroth-order, n-th order, 81
approximation of cos, 81
approximation of fractions, 81
approximation of general power, 81
approximation of ln, 81
approximation of sin, 81
approximation of square root, 81
approximation of tan, 81
Äquator $\stackrel{\mathbf{E}}{=}$ equator, 14
Äquivalenzklasse $\stackrel{\mathbf{E}}{=}$ equivalence class, 272
arbitrarily $\stackrel{\mathbf{G}}{=}$ willkürlich, 14
arbitrary $\stackrel{\mathbf{G}}{=}$ beliebig, 34
arbitrary $\stackrel{\mathbf{G}}{=}$ beliebiger, 243
arbitrary $\stackrel{\mathbf{G}}{=}$ willkürlichen, 239
arc, *see* arcus
arc $\stackrel{\mathbf{G}}{=}$ Bogen, 2
arc cos, multiple values of, 47
arc sin, 13
arc's length, 2
arc-arrow $\stackrel{\mathbf{G}}{=}$ Winkelböglein, 6
arcus of a complex number, 363
arcus-function, *see* arc sin, arc cos, arc tan
area
    integral, *see* integral, area
area $\stackrel{\mathbf{G}}{=}$ Fläche, 34, 155
area of circle, 34
area of parallelogram expressed by vector product, 316
arrangements $\stackrel{\mathbf{G}}{=}$ Anordnungen, 73
arrow $\stackrel{\mathbf{G}}{=}$ Pfeil, 272
associative law for scalar multiplication of vectors, 276
associative law for scalar products, 306
associative law for vector addition, 277
Ast $\stackrel{\mathbf{E}}{=}$ branch, 13

Ast, Zweig $\stackrel{\mathbf{E}}{=}$ branch, 224
atomic units, 100
attached $\stackrel{\mathbf{G}}{=}$ befestigt, 23
Aufgabe $\stackrel{\mathbf{E}}{=}$ task, 159
aufgemalt $\stackrel{\mathbf{E}}{=}$ plotted, 112
aufgespannt $\stackrel{\mathbf{E}}{=}$ spanned, 300
Aufleitung, *see* Stammfunktion
aufschneiden $\stackrel{\mathbf{E}}{=}$ cutting off, 203
Aufzählungsindex $\stackrel{\mathbf{E}}{=}$ enumeration index, 280
ausarbeiten $\stackrel{\mathbf{E}}{=}$ Elaborate, 34
ausdenken $\stackrel{\mathbf{E}}{=}$ devise, 60
Auslenkung $\stackrel{\mathbf{E}}{=}$ elongation, 24
ausreißen $\stackrel{\mathbf{E}}{=}$ tear out, 198
austauschen $\stackrel{\mathbf{E}}{=}$ Interchanging, 120
auswerten, vereinfachen $\stackrel{\mathbf{E}}{=}$ Evaluate, 228
automorphism $\stackrel{\mathbf{G}}{=}$ strukturerhaltende Abbildung, 366
automorphism for complex numbers, 366
auxiliary $\stackrel{\mathbf{G}}{=}$ Hilfs-, 29
auxiliary condition $\stackrel{\mathbf{G}}{=}$ Nebenbedingung, 226
auxiliary variable $\stackrel{\mathbf{G}}{=}$ Hilfvariable, 6
available $\stackrel{\mathbf{G}}{=}$ zur Verfügung stehend, 155
average $\stackrel{\mathbf{G}}{=}$ Durchschnitt, 105, 158, 193
average height $\stackrel{\mathbf{G}}{=}$ durchschnittliche Höhe, 193
average, definition of, 158

Bahnkurve $\stackrel{\mathbf{E}}{=}$ trajectory, 268
Balken $\stackrel{\mathbf{E}}{=}$ beam, 196, 238
Balken $\stackrel{\mathbf{E}}{=}$ upper bar, 193
base $\stackrel{\mathbf{G}}{=}$ An-Kathete, 10, 25, 26
Basic trigonometric identity, 30
Baustelle $\stackrel{\mathbf{E}}{=}$ building site, 63
Bewegungsgleichung $\stackrel{\mathbf{E}}{=}$ equation of motion, 263
beam $\stackrel{\mathbf{G}}{=}$ Balken, 196, 238
beating (of oscillations) $\stackrel{\mathbf{G}}{=}$ Schwebung, 58
Becher $\stackrel{\mathbf{E}}{=}$ cup, 154
beeinflussen $\stackrel{\mathbf{E}}{=}$ affect, 119
befestigt $\stackrel{\mathbf{E}}{=}$ attached, 23
befördert $\stackrel{\mathbf{E}}{=}$ promoted, 5
Beginn $\stackrel{\mathbf{E}}{=}$ onset, 38
begründe $\stackrel{\mathbf{E}}{=}$ Justify, 189
Behauptung $\stackrel{\mathbf{E}}{=}$ statement, 51
beiderlei Vorzeichen $\stackrel{\mathbf{E}}{=}$ both signs, 159
Beitrag $\stackrel{\mathbf{E}}{=}$ contribution, 81
bekräftigen $\stackrel{\mathbf{E}}{=}$ corrobarate, 198
beliebig $\stackrel{\mathbf{E}}{=}$ arbitrary, 34
beliebiger $\stackrel{\mathbf{E}}{=}$ arbitrary, 243
bend $\stackrel{\mathbf{G}}{=}$ biegen, 198
bequem $\stackrel{\mathbf{E}}{=}$ convenient, 116
Beschleunigung $\stackrel{\mathbf{E}}{=}$ acceleration, 267
Besonderheit $\stackrel{\mathbf{E}}{=}$ peculiarity, 36
Bezeichnungen $\stackrel{\mathbf{E}}{=}$ denotations, 111
Bezeichnungsweise $\stackrel{\mathbf{E}}{=}$ notation, 29, 36
Bezeichnungsweisen $\stackrel{\mathbf{E}}{=}$ notations, 91
Bezugssystem $\stackrel{\mathbf{E}}{=}$ reference frame, 320, 354
bi = twice, 270
biegen $\stackrel{\mathbf{E}}{=}$ bend, 198
big hand $\stackrel{\mathbf{G}}{=}$ großer Zeiger, 19
bijective mapping, 366
Bilder $\stackrel{\mathbf{E}}{=}$ images, 367
bildlich $\stackrel{\mathbf{E}}{=}$ picturesque, 118, 138
bilinearity of the scalar product, 306
bilinearity of vector product, 318
binomial, 270
binomial coefficient, 270
binomial formula, first, second, third, 270
binomial theorem, 270
bisection of angles $\stackrel{\mathbf{G}}{=}$ Winkelhalbierende, 14
bisectors of the sides $\stackrel{\mathbf{G}}{=}$ Seitenhalbierenden, 292
bisectrix of the angle $\stackrel{\mathbf{G}}{=}$ Winkelhalbierende, 35
blob $\stackrel{\mathbf{G}}{=}$ Tropfen, 5
Bogen $\stackrel{\mathbf{E}}{=}$ arc, 2
Bogenmaß $\stackrel{\mathbf{E}}{=}$ radian measure, 2

Bohr's radius, 102
bold $\stackrel{\mathbf{G}}{=}$ fett, 6, 11, 20
bold line $\stackrel{\mathbf{G}}{=}$ fette Kurve, 224
both signs $\stackrel{\mathbf{G}}{=}$ beiderlei Vorzeichen, 159
boundary, *see* integral, boundary
boundary term $\stackrel{\mathbf{G}}{=}$ Randterm, 208
brackets $\stackrel{\mathbf{G}}{=}$ Klammern, 63
branch $\stackrel{\mathbf{G}}{=}$ Ast, 13
branch $\stackrel{\mathbf{G}}{=}$ Ast, Zweig, 224
Brechungsindex $\stackrel{\mathbf{E}}{=}$ diffraction index, 159
Brennpunkte $\stackrel{\mathbf{E}}{=}$ focal points, 51
Bruch kürzen $\stackrel{\mathbf{E}}{=}$ cancel, 135
Bruchstrich $\stackrel{\mathbf{E}}{=}$ line of the fraction, 64
Bruttosozialprodukt $\stackrel{\mathbf{E}}{=}$ gross national product, 168
building site $\stackrel{\mathbf{G}}{=}$ Baustelle, 63

$\mathbb{C}$, *see* complex numbers
c, *see* velocity of light
calculator $\stackrel{\mathbf{G}}{=}$ Taschenrechner, 21, 44
calculus $\stackrel{\mathbf{G}}{=}$ Infinitesimalrechnung, 174
calculus of variations $\stackrel{\mathbf{G}}{=}$ Variationsrechnung, 229
cancel $\stackrel{\mathbf{G}}{=}$ Bruch kürzen, 135
cancel each other out $\stackrel{\mathbf{G}}{=}$ sich gegenseitig auslöschen, 247
canceling $\stackrel{\mathbf{G}}{=}$ sich aufheben, 190
cancels $\stackrel{\mathbf{G}}{=}$ herausfallen, 15
Cardano, Hieronimo, 358
cardboard $\stackrel{\mathbf{G}}{=}$ Karton, 240
cardioid $\stackrel{\mathbf{G}}{=}$ Herzkurve, 243
cartesian coordiantes, 8
Cartesius, 8
cavity $\stackrel{\mathbf{G}}{=}$ Hohlraum, 100
center of mass $\stackrel{\mathbf{G}}{=}$ Schwerpunkt, 239, 287, 296
central force $\stackrel{\mathbf{G}}{=}$ Zentralkraft, 357
central-symmetry $\stackrel{\mathbf{G}}{=}$ Zentralsymmetrie, 326
centriangle $\stackrel{\mathbf{G}}{=}$ Zentriwinkel, 2
centripetal acceleration, 347
certificates $\stackrel{\mathbf{G}}{=}$ Scheine, 120
cgs, 100
chain rule, *see* derivative, chain rule
chaos, deterministic, 88
Check $\stackrel{\mathbf{G}}{=}$ überprüfen, 51
circle $\stackrel{\mathbf{G}}{=}$ Kreis, 34
circle, area of, 34
circle, circumference of, 1
circumference, 1
circumference $\stackrel{\mathbf{G}}{=}$ Umfang, 1
circumscribed $\stackrel{\mathbf{G}}{=}$ umschrieben, 33
coalesced $\stackrel{\mathbf{G}}{=}$ zusammengefallen, 360
coefficient, 250
coercion force $\stackrel{\mathbf{G}}{=}$ Zwangskraft, 354
coincidence $\stackrel{\mathbf{G}}{=}$ Zufall, 87
coined $\stackrel{\mathbf{G}}{=}$ geprägt, 40
columns $\stackrel{\mathbf{G}}{=}$ Spalten, 75
commutative law for scalar multiplication of vectors, 276
commutative law for scalar products, 306
commutative law for vector addition, 277
compass $\stackrel{\mathbf{G}}{=}$ Zirkel, 51, 283
complementary angle $\stackrel{\mathbf{G}}{=}$ Komplementärwinkel, 39
complete differential, 219
complex conjugate $\stackrel{\mathbf{G}}{=}$ komplexkonjugiertes, 365
complex conjugation of an exponential, 377
complex number
  polar representation, 364
complex number represented by its absolute value and its arcus, 363
complex numbers, 358
  exponential representation, 373
complex plane $\stackrel{\mathbf{G}}{=}$ komplexe Zahlenebene, 361
complex valued function $\stackrel{\mathbf{G}}{=}$ komplexwertige Funktion, 371
component of a vector, 272
component of a vector in a certain direction, 301
component of vector expressed by a scalar product, 304

componentwise $\stackrel{\mathbf{G}}{=}$ komponentenweise, 274
composition $\stackrel{\mathbf{G}}{=}$ Hintereinanderausführen, 174
composition of functions $\stackrel{\mathbf{G}}{=}$ Zusammensetzung von Funktionen, 36
cone $\stackrel{\mathbf{G}}{=}$ Kegel, 200
conical $\stackrel{\mathbf{G}}{=}$ kegelförmig, 329
conservation
  angular momentum, 356
continuous $\stackrel{\mathbf{G}}{=}$ kontinuierliche, 240
contour line $\stackrel{\mathbf{G}}{=}$ Höhenlinie, 225
contribution $\stackrel{\mathbf{G}}{=}$ Beitrag, 81
contribution of n-th order, 80
convenient $\stackrel{\mathbf{G}}{=}$ bequem, 116
convention $\stackrel{\mathbf{G}}{=}$ Verabredung, 38
coordinate line, 244
coordinate-independent, 334
coordinates, 8
corkscrew $\stackrel{\mathbf{G}}{=}$ Korkenzieher, 315
corrobarate $\stackrel{\mathbf{G}}{=}$ bekräftigen, 198
cos, 9, **14**
  derivative, 141
cos as an even function, 39
sin has half-period π, 40
sin has period 2π, 40
cos of complementary angles, 39
cos of double angle, 42
$\cos x$, power series of $\cos x$, 72
cos, addition theorem, 42
cos, approximation of, 81
cos, integral of, 185
cos, power series of, 81
cosine, 9
cot, 26
cot as odd function, 41
cot has period π, 41
cot of complementary angles, 41
cotangent, 26
counter-clockwise $\stackrel{\mathbf{G}}{=}$ gegen den Uhrzeiger, 4
covariant, 334
credibility $\stackrel{\mathbf{G}}{=}$ Glaubwürdigkeit, 107
cross product, *see* vector product
cross-section $\stackrel{\mathbf{G}}{=}$ Querschnitt, 200
ctg, *see* cot
cube $\stackrel{\mathbf{G}}{=}$ Würfel, 68, 229
cuboid $\stackrel{\mathbf{G}}{=}$ Quader, 204, 226
cup $\stackrel{\mathbf{G}}{=}$ Becher, 154
curve, length of, 210
custard $\stackrel{\mathbf{G}}{=}$ Pudding, 295
cutting off $\stackrel{\mathbf{G}}{=}$ aufschneiden, 203
cycle $\stackrel{\mathbf{G}}{=}$ Umlauf, 100
cyclic permutation rule for wedge product, 331
cyclic symmetry, 334

$\mathcal{D}$ differential operator, 174
d, *see* differential
Δ, *see* increment
∂=see derivative
  partial, 217
$\mathcal{D}$, 12
damped $\stackrel{\mathbf{G}}{=}$ gedämpft, 151
dann und nur dann = genau dann $\stackrel{\mathbf{E}}{=}$ if and only if, 128
dark-shaded $\stackrel{\mathbf{G}}{=}$ dunkel-schraffiert, 203
dazwischenliegend $\stackrel{\mathbf{E}}{=}$ intermediate, 103
decadic logarithm $\stackrel{\mathbf{G}}{=}$ dekadischer Logarithmus, 91
decay $\stackrel{\mathbf{G}}{=}$ Zerfall, 94
decay-constant $\stackrel{\mathbf{G}}{=}$ Zerfallskonstante, 67
decay-constant $\stackrel{\mathbf{G}}{=}$ Zerfallskonstante, 253
decay-equation $\stackrel{\mathbf{G}}{=}$ Zerfallsgleichung, 253
decaying $\stackrel{\mathbf{G}}{=}$ zerfallende, abnehmende, 151
decimal logarithm $\stackrel{\mathbf{G}}{=}$ Zehner-Logarithmus, 91
decomposing a vector into components according to certain directions, 301
definitely divergent, 118
Definitionsbereich $\stackrel{\mathbf{E}}{=}$ domain, 12
degree, 1
dekadischer Logarithmus $\stackrel{\mathbf{E}}{=}$ decadic logarithm, 91
denominator $\stackrel{\mathbf{G}}{=}$ Nenner, 209

denotations $\stackrel{\mathbf{G}}{=}$ Bezeichnungen, 111
dense $\stackrel{\mathbf{G}}{=}$ dicht, 99
density $\stackrel{\mathbf{G}}{=}$ Dichte, 240
dependent-variable $\stackrel{\mathbf{G}}{=}$ abhängige Variable, 25
derivative, 136
  chain rule, 142, 149
  composite functions, 141
  constant can be pulled before the derivative, 141
  elementary functions, 140
  implicit, 224
  Leibniz's product rule, *see* derivative, product rule
  of a constant is zero, 140
  partial, 216
  power rule, 140
  product rule, 142
  quotient, 142
  sum, 141
derivatives
  higher, 162
Descartes, 8
determinant
  2 by 2, 322
determinant, 3 by 3, 340
deterministic chaos, 88
development of a function, *see* Taylor's formula
devise $\stackrel{\mathbf{G}}{=}$ ausdenken, 60
diameter $\stackrel{\mathbf{G}}{=}$ Durchmesser, 1
dicht $\stackrel{\mathbf{E}}{=}$ dense, 99
Dichte $\stackrel{\mathbf{E}}{=}$ density, 240
difference quotient $\stackrel{\mathbf{G}}{=}$ Differenzenquotient, 137
difference vector, 278
differential, 136
differential equation, order, 252
differential equations, 250
differential operator, 174
differential vector, 297
differential, complete, 219
differential, total, 219
differentiation, *see* derivative
Differenzenquotient $\stackrel{\mathbf{E}}{=}$ difference quotient, 137
diffraction index $\stackrel{\mathbf{G}}{=}$ Brechungsindex, 159
dimension of a vector, 272
dimensioned quantity $\stackrel{\mathbf{G}}{=}$ dimensionsbehaftete Größe, 99
dimensionless $\stackrel{\mathbf{G}}{=}$ dimensionslos, 100
dimensionsbehaftete Größe $\stackrel{\mathbf{E}}{=}$ dimensioned quantity, 99
dimensionslos $\stackrel{\mathbf{E}}{=}$ dimensionless, 100
discarded $\stackrel{\mathbf{G}}{=}$ fallenlassen, 100
discontinuous $\stackrel{\mathbf{G}}{=}$ unstetig, 127
discrepancies $\stackrel{\mathbf{G}}{=}$ Abweichungen, 105
displacement $\stackrel{\mathbf{G}}{=}$ Verschiebung, 291
displacement $\stackrel{\mathbf{G}}{=}$ Verschiebung, Verrückung, 136
distance from a line or plane, 308
distinction $\stackrel{\mathbf{G}}{=}$ Fallunterscheidung, 130
distinguish $\stackrel{\mathbf{G}}{=}$ kennzeichen, 166
distributing $\stackrel{\mathbf{G}}{=}$ verteilen, 73
distributive law for vectors, 278
divergent, definitely, 118
division of a vector by a scalar, 276
domain $\stackrel{\mathbf{G}}{=}$ Definitionsbereich, 12
doppeldeutig $\stackrel{\mathbf{E}}{=}$ double valued, 60
dot $\stackrel{\mathbf{G}}{=}$ Tupfen, 303
dot product = scalar product, 303
dotted $\stackrel{\mathbf{G}}{=}$ punktiert, 14, 69
dotted $\stackrel{\mathbf{G}}{=}$ punktierte, 214
double angle formula for sin and cos, 42
double valued $\stackrel{\mathbf{G}}{=}$ doppeldeutig, 60
Draht $\stackrel{\mathbf{E}}{=}$ wire, 16
Drehimpuls $\stackrel{\mathbf{E}}{=}$ angular momentum, 356
Drehimpulssatz $\stackrel{\mathbf{E}}{=}$ angular momentum law, 356
Drehmoment $\stackrel{\mathbf{E}}{=}$ torque, 239, 296, 356
dual logarithm $\stackrel{\mathbf{G}}{=}$ Zweier-Logarithmus, 91
dunkel-schraffiert $\stackrel{\mathbf{E}}{=}$ dark-shaded, 203
Durchmesser $\stackrel{\mathbf{E}}{=}$ diameter, 1
Durchschnitt $\stackrel{\mathbf{E}}{=}$ average, 105, 158, 193

durchschnittliche Höhe $\stackrel{\mathbf{E}}{=}$ average height, 193

e, *see* exponential function
$\vec{e}$ = unit vector, 276
eckige Klammern $\stackrel{\mathbf{E}}{=}$ square brackets, 103
economist $\stackrel{\mathbf{G}}{=}$ Wirtschaftswissenschaftler, 168
edges $\stackrel{\mathbf{G}}{=}$ Kanten, 289, 334
ϵ-environment $\stackrel{\mathbf{G}}{=}$ ϵ-Umgebung, 117
eigentlich $\stackrel{\mathbf{E}}{=}$ proper, 29
Eindeutigkeit der Potenzreihenentwicklung $\stackrel{\mathbf{E}}{=}$ uniqueness of power series, 371
eingebettet $\stackrel{\mathbf{E}}{=}$ immersed, 5
eingeschrieben $\stackrel{\mathbf{E}}{=}$ inscribed, 33
Einheit, Maßeinheit $\stackrel{\mathbf{E}}{=}$ Unit, 99
Einheits-Vektor $\stackrel{\mathbf{E}}{=}$ unit vector, 276
Einheitskreis $\stackrel{\mathbf{E}}{=}$ unit circle, 221, 373
Einschaltfunktion $\stackrel{\mathbf{E}}{=}$ switching-function, 128
Einstein's summation convention, 306
eintauchen $\stackrel{\mathbf{E}}{=}$ immersed, 160
Elaborate $\stackrel{\mathbf{G}}{=}$ ausarbeiten, 34
elctron
  charge, in gaussian electrostatic units, 102
electron
  mass, 102
ellipse, parametric representation of, 49
elongation $\stackrel{\mathbf{G}}{=}$ Auslenkung, 24
entspannte Länge $\stackrel{\mathbf{E}}{=}$ slack length, 263
Entwicklungssatz $\stackrel{\mathbf{E}}{=}$ formula for multiple vector products, 332
enumeration index $\stackrel{\mathbf{G}}{=}$ Aufzählungsindex, 280
equation of motion $\stackrel{\mathbf{G}}{=}$ Bewegungsgleichung, 263
equation, algebraic, 250
equation, differential, 250
equation, quadratic, 250
equator $\stackrel{\mathbf{G}}{=}$ Äquator, 14
equilateral $\stackrel{\mathbf{G}}{=}$ gleichschenklig, 10
equilateral triangle $\stackrel{\mathbf{G}}{=}$ gleichseitiges Dreieck, 287, 307
equivalence
  logical, 324
  negation of, 324
equivalence class $\stackrel{\mathbf{G}}{=}$ Äquivalenzklasse, 272
equivalence classes, 175
equivalent, 175
Erdbeschleunigung $\stackrel{\mathbf{E}}{=}$ gravitational acceleration due to the earth, 351
Erdbeschleunigung $\stackrel{\mathbf{E}}{=}$ gravitational acceleration on earth, 181
Ereignisse $\stackrel{\mathbf{E}}{=}$ events, 25
erhalten $\stackrel{\mathbf{E}}{=}$ obtained, 91
error
  absolute, 77
  relative, 77
error (absolute, relative), 81
error of a product: relative errors are additive, 87
error propagation
  multiple error sources, 225
Error propagation $\stackrel{\mathbf{G}}{=}$ Fehlerfortpflanzung, 159
ersetzen $\stackrel{\mathbf{E}}{=}$ substitute, 206
estimate $\stackrel{\mathbf{G}}{=}$ abschätzen, 160
Euler's formula $e^{iz} = \cos z + i \sin z$, 372
Euler's formula for sin and cos, 376
Euler's number, *see* exponential function
  defined by a limit, 125
ϵ-Umgebung $\stackrel{\mathbf{E}}{=}$ ϵ-environment, 117
Evaluate $\stackrel{\mathbf{G}}{=}$ auswerten, vereinfachen, 228
even $\stackrel{\mathbf{G}}{=}$ gerade, 39, 61
even function $\stackrel{\mathbf{G}}{=}$ gerade Funktion, 39, 167
events $\stackrel{\mathbf{G}}{=}$ Ereignisse, 25
e^x, power series of e^x, 65
exp, *see* exponential function
$\exp x$, power series of $\exp x$, 65
exponential function, 65
  derivative, 141

period, 377
exponential representation of complex numbers, 373
extrapolation, 170

f^{-1}, 35
fabelhaft $\stackrel{\mathbf{E}}{=}$ marvelous, 362
faces $\stackrel{\mathbf{G}}{=}$ Flächen, 290
factorial
0!, 270
factorials $\stackrel{\mathbf{G}}{=}$ Fakultäten, 65
Faden $\stackrel{\mathbf{E}}{=}$ string, 51
Fakultäten $\stackrel{\mathbf{E}}{=}$ factorials, 65
fallenlassen $\stackrel{\mathbf{E}}{=}$ discarded, 100
Fallunterscheidung $\stackrel{\mathbf{E}}{=}$ distinction, 130
fast alle $\stackrel{\mathbf{E}}{=}$ almost all, 117
Feder $\stackrel{\mathbf{E}}{=}$ spring, 23
Federgesetz $\stackrel{\mathbf{E}}{=}$ spring law, 263
Federkonstante $\stackrel{\mathbf{E}}{=}$ spring constant, 263
Fehler-Fortpflanzung $\stackrel{\mathbf{E}}{=}$ propagation of errors, 86
Fehlerfortpflanzung $\stackrel{\mathbf{E}}{=}$ Error propagation, 159
festgelegt $\stackrel{\mathbf{E}}{=}$ particular, 93
fett $\stackrel{\mathbf{E}}{=}$ bold, 6, 11, 20
fette Kurve $\stackrel{\mathbf{E}}{=}$ bold line, 224
first integral, 267
fit $\stackrel{\mathbf{G}}{=}$ passen, 54
fixed vectors, 296
Fläche $\stackrel{\mathbf{E}}{=}$ area, 34
Fläche $\stackrel{\mathbf{E}}{=}$ area, 155
Flächen $\stackrel{\mathbf{E}}{=}$ faces, 290
flex-point $\stackrel{\mathbf{G}}{=}$ Wendepunkt, 168
flex-points $\stackrel{\mathbf{G}}{=}$ Wendepunkte, 192
focal points $\stackrel{\mathbf{G}}{=}$ Brennpunkte, 51
Folgen $\stackrel{\mathbf{E}}{=}$ sequences, 115
force is mass times acceleration, 348
forces as vectors, 296
Form $\stackrel{\mathbf{E}}{=}$ shape, 154
form $\stackrel{\mathbf{E}}{=}$ shape, 229
formal solution, 252
formal symmetry, 228
formelmäßig $\stackrel{\mathbf{E}}{=}$ analytical, 38
Formelsammlung $\stackrel{\mathbf{E}}{=}$ formulary, 34
formula for multiple vector products $\stackrel{\mathbf{G}}{=}$ Entwicklungssatz, 332
formulary $\stackrel{\mathbf{G}}{=}$ Formelsammlung, 34
fractions, *see* numbers, rational
fractions, approximation of, 81
fractions, power series of, 81
frame $\stackrel{\mathbf{G}}{=}$ Rahmen, 195, 196
free fall $\stackrel{\mathbf{G}}{=}$ freier Fall, 267
free vectors, 296
freier Fall $\stackrel{\mathbf{E}}{=}$ free fall, 267
freiwillig $\stackrel{\mathbf{E}}{=}$ optional, 6
frequency $\stackrel{\mathbf{G}}{=}$ Häufigkeit, 25, 88
frequency and period, 53
fully-fledged $\stackrel{\mathbf{G}}{=}$ vollwertig, 214
function
implicit, 220
inverse, 222
linear, 143
functional analysis $\stackrel{\mathbf{G}}{=}$ Funktionalanalysis, 229
fundamental theorem of algebra $\stackrel{\mathbf{G}}{=}$ Fundamentalsatz der Algebra, 360
Fundamentalsatz der Algebra $\stackrel{\mathbf{E}}{=}$ fundamental theorem of algebra, 360
Funktionalanalysis $\stackrel{\mathbf{E}}{=}$ functional analysis, 229
Funktionswert $\stackrel{\mathbf{E}}{=}$ value of the function, 23

ganze Zahlen $\stackrel{\mathbf{E}}{=}$ integers, 11
Gauss, 158
gaussian bell-shaped curve $\stackrel{\mathbf{G}}{=}$ Gausssche Glockenkurve, 166
Gaussian plane $\stackrel{\mathbf{G}}{=}$ Gaußsche Zahlenebene, 362
Gausssche Glockenkurve $\stackrel{\mathbf{E}}{=}$ gaussian bell-shaped curve, 166
Gaußsche Zahlenebene $\stackrel{\mathbf{E}}{=}$ Gaussian plane, 362
Gebirge $\stackrel{\mathbf{E}}{=}$ mountain range, 193
gedämpft $\stackrel{\mathbf{E}}{=}$ damped, 151

geeignet $\stackrel{\mathbf{E}}{=}$ suitable, 56, 178
gegen den Uhrzeiger $\stackrel{\mathbf{E}}{=}$ counterclockwise, 4
Gegen-Kathete $\stackrel{\mathbf{E}}{=}$ perpendicular, 10, 25, 26
geheilt $\stackrel{\mathbf{E}}{=}$ remedied, 133
genau dann = dann und nur dann = iff = $\Leftrightarrow$, 128
Genauigkeit $\stackrel{\mathbf{E}}{=}$ precision, 34
general power, approximation of, 81
general power, power series of, 81
general solution, 250
geometric series, *see* series, geometric
geometry, analytic, 279
geprägt $\stackrel{\mathbf{E}}{=}$ coined, 40
Gerade $\stackrel{\mathbf{E}}{=}$ straigth line, 137
gerade $\stackrel{\mathbf{E}}{=}$ even, 39, 61
gerade Funktion $\stackrel{\mathbf{E}}{=}$ even function, 39, 167
Geschwindigkeit $\stackrel{\mathbf{E}}{=}$ velocity, 105
gewaltig $\stackrel{\mathbf{E}}{=}$ tremendous, 196
gewöhnlich $\stackrel{\mathbf{E}}{=}$ ordinary, 116
gist $\stackrel{\mathbf{G}}{=}$ Knackpunkt, 163, 218
glatt $\stackrel{\mathbf{E}}{=}$ smooth, 5
Glaubwürdigkeit $\stackrel{\mathbf{E}}{=}$ credibility, 107
gleichschenklig $\stackrel{\mathbf{E}}{=}$ equilateral, 10
gleichschenklig $\stackrel{\mathbf{E}}{=}$ unilateral, 28
gleichseitiges Dreieck $\stackrel{\mathbf{E}}{=}$ equilateral triangle, 287, 307
gleichzeitig $\stackrel{\mathbf{E}}{=}$ simultaneously, 41
gliding vectors, 296
grade, 1
gradient, *see* slope
gradient $\stackrel{\mathbf{G}}{=}$ Steigung, 137
graph, 11
graph paper $\stackrel{\mathbf{G}}{=}$ kariertes Papier, 282
gravitational acceleration due to the earth $\stackrel{\mathbf{G}}{=}$ Erdbeschleunigung, 351
gravitational acceleration on earth $\stackrel{\mathbf{G}}{=}$ Erdbeschleunigung, 181
gravitational attraction, Newton's law, 348
gravitational constant, 349
great diameter $\stackrel{\mathbf{G}}{=}$ große Halbachse, 49
Grenzwert $\stackrel{\mathbf{E}}{=}$ the limit, 115
gross national product $\stackrel{\mathbf{G}}{=}$ Bruttosozialprodukt, 168
große Halbachse $\stackrel{\mathbf{E}}{=}$ great diameter, 49
großer Zeiger $\stackrel{\mathbf{E}}{=}$ big hand, 19
ground state $\stackrel{\mathbf{G}}{=}$ Grundzustand, 100
growth equation $\stackrel{\mathbf{G}}{=}$ Wachstumsgleichung, 253
growth-constant $\stackrel{\mathbf{G}}{=}$ Wachstumskonstante, 67
Grundzustand $\stackrel{\mathbf{E}}{=}$ ground state, 100
guess $\stackrel{\mathbf{G}}{=}$ Vermutung, 107
guess $\stackrel{\mathbf{G}}{=}$ Voraussage, 158

$\hbar$, 102
Höhe $\stackrel{\mathbf{E}}{=}$ height, 28
Halbwertszeit $\stackrel{\mathbf{E}}{=}$ half life time, 94
half angle line $\stackrel{\mathbf{G}}{=}$ Winkelhalbierende, 223
half axis of an ellipse, 49
half life time $\stackrel{\mathbf{G}}{=}$ Halbwertszeit, 94
half-logarithmic, 112
half-period, 40
harmonic oscillator, 23
Häufigkeit $\stackrel{\mathbf{E}}{=}$ frequency, 25, 88
Hauptast $\stackrel{\mathbf{E}}{=}$ principal branch, 224
Hauptwert $\stackrel{\mathbf{E}}{=}$ principal value, 224
hebbare Singularität $\stackrel{\mathbf{E}}{=}$ removable singularity, 133
Hebelgesetz $\stackrel{\mathbf{E}}{=}$ lever principle, 239
height $\stackrel{\mathbf{G}}{=}$ Höhe, 28
herausfallen $\stackrel{\mathbf{E}}{=}$ cancels, 15
Hertz, 38
Herzkurve $\stackrel{\mathbf{E}}{=}$ cardioid, 243
Hilfs- $\stackrel{\mathbf{E}}{=}$ auxiliary, 29
Hilfvariable $\stackrel{\mathbf{E}}{=}$ auxiliary variable, 6
Hintereinanderausführen $\stackrel{\mathbf{E}}{=}$ composition, 174
Höhenlinie $\stackrel{\mathbf{E}}{=}$ contour line, 225
Hohlraum $\stackrel{\mathbf{E}}{=}$ cavity, 100
hydrogen $\stackrel{\mathbf{G}}{=}$ Wasserstoff, 100

hydrogen atom $\overset{\mathbf{G}}{=}$ Wasserstoffatom, 102
Hyperbel $\overset{\mathbf{E}}{=}$ hyperbola, 213
hyperbola $\overset{\mathbf{G}}{=}$ Hyperbel, 213
hypotenuse, 10
Hz, 38

i, 358
id, 36, *see* identical operation
id (identical operator), 174
identical function, 36
identical operation, 366
identical operator, 174
identity, 36
if and only if $\overset{\mathbf{G}}{=}$ dann und nur dann = genau dann, 128
$\Im$, *see* imaginary part
im Wesentlichen $\overset{\mathbf{E}}{=}$ in essence, 13
images $\overset{\mathbf{G}}{=}$ Bilder, 367
imaginär, frei erfunden $\overset{\mathbf{E}}{=}$ imaginary, 358
Imaginärteil $\overset{\mathbf{E}}{=}$ imaginary part, 358
imaginary $\overset{\mathbf{G}}{=}$ imaginär, frei erfunden, 358
imaginary part $\overset{\mathbf{G}}{=}$ Imaginärteil, 358
Imaginary part expressed by complex conjugation, 368
immersed $\overset{\mathbf{G}}{=}$ eingebettet, 5
immersed $\overset{\mathbf{G}}{=}$ eintauchen, 160
implicit function, *see* function, implicit
implied $\overset{\mathbf{G}}{=}$ impliziert, unterstellt, angenommen, 61
impliziert, unterstellt, angenommen $\overset{\mathbf{E}}{=}$ implied, 61
improper integrals $\overset{\mathbf{G}}{=}$ uneigentliche Integrale, 214
in essence $\overset{\mathbf{G}}{=}$ im Wesentlichen, 13
inclination $\overset{\mathbf{G}}{=}$ Neigung, 30
increment, 136
increment vector, 281
increments $\overset{\mathbf{G}}{=}$ Zuwächse, 136
indefinite integral, *see* integral, indefinite
independent variable $\overset{\mathbf{G}}{=}$ unabhängige Variable, 24
index, 75
infinitely multivalued $\overset{\mathbf{G}}{=}$ unendlich vieldeutig, 13
Infinitesimalrechnung $\overset{\mathbf{E}}{=}$ calculus, 174
infinity $\overset{\mathbf{G}}{=}$ unendlich, 116
inherent $\overset{\mathbf{G}}{=}$ innewohnend, 5
initial $\overset{\mathbf{G}}{=}$ Anfangs-, 94
initial condition $\overset{\mathbf{G}}{=}$ Anfangsbedingung, 253, 265
innewohnend $\overset{\mathbf{E}}{=}$ inherent, 5
inscribed $\overset{\mathbf{G}}{=}$ eingeschrieben, 33
instantaneous velocity $\overset{\mathbf{G}}{=}$ Momentangeschwindigkeit, 140
integers, 98
integers $\overset{\mathbf{G}}{=}$ ganze Zahlen, 11
integral
  additivity in the integration range, 179
  area, 171
  area, counted negative, 179
  boundary, 171
  constant can be pulled before the integral, 178
  double, 235
  improper, 213
  indefinite, 173
  integration range, 179
  linear combinations of, 179
  multiple, 235
  substitution method, 206
  sum, 178
integral of 1/x, 185
integral of cos, 185
integral of sin, 186
integral operator, 174
integral, first, 267
integral, second, 267
integrand, 171
integration
  logarithmic, 209
  partial, 208
integration as the inverse of differentiation, 174
integration constant, 176
integration interval, *see* integral, inte-

gration range
integration range, *see* integral, integration range
integration variable, 173
interchange $\overset{\mathbf{G}}{=}$ vertauschen, 35
Interchanging $\overset{\mathbf{G}}{=}$ austauschen, 120
intermediate $\overset{\mathbf{G}}{=}$ dazwischenliegend, 103
invariance of scalar products, lengths and angles, 311
invariant= scalar, 274
inverse function $\overset{\mathbf{G}}{=}$ Umkehrfunktion, 13
inverse function $\overset{\mathbf{G}}{=}$ Umkehrfunktion, inverse Funktion, 222
inverse mapping $\overset{\mathbf{G}}{=}$ Umkehrabbildung, 366
irrational number, 2

$\mathcal{J}$ integral operator, 174
Jahrtausende $\overset{\mathbf{E}}{=}$ millenniums, 115
Justify $\overset{\mathbf{G}}{=}$ begründe, 189

Kanten $\overset{\mathbf{E}}{=}$ edges, 289, 334
kariertes Papier $\overset{\mathbf{E}}{=}$ graph paper, 282
Karton $\overset{\mathbf{E}}{=}$ cardboard, 240
Kegel $\overset{\mathbf{E}}{=}$ cone, 200
kegelförmig $\overset{\mathbf{E}}{=}$ conical, 329
Keil $\overset{\mathbf{E}}{=}$ wedge, 331
kennzeichen $\overset{\mathbf{E}}{=}$ distinguish, 166
kernel symbol $\overset{\mathbf{G}}{=}$ Kernsymbol, 272
Kernsymbol $\overset{\mathbf{E}}{=}$ kernel symbol, 272
Klammern $\overset{\mathbf{E}}{=}$ brackets, 63
kleine Halbachse $\overset{\mathbf{E}}{=}$ small diameter, 49
Knackpunkt $\overset{\mathbf{E}}{=}$ gist, 163, 218
Komplementärwinkel $\overset{\mathbf{E}}{=}$ complementary angle, 39
komplex-konjugiertes $\overset{\mathbf{E}}{=}$ complex conjugate, 365
komplexe Zahlenebene $\overset{\mathbf{E}}{=}$ complex plane, 361
komplexwertige Funktion $\overset{\mathbf{E}}{=}$ complex valued function, 371
komponentenweise $\overset{\mathbf{E}}{=}$ componentwise, 274
kontinuierliche $\overset{\mathbf{E}}{=}$ continuous, 240
Korkenzieher $\overset{\mathbf{E}}{=}$ corkscrew, 315
Kreis $\overset{\mathbf{E}}{=}$ circle, 34
Kreisel $\overset{\mathbf{E}}{=}$ spinning top, 329
Kreisfrequenz $\overset{\mathbf{E}}{=}$ angular frequency, 264
Kreisfrequenz $\overset{\mathbf{E}}{=}$ angular-frequency, 25
Kronecker, 98

lapse $\overset{\mathbf{G}}{=}$ Zeit-Spanne, 94
law of the center of mass $\overset{\mathbf{G}}{=}$ Schwerpunktsatz, 296
ld, *see* log
leg $\overset{\mathbf{G}}{=}$ Schenkel, 10
legs $\overset{\mathbf{G}}{=}$ Schenkel, 33, 352
Leibniz's product rule
  for vectors, 343
Leibniz's product rule for derivatives, *see* derivative, product rule
length of a curve, 210
length of a vector, 272
length of vector, 273
length of vector expressed by scalar product, 304
lever principle $\overset{\mathbf{G}}{=}$ Hebelgesetz, 239
lg, *see* log
limit from the left $\overset{\mathbf{G}}{=}$ linksseitiger Limes, 130
limit from the right $\overset{\mathbf{G}}{=}$ rechtsseitiger Limes, 130
line of action $\overset{\mathbf{G}}{=}$ Wirkungslinie, 296
line of the fraction $\overset{\mathbf{G}}{=}$ Bruchstrich, 64
linear, 2
linear combination of vectors, 299
linear dependence expressed by vector product, 323
linear function, 143
linear independence expressed by vector product, 323
linear momentum $\overset{\mathbf{G}}{=}$ Linearimpuls, 356
Linearimpuls $\overset{\mathbf{E}}{=}$ linear momentum, 356
linearly dependent, 300
linearly independent, 300
linksseitiger Limes $\overset{\mathbf{E}}{=}$ limit from the left, 130
Lipshitz-condition, 253

ln, *see* log
local minimum or maximum, 154
log, 91
  derivative, 141
logarithm, *see* log
logarithm, approximation of, 81
logarithm, power series of, 81
logarithmic integration, 209
logarithmic paper, 108
logarithmic scale, 108
logical OR $\stackrel{\mathbf{G}}{=}$ logisches ODER, 305
logisches ODER $\stackrel{\mathbf{E}}{=}$ logical OR, 305
lower boundary $\stackrel{\mathbf{G}}{=}$ untere Grenze, 171
lower sum $\stackrel{\mathbf{G}}{=}$ Untersumme, 173

main theorem
  calculus, 174
mankind $\stackrel{\mathbf{G}}{=}$ Menschheit, 115
mantissa, 102
mapping $\stackrel{\mathbf{G}}{=}$ Abbildung, 280, 326
marvelous $\stackrel{\mathbf{G}}{=}$ fabelhaft, 362
Maßzahl $\stackrel{\mathbf{E}}{=}$ Measure number, 100
mathematically positive, 4
matrices = plural of matrix, 75
matrix, 75
maximum, 164
  absolute, 154
  local, 154
Measure number $\stackrel{\mathbf{G}}{=}$ Maßzahl, 100
Mehrfachlösungen $\stackrel{\mathbf{E}}{=}$ multiple solutions, 360
Menge $\stackrel{\mathbf{E}}{=}$ amount, 155
Menge $\stackrel{\mathbf{E}}{=}$ set, 11, 298
Menschheit $\stackrel{\mathbf{E}}{=}$ mankind, 115
mile, 105
millenniums $\stackrel{\mathbf{G}}{=}$ Jahrtausende, 115
minimum, 164
  absolute, 154
  local, 154
mirror symmetry $\stackrel{\mathbf{G}}{=}$ Spiegelsymmetrie, 14, 35
mirror symmetry $\stackrel{\mathbf{G}}{=}$ Spiegelsymmetrieachse, 50
mirror-symmetry $\stackrel{\mathbf{G}}{=}$ Spiegelung, 71
MKS, 100
modulo, 11
Moivre's formula, 377
momentaneous velocity, 150
Momentangeschwindigkeit $\stackrel{\mathbf{E}}{=}$ instantaneous velocity, 140
monomial, 270
mountain range $\stackrel{\mathbf{G}}{=}$ Gebirge, 193
mph, 105
multiple solutions $\stackrel{\mathbf{G}}{=}$ Mehrfachlösungen, 360
multivalued $\stackrel{\mathbf{G}}{=}$ vieldeutig, 13

$\mathbb{N}$, *see* numbers, natural
$\mathbb{N}^*$, *see* numbers, natural
$\mathbb{N}_o$, *see* numbers, natural
ν, *see* frequency and period
$\vec{n}$ = unit vector, 276
n = dimension of vector space, 274
n-tuple, 272
näher bestimmen $\stackrel{\mathbf{E}}{=}$ qualify, 23
näherungsweise $\stackrel{\mathbf{E}}{=}$ approximately, 77
Nährlösung $\stackrel{\mathbf{E}}{=}$ nutrient solution, 254
Nebenbedingung $\stackrel{\mathbf{E}}{=}$ auxiliary condition, 226
negation, 324
Neigung $\stackrel{\mathbf{E}}{=}$ inclination, 30
Nenner $\stackrel{\mathbf{E}}{=}$ denominator, 209
new grades, 1
Newton' s axiom, zeroth, 295
Newton's law of gravitation, 348
Newton's second law, 348
nicht-strenge $\stackrel{\mathbf{E}}{=}$ non-rigorous, 198
non-rigorous $\stackrel{\mathbf{G}}{=}$ nicht-strenge, 198
normal projection, 9
notation $\stackrel{\mathbf{G}}{=}$ Bezeichnungsweise, 29, 36
notations $\stackrel{\mathbf{G}}{=}$ Bezeichnungsweisen, 91
null sequence $\stackrel{\mathbf{G}}{=}$ Nullfolge, 118
null vector is parallel to any vector, 316
null-vector, 275
null-vector is orthogonal to any vector, 305
Nulldurchgang $\stackrel{\mathbf{E}}{=}$ zero crossing, 37
Nulldurchgang $\stackrel{\mathbf{E}}{=}$ zero-passage, 24
Nullfolge $\stackrel{\mathbf{E}}{=}$ null sequence, 118

Nullpunkt $\stackrel{\mathbf{E}}{=}$ origin, 38
Nullstellen $\stackrel{\mathbf{E}}{=}$ zeroes, 11
number systems, 98
numbers
  complex, *see* complex numbers
  integer, 98
  irrational, 99
  natural, 98
  pure, 100
  rational, 98
  real, 99
numerator $\stackrel{\mathbf{G}}{=}$ Zähler, 209
nutrient solution $\stackrel{\mathbf{G}}{=}$ Nährlösung, 254

$\circ$, 36
ω, *see* angular frequency
Oberfläche $\stackrel{\mathbf{E}}{=}$ surface, 69
Oberfläche $\stackrel{\mathbf{E}}{=}$ surface area, 226
Obersumme $\stackrel{\mathbf{E}}{=}$ upper sum, 173
obtained $\stackrel{\mathbf{G}}{=}$ erhalten, 91
obvious $\stackrel{\mathbf{G}}{=}$ offensichtlich, 206
odd $\stackrel{\mathbf{G}}{=}$ ungerade, 39, 61
odd function $\stackrel{\mathbf{G}}{=}$ ungerade Funktion, 39
offensichtlich $\stackrel{\mathbf{E}}{=}$ obvious, 206
onset $\stackrel{\mathbf{G}}{=}$ Beginn, 38
operand, 174
operator, 174
operator product, 174
operator, identical, 174
operator, trivial, 174
optional $\stackrel{\mathbf{G}}{=}$ freiwillig, 6
orbiting $\stackrel{\mathbf{G}}{=}$ umkreisen, 100
order (of an approximation), 81
order, of a differential equation, 252
ordinary $\stackrel{\mathbf{G}}{=}$ gewöhnlich, 116
Ordinate $\stackrel{\mathbf{E}}{=}$ ordinate, 11
ordinate $\stackrel{\mathbf{G}}{=}$ Ordinate, 11
orientation, 5
orientation of a vector, 272
oriented volume, *see* volume, orientation of
origin $\stackrel{\mathbf{G}}{=}$ Nullpunkt, 38
originals $\stackrel{\mathbf{G}}{=}$ Urbilder, 367
orthogonal = perpendicular, 305
orthogonality expressed by scalar product, 305
Ortsvektor $\stackrel{\mathbf{E}}{=}$ position vector, 280
Ortsvektor $\stackrel{\mathbf{E}}{=}$ radius vector, 280
oscillation equation $\stackrel{\mathbf{G}}{=}$ Schwingungsgleichung, 263
oscillator $\stackrel{\mathbf{G}}{=}$ Schwinger, 23
oscillator, harmonic, 23

pairwise orthogonal gives equal angles, 352
parallelepipedon $\stackrel{\mathbf{G}}{=}$ Spat, 331
parallelity expressed by vector product, 316
parallelogramm construction, 282
parallelogramm rule for vector addition, 277
parameter, 29
parameter representation
  plane, 310
  straight line, 308
Parameterdarstellung eines Kreises $\stackrel{\mathbf{E}}{=}$ parametric representation of a circle, 28
parametric representation of a circle $\stackrel{\mathbf{G}}{=}$ Parameterdarstellung eines Kreises, 28
Parität $\stackrel{\mathbf{E}}{=}$ parity, 376
parity $\stackrel{\mathbf{G}}{=}$ Parität, 376
partial derivative $\stackrel{\mathbf{G}}{=}$ partielle Ableitung, 216
partial integration, *see* integration, partial
Particular $\stackrel{\mathbf{G}}{=}$ spezielle, 7
particular $\stackrel{\mathbf{G}}{=}$ festgelegt, 93
particular solution $\stackrel{\mathbf{G}}{=}$ spezielle Lösung, 253
partielle Ableitung $\stackrel{\mathbf{E}}{=}$ partial derivative, 216
passen $\stackrel{\mathbf{E}}{=}$ fit, 54
peculiarity $\stackrel{\mathbf{G}}{=}$ Besonderheit, 36
penalty $\stackrel{\mathbf{G}}{=}$ Strafe, 158
pendulum
  mathematical, 349
percent, 30

perimeter, 1
period and frequency, 53
period, primitive, 12
permutation, 73
perpendicular $\stackrel{G}{=}$ Gegen-Kathete, 10, 25, 26
perpendicular $\stackrel{G}{=}$ senkrecht, 10, 203
perpendicular = orthogonal, 305
Pfeil $\stackrel{E}{=}$ arrow, 272
phase, 24
phase = complex number on the unit circle, 373
phase shift, 36
phase-shift $\stackrel{G}{=}$ Phasenverschiebung, 24
Phasenverschiebung $\stackrel{E}{=}$ phase-shift, 24
photon number flux density $\stackrel{G}{=}$ Photonenzahlflussdichte, 22
Photonenzahlflussdichte $\stackrel{E}{=}$ photon number flux density, 22
π, 1
π with 50 decimals, 88
pico, 100
picturesque $\stackrel{G}{=}$ bildlich, 118, 138
piecewise continuous $\stackrel{G}{=}$ stuckweise stetig, 129
Planck's constant, *see* $\hbar$
plane
  parameter representation, 310
plotted $\stackrel{G}{=}$ aufgemalt, 112
point
  stationary, 154
point-symmetry $\stackrel{G}{=}$ Punkt-Symmetrie, 326
polar coordinates $\stackrel{G}{=}$ Polarkoordinaten, 243
polar representation of complex number, 364
Polarkoordinaten $\stackrel{E}{=}$ polar coordinates, 243
poles $\stackrel{G}{=}$ Polstellen, 41
Polstellen $\stackrel{E}{=}$ poles, 41
polygons $\stackrel{G}{=}$ Vieleck, 34
position vector $\stackrel{G}{=}$ Ortsvektor, 280
position vector as an ordinary vector, 330
positive definite, 167
Potenz $\stackrel{E}{=}$ power, 62
Potenzen $\stackrel{E}{=}$ powers, 60
potenzieren $\stackrel{E}{=}$ raising to a power, 92
Potenzregel $\stackrel{E}{=}$ power rule, 140
Potenzreihenentwicklung $\stackrel{E}{=}$ power series, 65
Potenzreihenentwicklung von $\cos x$, 72
Potenzreihenentwicklung von $e^x = \exp x$, 65
Potenzreihenentwicklung der allgemeinen Potenz, 81
Potenzreihenentwicklung von Brüchen, 81
Potenzreihenentwicklung von cos, 81
Potenzreihenentwicklung von ln, 81
Potenzreihenentwicklung von sin, 81, 165
Potenzreihenentwicklung von tan, 81
Potenzreihenentwicklung von Wurzel, 81
power $\stackrel{G}{=}$ Potenz, 62
power rule, *see* derivative, power rule
power rule $\stackrel{G}{=}$ Potenzregel, 140
power series $\stackrel{G}{=}$ Potenzreihenentwicklung, 65
power series of $\cos x$, 72
power series of $e^x = \exp x$, 65
power series of cos, 81
power series of fractions, 81
power series of general power, 81
power series of ln, 81
power series of sin, 81, 165
power series of square root, 81
power series of tan, 81
powers $\stackrel{G}{=}$ Potenzen, 60
precision $\stackrel{G}{=}$ Genauigkeit, 34
prefactor $\stackrel{G}{=}$ Vorfaktor, 67
preliminary $\stackrel{G}{=}$ Vorbereitung, 227
previous $\stackrel{G}{=}$ vorhergehend, 56
principal branch $\stackrel{G}{=}$ Hauptast, 224
principal value $\stackrel{G}{=}$ Hauptwert, 224
principle of least squares $\stackrel{G}{=}$ Prinzip der

kleinsten Fehlerquadrate, 158
principle of uniqueness of quantum systems, 100
Prinzip der kleinsten Fehlerquadrate $\stackrel{\mathbf{E}}{=}$ principle of least squares, 158
priority
  addition and multiplication, 63
  exponents, 63
  functional binding, 374
  powers or exponentiation, 65
  subtraction and division, 64
probability $\stackrel{\mathbf{G}}{=}$ Wahrscheinlichkeit, 88
product of operators, 174
product rule, *see* Leibniz's product rule
product rule for derivatives, *see* derivative, product rule
projection, orthogonal, 9
promoted $\stackrel{\mathbf{G}}{=}$ befördert, 5
propagation of errors $\stackrel{\mathbf{G}}{=}$ Fehler-Fortpflanzung, 86
proper $\stackrel{\mathbf{G}}{=}$ eigentlich, 29
proportional, 2
ps, *see* pico
pseudo-probability, 88
pseudo-vector, 316, 328
Pudding $\stackrel{\mathbf{E}}{=}$ custard, 295
Punkt-Symmetrie $\stackrel{\mathbf{E}}{=}$ point-symmetry, 326
punktiert $\stackrel{\mathbf{E}}{=}$ dotted, 14, 69
punktierte $\stackrel{\mathbf{E}}{=}$ dotted, 214
Pythagoras, 27
Pythagorean theorem $\stackrel{\mathbf{G}}{=}$ Satz des Pythagoras, 27

$\mathbb{Q}$, *see* numbers, rational
Quader $\stackrel{\mathbf{E}}{=}$ cuboid, 204, 226
quadrant, 8
Quadrat $\stackrel{\mathbf{E}}{=}$ square, 33
quadratic, equation, 250
Quadratwurzel $\stackrel{\mathbf{E}}{=}$ square root, 34, 60
quadrieren $\stackrel{\mathbf{E}}{=}$ squaring, 34
qualify $\stackrel{\mathbf{G}}{=}$ näher bestimmen, 23
Querschnitt $\stackrel{\mathbf{E}}{=}$ cross-section, 200
quotient decomposed into real and imaginary parts, 369
quotient rule, *see* derivative, quotient

$\mathbb{R}$, *see* numbers, real
rad, 2
rad = 1, 101
radian, *see* rad, 3
radian measure $\stackrel{\mathbf{G}}{=}$ Bogenmaß, 2
radicand, 60
radioactive decay $\stackrel{\mathbf{G}}{=}$ radioaktiver Zerfall, 94
radioaktiver Zerfall $\stackrel{\mathbf{E}}{=}$ radioactive decay, 94
radius vector $\stackrel{\mathbf{G}}{=}$ Ortsvektor, 280
Rahmen $\stackrel{\mathbf{E}}{=}$ frame, 195, 196
raising to a power $\stackrel{\mathbf{G}}{=}$ potenzieren, 92
Rakete $\stackrel{\mathbf{E}}{=}$ rocket, 111
random errors $\stackrel{\mathbf{G}}{=}$ zufällige Fehler, 159
Randterm $\stackrel{\mathbf{E}}{=}$ boundary term, 208
range $\stackrel{\mathbf{G}}{=}$ Wertebereich, 12, 192
Rate $\stackrel{\mathbf{E}}{=}$ rate, 94
rate $\stackrel{\mathbf{G}}{=}$ Rate, 94
ratio $\stackrel{\mathbf{G}}{=}$ Verhältnis, 30
rationals, *see* numbers, rational
Raumdiagonale $\stackrel{\mathbf{E}}{=}$ (space-) diagonal, 68
$\Re$, *see* real part
real axis $\stackrel{\mathbf{G}}{=}$ reelle Achse, 362
real axis $\stackrel{\mathbf{G}}{=}$ Zahlengerade, 99
real part $\stackrel{\mathbf{G}}{=}$ Realteil, 358
Real part expressed by complex conjugation, 368
reals, *see* numbers, real
Realteil $\stackrel{\mathbf{E}}{=}$ real part, 358
Rechteck mit fetten Umrissen $\stackrel{\mathbf{E}}{=}$ solid rectangle, 193
rechtshändig $\stackrel{\mathbf{E}}{=}$ right-handed, 319
Rechtsschraube $\stackrel{\mathbf{E}}{=}$ right screw, 315
rechtsseitiger Limes $\stackrel{\mathbf{E}}{=}$ limit from the right, 130
rechtwinkliges Dreieck $\stackrel{\mathbf{E}}{=}$ right triangle, 9
rectification, 210

reelle Achse $\stackrel{E}{=}$ real axis, 362
reference frame $\stackrel{G}{=}$ Bezugssystem, 320, 354
Reihe $\stackrel{E}{=}$ series, 121
relative error $\stackrel{G}{=}$ relativer Fehler, 77
relativer Fehler $\stackrel{E}{=}$ relative error, 77
remedied $\stackrel{G}{=}$ geheilt, 133
removable singularity $\stackrel{G}{=}$ hebbare Singularität, 133
rest-position $\stackrel{G}{=}$ Ruhelage, 24
resting length $\stackrel{G}{=}$ Ruhelänge, 263
right screw $\stackrel{G}{=}$ Rechtsschraube, 315
right triangle $\stackrel{G}{=}$ rechtwinkliges Dreieck, 9
right-handed $\stackrel{G}{=}$ rechtshändig, 319
rocket $\stackrel{G}{=}$ Rakete, 111
rod $\stackrel{G}{=}$ Stab, 99, 158, 272
root, 61
rope, 15
rope $\stackrel{G}{=}$ Seil, 14
rotation as a succession of mirror-symmetries, 313
rotation in a plane expressed by complex multiplication, 374
rows $\stackrel{G}{=}$ Zeilen, 75
Ruhelage $\stackrel{E}{=}$ rest-position, 24
Ruhelänge $\stackrel{E}{=}$ resting length, 263
rutschen $\stackrel{E}{=}$ slipping, 6

$\int$ integral, 172
saddle point, 165
sample $\stackrel{G}{=}$ Stichprobe, 89
Satz des Pythagoras $\stackrel{E}{=}$ Pythagorean theorem, 27
save $\stackrel{G}{=}$ sparen, 156
scalar = invariant, 274
scalar multiplication of vector, 274
scalar product, 303
scale $\stackrel{G}{=}$ Skalierung = Maßstab, 108
Schallwellen $\stackrel{E}{=}$ sound waves, 55
schattiert $\stackrel{E}{=}$ shaded, 33
Schaukelbalken $\stackrel{E}{=}$ see-saw's bar, 239
Scheine $\stackrel{E}{=}$ certificates, 120
Scheitel $\stackrel{E}{=}$ vertices, 51
Schenkel $\stackrel{E}{=}$ leg, 10
Schenkel $\stackrel{E}{=}$ legs, 33, 352
Schenkel $\stackrel{E}{=}$ sides, 202
schluderig $\stackrel{E}{=}$ sloppy, 173
schraffierte $\stackrel{E}{=}$ shaded, 195
Schwebung $\stackrel{E}{=}$ beating (of oscillations), 58
Schwerpunkt $\stackrel{E}{=}$ center of mass, 239, 287, 296
Schwerpunktsatz $\stackrel{E}{=}$ law of the center of mass, 296
Schwinger $\stackrel{E}{=}$ oscillator, 23
Schwingungsgleichung $\stackrel{E}{=}$ oscillation equation, 263
secant, 137
second integral, 267
see-saw $\stackrel{G}{=}$ Wippe, Schaukel, 239
see-saw's bar $\stackrel{G}{=}$ Schaukelbalken, 239
Seil $\stackrel{E}{=}$ rope, 14
Seitenhalbierenden $\stackrel{E}{=}$ bisectors of the sides, 292
senkrecht $\stackrel{E}{=}$ perpendicular, 10, 203
sense of rotation, 4
separation of variables, 255
sequences
  infinite
    composite, 121
sequences $\stackrel{G}{=}$ Folgen, 115
series $\stackrel{G}{=}$ Reihe, 121
series, geometric, 124
set $\stackrel{G}{=}$ Menge, 11, 298
shaded $\stackrel{G}{=}$ schattiert, 33
shaded $\stackrel{G}{=}$ schraffierte, 195
shape $\stackrel{G}{=}$ Form, 154
shape $\stackrel{G}{=}$ form, 229
shifted $\stackrel{G}{=}$ verschieben, 55
shortest distance from a line or plane, 308
SI-units, 100
sich aufheben $\stackrel{E}{=}$ canceling, 190
sich gegenseitig auslöschen $\stackrel{E}{=}$ cancel each other out, 247

side-projection, 9
sides $\stackrel{\mathbf{G}}{=}$ Schenkel, 202
sign $\stackrel{\mathbf{G}}{=}$ Vorzeichen, 5
sigularity $\stackrel{\mathbf{G}}{=}$ Singularität, 133
simultaneously $\stackrel{\mathbf{G}}{=}$ gleichzeitig, 41
sin, 9
derivative, 141
sin as an odd function, 39
sin has half-period π, 40
sin has period 2π, 40
sin of complementary angles, 39
sin of double angle, 42
sin, addition theorem, 42
sin, approximation of, 81
sin, integral of, 186
sin, power series of, 81, 165
sin, sum of sines expressed as a product, 52
sin, zeroes of, 11
sine, 9
Singularität $\stackrel{\mathbf{E}}{=}$ sigularity, 133
Skalierung = Maßstab $\stackrel{\mathbf{E}}{=}$ scale, 108
Sketch $\stackrel{\mathbf{G}}{=}$ skizzieren, 56
skizzieren $\stackrel{\mathbf{E}}{=}$ Sketch, 56
slack length $\stackrel{\mathbf{G}}{=}$ entspannte Länge, 263
slipping $\stackrel{\mathbf{G}}{=}$ rutschen, 6
slope $\stackrel{\mathbf{G}}{=}$ Steigung, 30, 137
sloppy $\stackrel{\mathbf{G}}{=}$ schluderig, 173
small diameter $\stackrel{\mathbf{G}}{=}$ kleine Halbachse, 49
small quantity of first order, 77
smooth $\stackrel{\mathbf{G}}{=}$ glatt, 5
solid rectangle $\stackrel{\mathbf{G}}{=}$ Rechteck mit fetten Umrissen, 193
solution, formal, 252
solution, general, 250
sound waves $\stackrel{\mathbf{G}}{=}$ Schallwellen, 55
(space-) diagonal $\stackrel{\mathbf{G}}{=}$ Raumdiagonale, 68
Spalten $\stackrel{\mathbf{E}}{=}$ columns, 75
spanned $\stackrel{\mathbf{G}}{=}$ aufgespannt, 300
sparen $\stackrel{\mathbf{E}}{=}$ save, 156
Spat $\stackrel{\mathbf{E}}{=}$ parallelepipedon, 331
Spatprodukt $\stackrel{\mathbf{E}}{=}$ wedge product, 331
spezielle $\stackrel{\mathbf{E}}{=}$ Particular, 7
spezielle Lösung $\stackrel{\mathbf{E}}{=}$ particular solution, 253
sphere
vectorial equation of, 284
Spiegelsymmetrie $\stackrel{\mathbf{E}}{=}$ mirror symmetry, 14, 35
Spiegelsymmetrieachse $\stackrel{\mathbf{E}}{=}$ mirror symmetry, 50
Spiegelung $\stackrel{\mathbf{E}}{=}$ mirror-symmetry, 71
spinning top $\stackrel{\mathbf{G}}{=}$ Kreisel, 329
spitzfindig $\stackrel{\mathbf{E}}{=}$ subtle, 299
spring $\stackrel{\mathbf{G}}{=}$ Feder, 23
spring constant $\stackrel{\mathbf{G}}{=}$ Federkonstante, 263
spring law $\stackrel{\mathbf{G}}{=}$ Federgesetz, 263
square $\stackrel{\mathbf{G}}{=}$ Quadrat, 33
square brackets $\stackrel{\mathbf{G}}{=}$ eckige Klammern, 103
square root $\stackrel{\mathbf{G}}{=}$ Quadratwurzel, 34, 60
square root, approximation of, 81
square root, power series of, 81
squaring $\stackrel{\mathbf{G}}{=}$ quadrieren, 34
squashing $\stackrel{\mathbf{G}}{=}$ zerquetschen, 203
Stab $\stackrel{\mathbf{E}}{=}$ rod, 99, 158, 272
Stammfunktion, Aufleitung $\stackrel{\mathbf{E}}{=}$ antiderivative, 176
statement
logical, 324
statement $\stackrel{\mathbf{G}}{=}$ Behauptung, 51
stationary point, 154
steering wheel $\stackrel{\mathbf{G}}{=}$ Steuerrad, 168
Steigung $\stackrel{\mathbf{E}}{=}$ gradient, 137
Steigung $\stackrel{\mathbf{E}}{=}$ slope, 30, 137
Steuerrad $\stackrel{\mathbf{E}}{=}$ steering wheel, 168
Stichprobe $\stackrel{\mathbf{E}}{=}$ sample, 89
stillschweigend $\stackrel{\mathbf{E}}{=}$ tacitly, 372
Strafe $\stackrel{\mathbf{E}}{=}$ penalty, 158
straight line
parameter representation, 308
straigth line $\stackrel{\mathbf{G}}{=}$ Gerade, 137
streben nach $\stackrel{\mathbf{E}}{=}$ to strive for, 347
Streuung $\stackrel{\mathbf{E}}{=}$ variance, 89

string $\stackrel{\mathbf{G}}{=}$ Faden, 51
strukturerhaltende Abbildung $\stackrel{\mathbf{E}}{=}$ automorphism, 366
stuckweise stetig $\stackrel{\mathbf{E}}{=}$ piecewise continuous, 129
subdeterminant, 340
subdeterminant $\stackrel{\mathbf{G}}{=}$ Unterdeterminante, 340
subset relations $\stackrel{\mathbf{G}}{=}$ Untermengenbeziehungen, 99
substitute $\stackrel{\mathbf{G}}{=}$ ersetzen, 206
substitution method, *see* integral, substitution method
substitution of variables, *see* integral, substitution method
substitution of variables $\stackrel{\mathbf{G}}{=}$ Variablensubstitution, 207
subtle $\stackrel{\mathbf{G}}{=}$ spitzfindig, 299
subtraction of vectors, 278
suitable $\stackrel{\mathbf{G}}{=}$ geeignet, 56, 178
sum
  infinite, 121
  partial, 121
summation convention, *see* Einstein's summation convention
superfluous $\stackrel{\mathbf{G}}{=}$ überflüssig, 207
superposition $\stackrel{\mathbf{G}}{=}$ Überlagerung, 55
superposition of waves, 55
surface $\stackrel{\mathbf{G}}{=}$ Oberfläche, 69
surface area $\stackrel{\mathbf{G}}{=}$ Oberfläche, 226
sustained $\stackrel{\mathbf{G}}{=}$ unterstützt, 239
switching-function $\stackrel{\mathbf{G}}{=}$ Einschaltfunktion, 128
symmetry
  cyclic, 334
  formal, 228
    cyclic, 334

tacitly $\stackrel{\mathbf{G}}{=}$ stillschweigend, 372
tan, 25
tan as odd function, 41
tan has period π, 41
tan of complementary angles, 41
tan, addition theorem, 52
tan, approximation of, 81
tan, poles of, 41
tan, power series of, 81
tan, zeros of, 41
Tangens $\stackrel{\mathbf{E}}{=}$ tangent, 138
tangent, 25, 136
tangent $\stackrel{\mathbf{G}}{=}$ Tangens, 138
tangent $\stackrel{\mathbf{G}}{=}$ Tangente, 138
Tangente $\stackrel{\mathbf{E}}{=}$ tangent, 138
tangential mapping $\stackrel{\mathbf{G}}{=}$ Tangentialabbildung, 139
tangential plane $\stackrel{\mathbf{G}}{=}$ Tangentialebene, 220
Tangentialabbildung $\stackrel{\mathbf{E}}{=}$ tangential mapping, 139
Tangentialebene $\stackrel{\mathbf{E}}{=}$ tangential plane, 220
Taschenrechner $\stackrel{\mathbf{E}}{=}$ calculator, 21, 44
task $\stackrel{\mathbf{G}}{=}$ Aufgabe, 159
Taylor's formula, 162
  several variables, 217
tear out $\stackrel{\mathbf{G}}{=}$ ausreißen, 198
tearing $\stackrel{\mathbf{G}}{=}$ zerreißen, 203
term of n-th order, 80
tetrahedron, 333
tg, *see* tan, 26
the limit $\stackrel{\mathbf{G}}{=}$ Grenzwert, 115
theta-function, 128
time-lag $\stackrel{\mathbf{G}}{=}$ Zeitverschiebung, 24
tip of a vector, 272
to strive for $\stackrel{\mathbf{G}}{=}$ streben nach, 347
torque $\stackrel{\mathbf{G}}{=}$ Drehmoment, 239, 296, 356
total differential, 219
trajectory $\stackrel{\mathbf{G}}{=}$ Bahnkurve, 268
translation invariance, 299
tremendous $\stackrel{\mathbf{G}}{=}$ gewaltig, 196
trigonometric function, 10
trivial operator, 174
Tropfen $\stackrel{\mathbf{E}}{=}$ blob, 5
Truncate $\stackrel{\mathbf{G}}{=}$ abschneiden, 166
truncated $\stackrel{\mathbf{G}}{=}$ abgeschnitten, 77, 164
Tupfen $\stackrel{\mathbf{E}}{=}$ dot, 303
tuple, 272
tuple bracket, 280

überflüssig $\stackrel{E}{=}$ superfluous, 207
Überlagerung $\stackrel{E}{=}$ superposition, 55
überprüfen $\stackrel{E}{=}$ Check, 51
Umfang $\stackrel{E}{=}$ circumference, 1
Umkehrabbildung $\stackrel{E}{=}$ inverse mapping, 366
Umkehrfunktion $\stackrel{E}{=}$ inverse function, 13
Umkehrfunktion, inverse Funktion $\stackrel{E}{=}$ inverse function, 222
umkreisen $\stackrel{E}{=}$ orbiting, 100
Umlauf $\stackrel{E}{=}$ cycle, 100
umschrieben $\stackrel{E}{=}$ circumscribed, 33
unabhängige Variable $\stackrel{E}{=}$ independent variable, 24
unambiguous $\stackrel{G}{=}$ unzweideutig, 70
unbending $\stackrel{G}{=}$ abwickeln, 203
uneigentliche Integrale $\stackrel{E}{=}$ improper integrals, 214
unendlich $\stackrel{E}{=}$ infinity, 116
unendlich vieldeutig $\stackrel{E}{=}$ infinitely multivalued, 13
ungerade $\stackrel{E}{=}$ odd, 39, 61
ungerade Funktion $\stackrel{E}{=}$ odd function, 39
unilateral $\stackrel{G}{=}$ gleichschenklig, 28
uniqueness of power series $\stackrel{G}{=}$ Eindeutigkeit der Potenzreihenentwicklung, 371
Unit $\stackrel{G}{=}$ Einheit, Maßeinheit, 99
unit circle $\stackrel{G}{=}$ Einheitskreis, 221, 373
unit vector, 276
unit vector $\stackrel{G}{=}$ Einheits-Vektor, 276
unstetig $\stackrel{E}{=}$ discontinuous, 127
Unterdeterminante $\stackrel{E}{=}$ subdeterminant, 340
untere Grenze $\stackrel{E}{=}$ lower boundary, 171
Untermengenbeziehungen $\stackrel{E}{=}$ subset relations, 99
unterstützt $\stackrel{E}{=}$ sustained, 239
Untersumme $\stackrel{E}{=}$ lower sum, 173
unzweideutig $\stackrel{E}{=}$ unambiguous, 70
upper bar $\stackrel{G}{=}$ Balken, 193
upper sum $\stackrel{G}{=}$ Obersumme, 173
Urbilder $\stackrel{E}{=}$ originals, 367

value $\stackrel{G}{=}$ Wert, 11
value of the function $\stackrel{G}{=}$ Funktionswert, 23
vanish $\stackrel{G}{=}$ verschwinden, 78
Variablensubstitution $\stackrel{E}{=}$ substitution of variables, 207
variance $\stackrel{G}{=}$ Streuung, 89
Variationsrechnung $\stackrel{E}{=}$ calculus of variations, 229
vector, 272
- axial, 330
- multiplication by a scalar, 274
- ordinary, 330
- polar, 330
- pseudovector, 330

vector addition by parallelogramm construction, 282
vector field, 297
vector index, 280
vector product, 315
- multiple, 332

vector product in components, 318
vector products of coordinate unit vectors, 321
vector, addition, 277
vector, represented as its length times a unit vector, 276
vector-space $\stackrel{G}{=}$ Vektor-Raum, 298
vectorial variable $\stackrel{G}{=}$ Vektorvariable = vektorwertige Variable, 280
vectors depending on a scalar variable, 296
Vektor-Raum $\stackrel{E}{=}$ vector-space, 298
Vektorvariable = vektorwertige Variable $\stackrel{E}{=}$ vectorial variable, 280
velocity $\stackrel{G}{=}$ Geschwindigkeit, 105
velocity as a vector, 297
velocity as the derivative with respect to t, 150
velocity of circular motion, 344
velocity of light, 100
Verabredung $\stackrel{E}{=}$ convention, 38
Verhältnis $\stackrel{E}{=}$ ratio, 30

Vermutung $\stackrel{\mathbf{E}}{=}$ guess, 107
verschieben $\stackrel{\mathbf{E}}{=}$ shifted, 55
Verschiebung $\stackrel{\mathbf{E}}{=}$ displacement, 291
Verschiebung, Verrückung $\stackrel{\mathbf{E}}{=}$ displacement, 136
verschwinden $\stackrel{\mathbf{E}}{=}$ vanish, 78
vertauschen $\stackrel{\mathbf{E}}{=}$ interchange, 35
verteilen $\stackrel{\mathbf{E}}{=}$ distributing, 73
vertices $\stackrel{\mathbf{G}}{=}$ Scheitel, 51
vieldeutig $\stackrel{\mathbf{E}}{=}$ ambiguous, 176
vieldeutig $\stackrel{\mathbf{E}}{=}$ multivalued, 13
Vieleck $\stackrel{\mathbf{E}}{=}$ polygons, 34
vollwertig $\stackrel{\mathbf{E}}{=}$ fully-fledged, 214
volume
    hyperparallelepipedon, 322
    orientation of, 331
    parallelepipedon, 322
volume expressed by a wedge product, 331
Voraussage $\stackrel{\mathbf{E}}{=}$ guess, 158
Vorbereitung $\stackrel{\mathbf{E}}{=}$ preliminary, 227
Vorfaktor $\stackrel{\mathbf{E}}{=}$ prefactor, 67
vorhergehend $\stackrel{\mathbf{E}}{=}$ previous, 56
Vorzeichen $\stackrel{\mathbf{E}}{=}$ sign, 5

Wachstumsgleichung $\stackrel{\mathbf{E}}{=}$ growth equation, 253
Wachstumskonstante $\stackrel{\mathbf{E}}{=}$ growth-constant, 67
Wahrscheinlichkeit $\stackrel{\mathbf{E}}{=}$ probability, 88
Wasserstoff $\stackrel{\mathbf{E}}{=}$ hydrogen, 100
Wasserstoffatom $\stackrel{\mathbf{E}}{=}$ hydrogen atom, 102
wedge $\stackrel{\mathbf{G}}{=}$ Keil, 331
wedge product $\stackrel{\mathbf{G}}{=}$ Spatprodukt, 331
wedge product and determinants, 332
Wendepunkt $\stackrel{\mathbf{E}}{=}$ flex-point, 168
Wendepunkte $\stackrel{\mathbf{E}}{=}$ flex-points, 192
Wert $\stackrel{\mathbf{E}}{=}$ value, 11
Wertebereich $\stackrel{\mathbf{E}}{=}$ range, 12, 192
willkürlich $\stackrel{\mathbf{E}}{=}$ arbitrarily, 14
willkürlichen $\stackrel{\mathbf{E}}{=}$ arbitrary, 239
Winkelböglein $\stackrel{\mathbf{E}}{=}$ arc-arrow, 6
Winkelgeschwindigkeit $\stackrel{\mathbf{E}}{=}$ angular velocity, 29, 347
Winkelgeschwindigkeitsvektor $\stackrel{\mathbf{E}}{=}$ angular velocity vector, 328
Winkelhalbierende $\stackrel{\mathbf{E}}{=}$ bisection of angles, 14
Winkelhalbierende $\stackrel{\mathbf{E}}{=}$ bisectrix of the angle, 35
Winkelhalbierende $\stackrel{\mathbf{E}}{=}$ half angle line, 223
Wippe, Schaukel $\stackrel{\mathbf{E}}{=}$ see-saw, 239
wire $\stackrel{\mathbf{G}}{=}$ Draht, 16
Wirkungslinie $\stackrel{\mathbf{E}}{=}$ line of action, 296
Wirtschaftswissenschaftler $\stackrel{\mathbf{E}}{=}$ economist, 168
Würfel $\stackrel{\mathbf{E}}{=}$ cube, 68, 229
Wurzel, Approximationen, 81
Wurzel, Potenzreihen, 81

$\mathbb{Z}$, 12, integers98
Zahlengerade $\stackrel{\mathbf{E}}{=}$ real axis, 99
Zähler $\stackrel{\mathbf{E}}{=}$ numerator, 209
Zehner-Logarithmus $\stackrel{\mathbf{E}}{=}$ decimal logarithm, 91
Zeiger as complex numbers, 364
Zeilen $\stackrel{\mathbf{E}}{=}$ rows, 75
Zeit-Spanne $\stackrel{\mathbf{E}}{=}$ lapse, 94
Zeitverschiebung $\stackrel{\mathbf{E}}{=}$ time-lag, 24
Zentralkraft $\stackrel{\mathbf{E}}{=}$ central force, 357
Zentralsymmetrie $\stackrel{\mathbf{E}}{=}$ central-symmetry, 326
Zentriwinkel $\stackrel{\mathbf{E}}{=}$ centriangle, 2
Zerfallskonstante $\stackrel{\mathbf{E}}{=}$ decay-constant, 67
Zerfall $\stackrel{\mathbf{E}}{=}$ decay, 94
zerfallende, abnehmende $\stackrel{\mathbf{E}}{=}$ decaying, 151
Zerfallsgleichung $\stackrel{\mathbf{E}}{=}$ decay-equation, 253
Zerfallskonstante $\stackrel{\mathbf{E}}{=}$ decay-constant, 253
zero crossing $\stackrel{\mathbf{G}}{=}$ Nulldurchgang, 37
zero-passage $\stackrel{\mathbf{G}}{=}$ Nulldurchgang, 24
zeroes $\stackrel{\mathbf{G}}{=}$ Nullstellen, 11
zeroes of sin, 11

zerquetschen $\stackrel{\mathbf{E}}{=}$ squashing, 203
zerreißen $\stackrel{\mathbf{E}}{=}$ tearing, 203
Zirkel $\stackrel{\mathbf{E}}{=}$ compass, 51, 283
Zufall $\stackrel{\mathbf{E}}{=}$ coincidence, 87
zufällige Fehler $\stackrel{\mathbf{E}}{=}$ random errors, 159
Zuordnung $\stackrel{\mathbf{E}}{=}$ allocation, 14
zur Verfügung stehend $\stackrel{\mathbf{E}}{=}$ available, 155
zusammengefallen $\stackrel{\mathbf{E}}{=}$ coalesced, 360
Zusammensetzung von Funktionen $\stackrel{\mathbf{E}}{=}$ composition of functions, 36
Zuwächse $\stackrel{\mathbf{E}}{=}$ increments, 136
Zwangskraft $\stackrel{\mathbf{E}}{=}$ coercion force, 354
zweideutig $\stackrel{\mathbf{E}}{=}$ ambiguous, 91
Zweideutigkeit $\stackrel{\mathbf{E}}{=}$ ambiguity, 276
Zweier-Logarithmus $\stackrel{\mathbf{E}}{=}$ dual logarithm, 91